

# MSM3A05a/MSM4A05a Problem Sheet 4. Nonlinear systems and Chaos

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## Question 1.

Consider the following system of equations

$$\frac{dx}{dt} = ay - y^2, \quad \frac{dy}{dt} = x - 2y + \frac{x^2}{2}.$$

(a) Clearly  $(0, 0)$  is an equilibrium point and we can determine its stability by linearizing about the origin, i.e., we consider the following system

$$\frac{dx}{dt} = ay, \quad \frac{dy}{dt} = x - 2y,$$

which leads to the Jacobian

$$J = \begin{pmatrix} 0 & a \\ 1 & -2 \end{pmatrix},$$

leading to the eigenvalues

$$\lambda_{\pm} = -1 \pm (1 + a)^{\frac{1}{2}}.$$

If  $a > 0$  then  $\lambda_+ \lambda_- < 0$  and hence we have a saddle and this is unstable. The unstable normal mode may be determined by finding the eigenvector corresponding to the eigenvalue  $\lambda_+$  and thus using  $\lambda_+ = a/2$  as  $a \rightarrow 0$  we have  $J(u, v)^T = \lambda_+(u, v)^T$  which gives  $2v = u$  and so the unstable normal mode behaves like

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{\frac{at}{2}},$$

as  $a \rightarrow 0$ .

(b) We consider the three cases separately, they are

(i)  $a < -1$ , here we have a stable spiral,

(ii)  $-1 < a < 0$ , this is a stable node,

(iii)  $a > 0$ , this is the saddle

(c) The other equilibrium points are given by solving  $y(a - y) = 0$  and  $x - 2y + x^2/2$  simultaneously and so we have  $(-2, 0)$  is an equilibrium point and so are  $(\Omega_{\pm}, a)$  where  $\Omega_{\pm} = -1 \pm \sqrt{1 + 4a}$ . The first of these exists for all  $a$  whilst the latter two only exist when  $a > -1/4$ . In order to determine stability of these points we need to linearize about each point. Let us consider the linearized version around  $(-2, 0)$  we then have the system

$$\dot{y} = -2y - s, \quad \dot{s} = ay,$$

where  $s = x + 2$ . Thus the Jacobian is given by

$$J = \begin{pmatrix} 0 & a \\ -1 & -2 \end{pmatrix},$$

whence we have  $\lambda_{\pm} = -1 \pm \sqrt{1 - a}$ . This equilibrium will be unstable if one of the eigenvalues is unstable - that is if  $\sqrt{1 - a} > 1$  and thus if  $a < 0$ . Next we consider the equilibrium points  $(\Omega_{\pm}, a)$ . Given the form for  $\Omega_{\pm}$  we see that these equilibrium points only exist when  $a > -1/4$ . Linearizing about these points we have  $s = x - \Omega_{\pm}$  and  $r = y - a$  and we then have

$$\dot{r} = s - 2r + \Omega_{\pm}s, \quad \dot{s} = -ar,$$

where we have neglected higher order terms. The Jacobian is thus given by

$$J = \begin{pmatrix} 0 & -a \\ 1 + \Omega_{\pm} & -2 \end{pmatrix},$$

and hence the characteristic equation is given by

$$\lambda^2 + 2\lambda + a(1 + \Omega_{\pm}) = 0.$$

The eigenvalues are thus given by

$$\lambda_{\pm} = -1 \pm \sqrt{1 - a(1 + \Omega_{\pm})}.$$

We seek values of  $a$  such that the above equilibrium point is unstable. Thus we require solutions of

$$\sqrt{1 - a(1 + \Omega_{\pm})} > 1 \Rightarrow a(1 + \Omega_{\pm}) < 0.$$

Hence for  $\Omega_+$  we seek  $a\sqrt{1 + 4a} < 0$  and so the valid range of instability is  $a \in (-1/4, 0)$ . For the case of  $\Omega_-$  we have  $-a\sqrt{1 + 4a} < 0$  and thus this equilibrium point is unstable if  $a > 0$ .

(d) We begin with the set of equations (these can easily be derived from the original set of equations by substituting in  $u = at$  and noting that only time derivatives change)

$$\begin{aligned} a \frac{dx}{du} - ay + y^2 &= 0, \\ x - 2y &= a \frac{dy}{du} - \frac{x^2}{2}, \\ 2x + y &= aA, \end{aligned}$$

and using the expansion given we have at leading order

$$\frac{dx_1}{du} - y_1 + y_1^2 = 0, \tag{1}$$

$$x_1 - 2y_1 = 0, \tag{2}$$

$$2x_1 + y_1 = A, \tag{3}$$

using (3) and (2) we can see that  $5x_1 = 2A$  and hence  $x_1 = 2A/5$  and so  $y_1 = A/5$ . Therefore using (1) we have

$$\frac{d}{du} \left( \frac{2A}{5} \right) + \frac{A^2}{25} - \frac{A}{5} = 0,$$

and since

$$\frac{dA}{du} = F_0(A) + aF_1(A) + \dots$$

we have

$$\left( \frac{2F_0}{5} \right) + \frac{A^2}{25} - \frac{A}{5} = 0, \Rightarrow F_0 = \frac{A}{2} - \frac{A^2}{10}.$$

At next order we have

$$\frac{dx_2}{du} - y_2 + 2y_2y_1 = 0, \tag{4}$$

$$x_2 - 2y_2 = \frac{dy_1}{du} - \frac{1}{2}x_1^2, \tag{5}$$

$$2x_2 + y_2 = 0. \tag{6}$$

Using the expression for  $x_1, y_1$  we have

$$\frac{dy_1}{du} = \frac{1}{5}F_0 = \frac{A}{10} - \frac{A^2}{50},$$

and so using (5) and (6) we have

$$5x_2 = \frac{A}{10} - \frac{A^2}{50} - \frac{4A^2}{50} = \frac{A}{10} - \frac{A^2}{10} \Rightarrow x_2 = \frac{1}{50}(A - A^2),$$

and so  $y_2 = -2/50(A - A^2)$ .

## Question 2.

The system of first order differential equations is given by

$$\dot{x} = y, \tag{7}$$

$$\dot{y} = x - x^3 - cy + \Gamma \sin(\Omega t). \tag{8}$$

With  $\Gamma = 0$  the fixed points are given by  $(-1, 0)$ ,  $(0, 0)$  and  $(1, 0)$ . The Jacobian matrix is given by

$$J = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & -c \end{pmatrix}, \tag{9}$$

with eigenvalues

$$\lambda_{\pm} = \frac{1}{2} \left( -c \pm \sqrt{c^2 - 4(3x^2 - 1)} \right).$$

We see that for  $x = 0$ , i.e.  $(0, 0)$  we have  $\lambda_+ > 0 > \lambda_-$  and hence the origin is a saddle. When  $x = \pm 1$  we have  $\lambda_{\pm} = -c \pm \sqrt{c^2 - 8}$ . We see therefore that for  $c > 2\sqrt{2}$  the eigenvalues are real and  $\lambda_+ < \lambda_- < 0$  and hence the points  $(\pm 1, 0)$  are stable nodes. When  $0 < c < 2\sqrt{2}$  the eigenvalues are complex with negative real part and hence they correspond to stable spirals. For  $c = 0$  the eigenvalues are purely imaginary and so the equilibrium points  $(\pm 1, 0)$  correspond to centres.