MSM3A05a/MSM4A05a Problem Sheet 2. Nonlinear systems and Chaos

QUESTION 1.

Let f be a map in R with fixed point x^*. Show that if $|f'(x^*)| < 1$ then x^* is a stable fixed point.
You may wish to begin by noting that

$$|f'(x^*)| = \lim_{x \to x^*} \frac{|f(x) - f(x^*)|}{|x - x^*|},$$

and then supposing that we have $|f'(x^*)| < c < 1$ and then by considering $\lim_{k\to\infty} |f^{(k)}(x) - x^*|$.

QUESTION 2.

Let x^* be a fixed point of a map $f(x)$
(a) Show that $(f^2)'(x^*) \geq 0$.
(b) Show that x^* is a stable (unstable) fixed point of $f(x)$ *if and only if* it is a stable (unstable) fixed point of $f^k(x)$ for all $k \geq 1$

QUESTION 3.

Consider the the Tent map given by $x_{n+1} = T_c(x_n)$, where

$$T_c(x) = c\min(x, 1-x) = \begin{cases} cx & if x \leq \frac{1}{2} \\ c(1-x) & if x > \frac{1}{2} \end{cases}$$

where $c > 0$.
(a) Show that when $0 < c < 1$ the Tent map has a fixed point $x^* = 0$ such that $|f'(x^*)| < 1$.
(b) Show that when $c > 1$ show that the origin is now a source (i.e. it is unstable) and show that there is another fixed point which is a source also. Draw a sketch in this case.
(c) Show that the Lyapunov exponent is given by $\lambda(x_0) = \ln c$.

QUESTION 4. Let $b = r/s$ be a rational number in the interval $(0, 1)$. Show that b is a periodic point of the Tent map $T_2(x)$ if and only if r is an even integer and s is an odd integer.

JU 24/01/13.