ON DEGREE SEQUENCES FORCING THE SQUARE OF A HAMILTON CYCLE

KATHERINE STADEN AND ANDREW TREGLOWN

Abstract. A famous conjecture of Pósa from 1962 asserts that every graph on n vertices and with minimum degree at least $2n/3$ contains the square of a Hamilton cycle. The conjecture was proven for large graphs in 1996 by Komlós, Sárközy and Szemerédi [25]. In this paper we prove a degree sequence version of Pósa's conjecture: Given any $\eta > 0$, every graph G of sufficiently large order n contains the square of a Hamilton cycle if its degree sequence $d_1 \leq \cdots \leq d_n$ satisfies $d_i \geq (1/3+\eta)n + i$ for all $i \leq n/3$. The degree sequence condition here is asymptotically best possible. Our approach uses a hybrid of the Regularity-Blow-up method and the Connecting-Absorbing method.

1. Introduction

One of the most fundamental results in extremal graph theory is Dirac's theorem [16] which states that every graph G on $n \geq 3$ vertices with minimum degree $\delta(G)$ at least $n/2$ contains a Hamilton cycle. It is easy to see that the minimum degree condition here is best possible. The *square* of a Hamilton cycle C is obtained from C by adding an edge between every pair of vertices of distance two on C. A famous conjecture of Pósa from 1962 (see [18]) provides an analogue of Dirac's theorem for the square of a Hamilton cycle.

Conjecture 1.1 (Pósa [18]). *Let G be a graph on n vertices. If $\delta(G) \geq 2n/3$, then G contains the square of a Hamilton cycle.*

Again, it is easy to see that the minimum degree condition in Pósa's conjecture cannot be lowered. The conjecture was intensively studied in the 1990s (see e.g. [19, 20, 21, 22, 23]), culminating in its proof for large graphs G by Komlós, Sárközy and Szemerédi [25]. The proof applies Szemerédi's Regularity lemma and as such the graphs G considered are extremely large. More recently, the lower bound on the size of G in this result has been significantly lowered (see [12, 31]).

Although the minimum degree condition is best possible in Dirac's theorem, this does not necessarily mean that one cannot significantly strengthen this result. Indeed, Ore [32] showed that a graph G of order $n \geq 3$ contains a Hamilton cycle if $d(x) + d(y) \geq n$ for all non-adjacent $x \neq y \in V(G)$. The following result of Pósa [33] provides a *degree sequence* condition that ensures Hamiltonicity.

Theorem 1.2 (Pósa [33]). *Let G be a graph on $n \geq 3$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$. If $d_i \geq i+1$ for all $i < (n-1)/2$ and if additionally $d_{\lceil n/2 \rceil} \geq \lceil n/2 \rceil$ when n is odd, then G contains a Hamilton cycle.*

Notice that Theorem 1.2 is significantly stronger than Dirac's theorem as it allows for almost half of the vertices of G to have degree less than $n/2$. A theorem of Chvátal [13] generalises Theorem 1.2 by characterising all those degree sequences which ensure the existence of a Hamilton cycle in a graph: Suppose that the degrees of a graph G are $d_1 \leq \cdots \leq d_n$. If $n \geq 3$ and $d_i \geq i+1$ or $d_{n-i} \geq n-i$ for all $i < n/2$ then G is Hamiltonian. Moreover, if $d_1 \leq \cdots \leq d_n$ is a degree sequence

Date: July 30, 2015.

that does not satisfy this condition then there exists a non-Hamiltonian graph G whose degree sequence $d'_1 \leq \cdots \leq d'_n$ is such that $d'_i \geq d_i$ for all $1 \leq i \leq n$.

Recently there has been an interest in generalising Pósa's conjecture. An 'Ore-type' analogue of Pósa's conjecture has been proven for large graphs in [11, 15]. A random version of Pósa's conjecture was proven by Kühn and Osthus in [29]. In [4], Allen, Böttcher and Hladký determined the minimum degree threshold that ensures a large graph contains a square cycle of a given length. The problem of finding the square of a Hamilton cycle in a pseudorandom graph has recently been studied in [3]. The focus of this paper is to investigate degree sequence conditions that guarantee a graph contains the square of a Hamilton cycle. This problem was raised in the arXiv version of [6]. The main result of this paper is the following degree sequence version of Pósa's conjecture.

Theorem 1.3. *Given any $\eta > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. If G is a graph on $n \geq n_0$ vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

$$d_i \geq n/3 + i + \eta n \quad \textit{for all} \quad i \leq n/3,$$

then G contains the square of a Hamilton cycle.

