Embeddings in graphs via degree sequence conditions

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Includes joint work with Joseph Hyde and Hong Liu; Fiachra Knox; Katherine Staden.

Minimum degree conditions

Question

What minimum degree condition forces a graph to contain a given spanning substructure?

Theorem (Dirac 1952)

$\delta(G) \geq |G|/2 \implies G$ *contains a Hamilton cycle.*

- Easy to see minimum degree is best-possible
- However, can significantly improve on Dirac...

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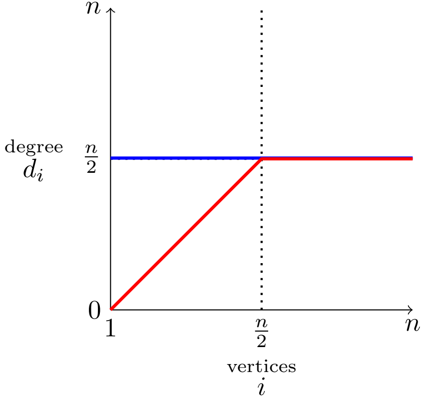
Theorem (Pósa 1963)

Let G be a graph with degree sequence $d_1 \leq \cdots \leq d_n$. G is *Hamiltonian* *if*

$$d_i \geq i + 1 \quad \forall\ i < n/2.$$



Pósa's theorem

Theorem (Pósa 1963)

Let G be a graph with degree sequence $d_1 \leq \cdots \leq d_n$. G is Hamiltonian if

$$d_i \geq i+1 \quad \forall\, i < n/2.$$

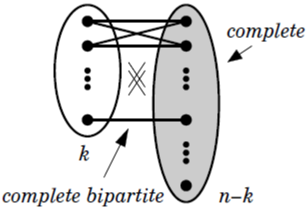
- Much stronger than Dirac's theorem
- Condition best-possible in sense cannot replace with $d_i \geq i$ even for a *single* value of i

Theorem (Chvátal 1972)

Let G be a graph with degree sequence $d_1 \leq \cdots \leq d_n$. G is Hamiltonian if

$$d_i \geq i+1 \quad \text{or} \quad d_{n-i} \geq n-i \qquad \forall\, i < n/2.$$

- Chvátal's theorem characterises all those 'Hamiltonian degree sequences'.



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Why study degree sequence conditions?

- Prove much more general analogues of classical minimum degree results
- Provides a useful setting to refine/develop methods (e.g. developing absorbing and regularity methods to deal with 'small' degree vertices)

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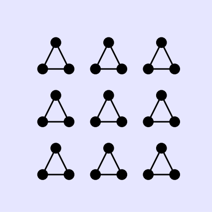
Perfect tilings in graphs

- An *H-tiling* in G is a collection of vertex-disjoint copies of H in G.
- An H-tiling is *perfect* if it covers all vertices in G.



Theorem (Hajnal, Szemerédi 1970)

G graph, $|G| = n$ where $r|n$ and

$$\delta(G) \geq (1 - 1/r)\, n$$

$\Longrightarrow$ *G contains a perfect K_r-tiling.*

Degree sequence conditions

Conjecture (Balogh, Kostochka and T. 2013)

G graph, $|G| = n$ where $r|n$ with degree sequence $d_1 \leq \cdots \leq d_n$ such that:

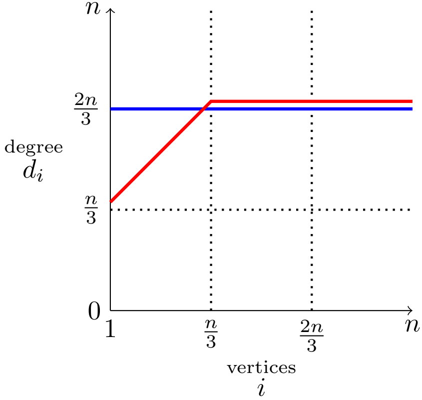
(α) $d_i \geq (1-2/r)n + i$ *for all* $i < n/r$;

(β) $d_{n/r+1} \geq (1-1/r)n$.

