MSM3A05a/MSM4A05a Solution Sheet 2. Nonlinear systems and Chaos

QUESTION 1.

We note that

$$|f'(x^*)| = \lim_{x\to x^*} \frac{|f(x) - f(x^*)|}{|x - x^*|},$$

and then we suppose that we have $|f'(x^*)| < c < 1$. Since $f(x)$ is smooth we must have aneighbourhood of x^*, $N(x^*)$, such that

$$\frac{|f(x) - f(x^*)|}{|x - x^*|} \le c \qquad for\ \ all \quad x \in N(x^*).$$

We know that $f(x^*) = x^*$ and so we must have $|f(x) - x^*| \le c|x - x^*|$ for all $x \in N(x^*)$. However, $0 < c < 1$ and so we deduce that $f(x)$ is *closer* to x^* than is x is close to x^*. We can extend this process, specifically by substituting $f(x)$ for x in the above expression to obtain

$$|f^{(2)}(x) - x^*| \le c|f(x) - x^*| \le c^2|x - x^*|.$$

We can then use induction to show that

$$|f^{(k)}(x) - x^*| \le c^k|x - x^*|,$$

and therefore we have

$$\lim_{k\to\infty} |f^{(k)}(x) - x^*| = 0,$$

as required.

QUESTION 2.

(a) We have that

$$(f^2)'(x) = f'(f(x))f'(x),$$

whence substituting $x = x^*$ we have

$$f'(f(x^*))f'(x^*) = f'(x^*)f'(x^*) = (f'(x^*))^2 \ge 0,$$

where we have used $f(x^*) = x^*$. (b) It should be clear that if x^* is a fixed point of $f(x)$ then $f^k(x^*) = x^*$ for all $k \ge 1$ and so a fixed point of the map $f(x)$ is a fixed point for any iteration of that map $f^k(x)$ for all $k \ge 1$. In order to determine the stability of this fixed point we need to determine $(f^k)'(x^*)$ which by repeated application of the chain rule (to see this try this for $(f^3)(x)$ and then $(f^4)(x)$ etc) is $(f'(x^*))^k$ and so

$$\left|(f^k)'(x^*)\right| = |f'(x^*)|^k.$$

It is trivial to show that if $k \ge 1$ then $|\alpha|^k < 1$ ($|\alpha|^k > 1$) if and only if $|\alpha| < 1$ ($|\alpha| > 1$) therefore we have $\left|(f^k)'(x^*)\right| = |f'(x^*)|^k < 1$ if and only if $|f'(x^*)| < 1$ and similarly for the case greater than 1. Thus a fixed point of f^k is stable (unstable) if and only if the fixed point is stable (unstable) for f.

Question 3. The Tent map is given by $x_{n+1} = T_c(x_n)$, where

$$T_c(x) = c\min(x, 1-x) = \begin{cases} cx & if x \leq \frac{1}{2} \\ c(1-x) & if x > \frac{1}{2} \end{cases}$$

where $c > 0$.

(a) When $0 < c < 1$ we have $T_c(x^*) = 0$ and $|T_c'(x)| = c < 1$ and so $|T_c'(0)| < 1$.

(b) In this case since $c > 1$ we have $|T_c'(0)| > 1$ which shows that the origin is now a source (i.e. it is unstable). There is another fixed point found by solving $T_c(x^*) = x^*$. When $c > 1$ we have $0.5 < x^* < 1$ so that $T_c(x^*) = c(1-x^*) = x^*$ and so

$$x^* = \frac{c}{c+1},$$

and since $|T_c'(x)| > 1$ for all x this is also a source.

(c) Since $|T_c'(x)| = c$ for all x we have

$$\sum_{j=0}^{n-1} \ln |T_c'(x_j)| = n \ln c$$

and so

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln |T_c'(x_j)| = \ln c.$$

Question 4. This question will be considered during the example class.

JU 29/01/13.