MSM3A05a/MSM4A05a Solution Sheet 3. Nonlinear systems and Chaos

Question 1. The system $\dot{x} = y$, $\dot{y} = x - 3x^5$ has an equilibrium point at $(0, 0)$ (which is a saddle point) and another at $(\pm 3^{\frac{1}{4}}, 0)$ which are centers. The differential equation for the phase paths are

$$\frac{dy}{dx} = \frac{x - 3x^5}{y},$$

which can be integrated to give the phase paths

$$\frac{y^2}{2} = \frac{x^2}{2} - \frac{x^6}{2} + C.$$

Homoclinic paths must therefore originate at the saddle and end at the saddle point. We therefore require that $C = 0$, that is

$$y^2 = x^2 - x^6,$$

which intersect the x-axis at $x = \pm 1$. Time solutions may be obtained by integrating

$$y = \frac{dx}{dt} = \pm x\sqrt{1 - x^4} \Rightarrow \int \frac{dx}{x\sqrt{1 - x^4}} = \pm \int dt = \pm(t - t_0).$$

If we use the substitution $u = 1/x^2$, the integral can be made more like the integral given as a hint, i.e.,

$$-\int \frac{du}{2\sqrt{u^2 - 1}} = \pm(t - t_0).$$

Hence for $x > 0$ we have

$$-\frac{1}{2}\cosh^{-1} u = \pm(t - t_0) \Rightarrow u = \cosh[\mp 2(t - t_0)] \Rightarrow x = \sqrt{\operatorname{sech}[2(t - t_0)]},$$

since *sech* is an even function. Similarly for $x < 0$ we have $x = -\sqrt{\operatorname{sech}[2(t - t_0)]}$.

Question 2.

Consider the system of ordinary differential equations

$$\begin{aligned} \frac{dx}{dt} &= x - y - y^3 - 2x^5 - 2x^3y^2 - xy^4, \\ \frac{dy}{dt} &= x + y + xy^2 - 2yx^4 - 2y^3x^2 - y^5. \end{aligned}$$

These can be converted into polars so that we have

$$\frac{dr}{dt} = \frac{\dot{x}x + \dot{y}y}{r} = \dot{x}\cos\theta + \dot{y}\sin\theta =$$
$$\cos\left(r\cos\theta - r\sin\theta - r^3\sin^3\theta - 2r^5 r\cos^5\theta - 2r^5\sin\theta\cos^4\theta - r^5\cos\theta\sin^4\theta\right) +$$
$$\sin\theta\left(r\cos\theta + r\sin\theta + r^3\sin^2\theta\cos\theta - 2r^5 r\cos^4\theta\sin\theta - 2r^5\sin^3\theta\cos^2\theta - r^5\sin^5\theta\right)$$
$$= r\left(1 - r^5(1+\cos^6\theta)\right).$$

Now $1 < (1+\cos^6\theta) < 2$ and so if we choose $r < 2^{\frac{1}{5}}$ we have $\dot{r} > 0$ whilst for $r > 1$ we have $\dot{r} < 0$ hence the annulus $2^{\frac{1}{5}} < r < 1$ has trajectories which enter into it (nothing can leave it). If we can show that there are no equilibrium points in this annulus we have the conditions of Poincare-Bendixson and so a limit cycle exists. To show that there are no equilibrium points in this region we need to consider $\dot{\theta}$ and show that in this region it is never zero (do this as an exercise).

QUESTION 3. The van der Pol equation is given by

$$\begin{aligned}
\dot{a} &= X(a,b) = \frac{\epsilon}{2}\left(1 - \frac{r^2}{2}\right)a - \frac{\omega^2 - 1}{2\omega}b, \\
\dot{b} &= Y(a,b) = \frac{\epsilon}{2}\left(1 - \frac{r^2}{2}\right)b + \frac{\omega^2 - 1}{2\omega}a + \frac{\Gamma}{2\omega},
\end{aligned}$$

where $r^2 = x^2 + y^2$. We use Bendixson's criterion to yield

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = \epsilon\left(1 - \frac{r^2}{2}\right).$$

Hence for $r < \sqrt{2}$ this is of one sign and so there can be no closed paths in the (a,b) plane.

JU 31/01/13.