$\implies$ *G contains a perfect K_r-tiling.*

- If true, stronger than Hajnal–Szemerédi since n/r vertices allowed 'small' degree.
- If true, best-possible.

T. (2016) asymptotically resolved the conjecture.



$$d_i \geq (1 - 2/r + \eta)\, n + i \quad \forall\ i < n/r$$

- Komlós (2000) asymptotically determined the minimum degree threshold that forces an *H-tiling covering an xth proportion* of the vertices of G for *all* graphs H and all $x \in (0, 1)$.
- Komlós's bound depends on the so-called *critical chromatic number* of H
- Very recently, Piguet and Saumell (2018+) and Hyde, Liu, T. (2018+) proved different types of degree sequence versions of this result.

Tilings in graphs

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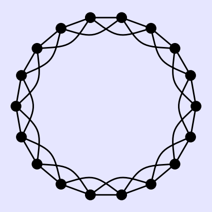
Powers of Hamilton cycles



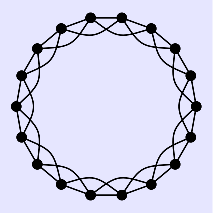
Conjecture (Pósa 1962)

G n-vertex and

$$\delta(G) \geq 2n/3$$

$\implies$ *G contains square of a Hamilton cycle*

Proved for large graphs by Komlós, Sárközy and Szemerédi (1996)



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Powers of Hamilton cycles

Theorem (Staden and T. 2017)

$\forall\ \eta > 0\ \exists\ n_0 \in \mathbb{N}$ s.t. if G on $n \geq n_0$ vertices with

$$d_i \geq \left(\frac{1}{3} + \eta\right) n + i \quad \textit{for all } i \leq \frac{n}{3}$$

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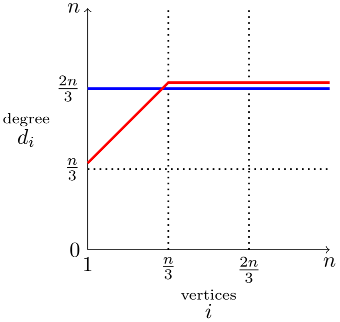

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$\Longrightarrow$ G contains the square of a Hamilton cycle.

- Doesn't quite imply Komlós–Sárközy–Szemerédi
- Up to error terms, the 'slope' is best-possible
- Perhaps surprisingly ηn cannot be replaced by $o(\sqrt{n})$ here!

Open problem

Prove a version for kth powers of Hamilton cycles

Embedding spanning trees

Theorem (Komlós, Sárközy and Szemerédi 1995)

$\forall\ \gamma > 0,\ \Delta \in \mathbb{N},\ \exists\ n_0 \in \mathbb{N}$ s.t. if G is n-vertex where $n \geq n_0$ and

$$\delta(G) \geq (1/2 + \gamma)n$$

$\implies$ G contains every spanning tree T with $\Delta(T) \leq \Delta$.

Theorem (Knox, T. 2013)

$\forall\ \gamma > 0,\ \Delta \in \mathbb{N},\ \exists\ n_0 \in \mathbb{N}$ s.t. if G is n-vertex where $n \geq n_0$ and

$$d_i \geq i + \gamma n \qquad \forall\ i < n/2$$

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In fact proved a much more general bipartite bandwidth theorem.

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Theorem (Komlós, Sárközy and Szemerédi 2001)

$\forall\ \gamma > 0$, $\exists\ n_0 \in \mathbb{N}$, $c > 0$ s.t. if G is n-vertex where $n \geq n_0$ and

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$\implies$ *G contains every spanning tree T with $\Delta(T) \leq cn/\log n$.*

- $\Delta(T)$ condition best-possible.

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Prove a degree sequence version of this result!

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Further research directions

- Prove a degree sequence version of the Bandwidth theorem (a special case of Knox-T. (2013) resolves the bipartite case)
- Directed graphs
  - e.g. the Nash–Williams conjecture for Hamilton cycles
  - Asymptotic results due to Kühn, Osthus, T. (2010); Christofides, Keevash, Kühn and Osthus (2010).
- Hypergraphs
  - e.g. perfect matching, Hamilton cycles, tilings...