Note that Theorem 1.3 allows for almost $n/3$ vertices in G to have degree substantially smaller than $2n/3$. However, it does not quite imply Pósa's conjecture for large graphs due to the term ηn. A surprising facet of the problem is that the term ηn in Theorem 1.3 cannot be omitted completely. Indeed, an example from the arXiv version of [6] shows that the term ηn cannot even be replaced by $o(\sqrt{n})$ for every $i \leq n/3$. So in this sense Theorem 1.3 is close to best possible. (Extremal examples for Theorem 1.3 are discussed in more detail in Section 3.) We suspect though that the degrees in Theorem 1.3 can be capped at $2n/3$.

Conjecture 1.4. *Given any $\eta > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. If G is a graph on $n \geq n_0$ vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

$$d_i \geq \min\{n/3 + i + \eta n, 2n/3\} \quad \textit{for all} \quad i,$$

then G contains the square of a Hamilton cycle.

It would be of considerable interest to establish an analogue of Chvátal's theorem for the square of a Hamilton cycle, i.e., to characterise those degree sequences which force the square of a Hamilton cycle. However, in general, we believe that it would be extremely difficult to strengthen Theorem 1.3, and it is likely that several further new ideas would be required.

A well-known result of Aigner and Brandt [2] and Alon and Fischer [5] states that if G is a graph on n vertices with minimum degree $\delta(G) \geq (2n-1)/3$ then G contains every graph H on n vertices with maximum degree $\Delta(H) \leq 2$. This resolves a special case of the famous Bollobás–Eldridge–Catlin Conjecture [7, 10]. (A conjecture of El-Zahar [17], that was proven for large graphs by Abbasi [1], implies that for many graphs H with $\Delta(H) \leq 2$, the minimum degree condition here can be substantially lowered.) Since a square path on n vertices contains any such graph H, an immediate consequence of Theorem 1.3 is the following degree sequence result.

Corollary 1.5. *Given any $\eta > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a graph on $n \geq n_0$ vertices such that $\Delta(H) \leq 2$. If G is a graph on n vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

$$d_i \geq n/3 + i + \eta n \quad \textit{for all} \quad i \leq n/3,$$

then G contains H.

The case when H is a triangle factor was proved in [38], and in fact this result is used as a tool in the proof of Theorem 1.3 (see Section 5).

A key component of the proof of Theorem 1.3 is a special structural embedding lemma (Lemma 7.2), which is likely to be of independent interest. In particular, we believe that it could have applications to other embedding problems (see Section 10 for further discussion).

The proof of Theorem 1.3 makes use of Szemerédi's Regularity lemma [37] and the Blow-up lemma [26]. In Section 2 we give a detailed sketch of the proof. We discuss extremal examples for Theorem 1.3 in Section 3. After introducing some notation and preliminary results in Section 4, we prove Theorem 1.3 in Sections 5–9.

2. Overview of the proof

Over the last few decades a number of powerful techniques have been developed for embedding problems in graphs. The Blow-up lemma [26], in combination with the Regularity lemma [37], has been used to resolve a number of long-standing open problems, including Pósa's conjecture for large graphs [25]. More recently, the so-called Connecting-Absorbing method developed by Rödl, Ruciński and Szemerédi [34] has also proven to be highly effective in tackling such embedding problems.

Typically, both these approaches have been applied to graphs with 'large' minimum degree. Our graph G in Theorem 1.3 may have minimum degree $(1/3+o(1))n$. In particular, this is significantly smaller than the minimum degree threshold that forces the square of a Hamilton cycle in a graph (namely, $2n/3$). As we describe below, having vertices of relatively small degree makes the proof of Theorem 1.3 highly involved and rather delicate. Indeed, our proof draws on ideas from both the Regularity-Blow-up method and the Connecting-Absorbing method. Further, we also develop a number of new ideas in order to deal with these vertices of small degree.

2.1. **An approximate version of Pósa's conjecture.** In order to highlight some of the difficulties in the proof of Theorem 1.3, we first give a sketch of a proof of an approximate version of Pósa's conjecture. This is based on the proof of Pósa's conjecture for large graphs given in [31].

Let $0 < \varepsilon \ll \gamma \ll \eta$. Suppose that G is a sufficiently large graph on n vertices with $\delta(G) \geq (2/3+\eta)n$. We wish to find the square of a Hamilton cycle in G. The proof splits into three main parts.

- **Step 1 (Absorbing path):** Find an 'absorbing' square path P_A in G such that $|P_A| \leq \gamma n$. P_A has the property that given *any* set $A \subseteq V(G) \setminus V(P_A)$ such that $|A| \leq 2\varepsilon n$, G contains a square path P with vertex set $V(P_A) \cup A$, where the first and last two vertices on P are the same as the first and last two vertices on P_A.
- **Step 2 (Reservoir set):** Let $G' := G \setminus V(P_A)$. Find a 'reservoir' set $\mathcal{R} \subseteq V(G')$ such that $|\mathcal{R}| \leq \varepsilon n$. $\mathcal{R}$ has the property that, given *arbitrary* disjoint ordered edges $ab, cd \in E(G)$, there are 'many' short square paths P in G so that: (i) The first two vertices on P are a, b respectively; (ii) The last two vertices on P are c, d respectively; (iii) $V(P) \setminus \{a, b, c, d\} \subseteq \mathcal{R}$.
- **Step 3 (Almost tiling with square paths):** Let $G'' := G' \setminus \mathcal{R}$. Find a collection $\mathcal{P}$ of a bounded number of vertex-disjoint square paths in G'' that together cover all but εn of the vertices in G''.

