Figure 1: A plot $y = |a\cos\theta|$ and $y = 1$.

Solutions to Assessed Problem Sheet 2. MSM3A05b/MSM4A05b. Nonlinear Systems and Chaos

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Question 1.

(a) We have the following system

$$\ddot{x} + \epsilon f(x)\dot{x} + x = 0, \tag{1}$$

where

$$f(x) = \begin{cases} 1 & : \quad |x| > 1 \\ -1 & : \quad |x| \leq 1 \end{cases} \tag{2}$$

Let us assume that a limit cycle exists of the form $a = a\cos t$ (we can just assume $t_0 = 0$). We then require evaluation of the integral (using the method of averaging)

$$I = \int_0^{2\pi} f(a\cos t) \cdot -a\sin t \cdot \sin t dt = 0.$$

To ease the algebra we may use the substitution $\theta = t - \pi$, whence $\cos t = \cos(\pi + \theta) = -\cos\theta$ and $\sin t = -\sin\theta$. We therefore have

$$I = \int_{-\pi}^{\pi} f(-a\cos\theta) \cdot \sin^2\theta d\theta = 0,$$

where we have taken out $-a$ and assumed $a \neq 0$. We note that $f(-a\cos\theta)$ is symmetrical about $\theta = 0$ and so we have

$$\int_0^{\pi} f(-a\cos\theta) \cdot \sin^2\theta d\theta = 0,$$

A plot of the graph between $\theta = 0$ and $\theta = \pi$ reveals that the curve $y = |a\cos\theta|$ crosses the line $y = 1$ at the two points $\theta = \alpha = \cos^{-1}(1/a)$ and $\theta = \pi - \alpha$ (see Fig. 1). It is clear that $|a\cos\theta > 1$ (and so $f = 1$) for $\theta \in [0, \alpha] \cup [\pi - \alpha, \pi]$ and $|a\cos\theta < 1$ (and so $f = -1$) for $\theta \in [\alpha, \pi - \alpha]$. We therefore have

$$\int_0^{\alpha} -1 \cdot \sin^2\theta d\theta + \int_{\alpha}^{\pi-\alpha} +1 \cdot \sin^2\theta d\theta + \int_{\pi-\alpha}^{\pi} -1 \cdot \sin^2\theta d\theta = 0.$$

Using the relation

$$\int \sin^2\theta d\theta = \frac{1}{4}\sin 2\theta - \frac{\theta}{2},$$

we have

$$-\left[\frac{1}{4}\sin 2\theta - \frac{\theta}{2}\right]_0^{\alpha} + \left[\frac{1}{4}\sin 2\theta - \frac{\theta}{2}\right]_{\alpha}^{\pi-\alpha} - \left[\frac{1}{4}\sin 2\theta - \frac{\theta}{2}\right]_{\pi-\alpha}^{2\pi} = 0.$$

This can be simplified to

$$4\alpha = 2\sin 2\alpha + \pi,$$

where

$$0 < \alpha = \cos^{-1}\left(\frac{1}{a}\right) < \pi/2.$$

(b) To show that the differential equations are equal we use the system

$$\frac{dw}{dt} = \frac{x}{\epsilon}, \qquad \frac{dx}{dt} = -\epsilon(w + F(x)),$$

where $F(x) = -x$ when $|x| \leq 1$ and $F(x) = x - 2sgn(x)$ if $|x| > 1$. We therefore differentiate the second equation to yield

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\epsilon\left(\frac{dw}{dt} + F'(x)\frac{dx}{dt}\right), \\ &= -\epsilon\left(\frac{x}{\epsilon}\right) - \epsilon\dot{x}F'(x). \end{aligned}$$

Now $F'(x) = -1$ if $|x| \leq 1$ whilst $F'(x) = 1$ if $|x| > 1$ (note the *sgn* term does not affect differentiation). Thus we see that if $|x| \leq 1$ we have $\ddot{x} = -x + \epsilon\dot{x}$ and if $|x| > 1$ we have $\ddot{x} = -x - \epsilon\dot{x}$ which is the form of the original equation (taking into account the different values of f).
(d) A sketch of the curve $w = -F(x)$ is shown in Fig. 2 (below) along with the locus of points describing the limit cycle.

(e) The period of the limit cycle

$$\begin{aligned} T(\epsilon) &= \int dt = \epsilon\int\frac{dw}{x}, \\ &\sim 2\epsilon\int_A^D \frac{dw}{x} = 2\epsilon\int_A^D -\frac{F'(x)}{x}dx, \\ &\sim 2\epsilon\int_A^B -\frac{1}{x}dx = -2\epsilon[\ln x]_3^1 = 2\epsilon\ln 3. \end{aligned}$$

JU 08/02/11.