MSM3A05a/MSM4A05a Problem Sheet 4. Nonlinear systems and Chaos

QUESTION 1.

This question is about a weakly nonlinear theory of a transcritical bifurcation. Consider the system

$$\frac{dx}{dt} = ay - y^2, \qquad \frac{dy}{dt} = x - 2y + \frac{1}{2}x^2,$$

(a) Show that $(x, y) = (0, 0)$ is a solution for all a. Show that it is unstable for $a > 0$ and an unstable normal mode behaving like $[2, 1]^T e^{\frac{at}{2}}$ as $a \to 0$.
(b) Sketch the phase portrait for solutions near $(x, y) = (0, 0)$ for the cases $a < -1$,$-1 < a < 0$ and $a > 0$ naming the equilibrium point in each case.
(c) Find the other equilibrium points stating the values of a for which they exist and the values of a for which they are unstable.
(d) To investigate the nonlinear solutions near one bifurcation, show that the system becomes

$$a\frac{dx}{dy} - ay + y^2 = 0, \qquad x - 2y = a\frac{dy}{du} - \frac{x^2}{2},$$

when $u = at$. Defining A such that $2x + y = aA$ and assuming a solution may expanded out as

$$x(a, t) = ax_1(u) + a^2x_2(t) + \cdots, \qquad y(a, t) = ay_1(u) + a^2y_2(t) + \cdots,$$

as $a \to 0$ where

$$\frac{dA}{du} = F_0(A) + aF_1(A) + \cdots$$

show that $(x_1, y_1) = (2A/5, A/5)$ and $F_0(A) = A/2 - A^2/10$. Find (x_2, y_2) in terms of A.

QUESTION 2. Consider the equation

$$\ddot{x} + c\dot{x} - x + x^3 = \Gamma \sin(\Omega t),$$

where $c > 0$ is a damping constant. Express the above equation as a system of two first order differential equations. In the case where $\Gamma = 0$ locate and identify all the fixed points. In particular you should consider all the cases where $c \geq 0$.

JU 01/02/13.