Solutions to Problem Sheet 5. MSM3A05b/MSM4A05b. Nonlinear Systems and Chaos

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Question 1.

(i)The Jacobian of the lorenz system is given by

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}.$$

At the points Q^+ and Q^- we have $(x, y, z) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1) = (\alpha, \alpha, r-1)$ we have

$$J - \lambda I = \begin{pmatrix} -\sigma-\lambda & \sigma & 0 \\ 1 & -1-\lambda & -\alpha \\ \alpha & \alpha & -b-\lambda \end{pmatrix},$$

and so

$$|J - \lambda I| = -(\lambda+\sigma)\left[(\lambda+1)(\lambda+b)+\alpha^2\right] - \sigma(\alpha^2 - b - \lambda) = 0.$$

This can be simplified to yield

$$\lambda^3 + (1+b+\sigma)\lambda^2 + (\sigma+r)b\lambda + 2\sigma b(r-1) = 0,$$

as required.
(ii) If we substitute in $\lambda = i\omega$ then equate real and imaginary parts we obtain the result -as shown in lectures.
(iii) The third eigenvalue can be found by inspection - that is if A, B and C are the eigenvalues then the characteristic equation will be

$$(\lambda - A)(\lambda - B)(\lambda - C) = 0,$$

where we know that $A = i\omega$ and $B = -i\omega$ and so we have

$$(\lambda^2 + \omega^2)(\lambda - C) = 0.$$

From part (ii) we have $\omega^2 = 2\sigma b(r-1)/(1+b+\sigma)$ and so the above equation becomes

$$\left(\lambda^2 + \frac{2\sigma b(r-1)}{(1+b+\sigma)}\right)(\lambda - C) = 0,$$

and thus comparing with the original characteristic equation we have

$$\frac{2\sigma b(r-1)C}{(1+b+\sigma)} = -2\sigma b(r-1),$$

and so $C = -(1+\sigma+b)$.

Question 2.

(a) To show that the Rikitake system is dissipative we consider

$$\dot{V} = \int_V \nabla \cdot \mathbf{f} dV.$$

From the system of equations we see that we have

$$\nabla \cdot \mathbf{f} = \frac{\partial}{\partial x}(-vx + zy) + \frac{\partial}{\partial y}(-vy + (z-a)x) + \frac{\partial}{\partial z}(1 - xy) = -v - v = -2v < 0.$$

We therefore have shown that this system is dissipative and furthermore we have

$$\dot{V} = -2vV \Rightarrow V(t) = V(0)e^{-2vt}.$$

(b) To find the fixed points we must solve

$$\begin{aligned} -vx + zy &= 0, & (1) \\ -vy + (z-a)x &= 0, & (2) \\ 1 - xy &= 0. & (3) \end{aligned}$$

From the last equations we have $x = k$ and $y = k^{-1}$ and from the first equation we have $z = vk^2$ with the second equation giving us a condition on k namely $v(k^2 - k^{-2}) = a$. Thus the fixed points are

$$x = \pm k, y = \pm k^{-1}, z = vk^2.$$

(c) In order to classify the stability of fixed points we can linearize about the equilibrium points etc but the Jacobian of the system is given by

$$J = \begin{pmatrix} -v & z & y \\ z-a & -v & x \\ -y & -x & 0 \end{pmatrix}.$$

We have that the trace of the above matrix (that is the sum of the diagonal elements) is $-2v < 0$ and also $\triangle = det(J)$ is given by

$$-2xyz - v(x^2 + y^2) + axy = -2vk^2 - v(k^2 + k^{-2}) + a = -2v(k^2 + k^{-2}) < 0.$$

These two quantities replace the p and $\triangle$ variables we used for the two dimensional case. Thus these points are saddle points.

JU 25/03/11.