Assuming that $\delta(G) \geq (2/3+\eta)n$, the proof of each of these three steps is not too involved. (Note though that the proof in [31] is more technical since there $\delta(G) \geq 2n/3$.)

After completing Steps 1–3, it is straightforward to find the square of a Hamilton cycle in G. Indeed, suppose ab is the last edge on a square path P_1 from $\mathcal{P}$ and cd is the first edge on a square path P_2 from $\mathcal{P}$. Then Step 2 implies that we can 'go through' $\mathcal{R}$ to join P_1 and P_2 into a single square path in G. Repeating this process we can obtain a square cycle C in G that contains all the square paths from $\mathcal{P}$. Further, we may also incorporate the absorbing square path P_A into C. C now covers almost all the vertices of G. We then use P_A to absorb all the vertices from $V(G) \setminus V(C)$ into C to obtain the square of a Hamilton cycle.

2.2. **A degree sequence version of Pósa's conjecture.** Suppose that G is a sufficiently large graph on n vertices as in the statement of Theorem 1.3. A result of the second author [38] guarantees that G contains a collection of $\lfloor n/3 \rfloor$ vertex-disjoint triangles (see Theorem 5.2). Further, this result together with a simple application of the Regularity lemma implies that G in fact contains a collection $\mathcal{P}$ of a bounded number of vertex-disjoint square paths that together cover almost all of the vertices in G. So we can indeed prove an analogue of Step 3 in this setting. In particular, if we could find a reservoir set $\mathcal{R}$ as above, then certainly we would be able to join together the square paths in $\mathcal{P}$ through $\mathcal{R}$, to obtain an almost spanning square cycle C in G.

Suppose that $ab, cd \in E(G)$ and we wish to find a square path P in G between ab and cd. If $d_G(a), d_G(b) < n/2$ then it may be the case that a and b have no common neighbours. Then it is clearly impossible to find such a square path P between ab and cd (since ab does not lie in a single square path!). The degree sequence condition on G is such that almost $n/6$ vertices in G may have degree less than $n/2$. Therefore we cannot hope to find a reservoir set precisely as in Step 2 above.

We overcome this significant problem as follows. We first show that G contains a reservoir set $\mathcal{R}$ that can *only* be used to find a square path between pairs of edges $ab, cd \in E(G)$ of *large degree* (namely, at least $(2/3+\eta)n$). This turns out to be quite involved (the whole of Section 6 is devoted to constructing $\mathcal{R}$). In order to use $\mathcal{R}$ to join together the square paths $P \in \mathcal{P}$ into an almost spanning square cycle, we now require that the first and last two vertices on each such P have large degree.

To find such a collection of square paths $\mathcal{P}$ we first find a special collection $\mathcal{F}$ of so-called 'folded paths' in a reduced graph R of G. Roughly speaking, folded paths are a generalisation of the notion of a square path. Each such folded path $F \in \mathcal{F}$ will act as a 'guide' for embedding one of the paths $P \in \mathcal{P}$ into G. More precisely, there is a homomorphism from a square path P into a folded path F. In particular, the structure of F will ensure that the first and last two vertices on P are 'mapped' to large degree vertices in G. This is achieved in Section 5.

Given our new reservoir set $\mathcal{R}$ and collection of square paths $\mathcal{P}$, we again can obtain an almost spanning square cycle C in G. Further, if we could construct an absorbing square path P_A as in Step 1, we would be able to absorb the vertices in $V(G) \setminus V(C)$ to obtain the square of a Hamilton cycle. However, we were unable to construct such an absorbing square path, and do not believe there is a 'simple' way to construct one. (Though, one could construct such a square path P_A if one only requires P_A to absorb vertices of large degree.) Instead, our method now turns towards the Regularity-Blow-up approach.

Using the results from Sections 5 and 6 we can obtain an almost spanning square cycle in the reduced graph R of G. In fact, we obtain a much richer structure Z_ℓ in R called a 'triangle cycle' (see Section 7). Z_ℓ is a special 6-regular graph on 3ℓ vertices that contains the square of a Hamilton cycle. In particular, Z_ℓ contains a collection of vertex-disjoint triangles T_ℓ that together cover all the vertices in Z_ℓ. We then show that G contains an almost spanning structure $\mathcal{C}$ that looks like the 'blow-up' of Z_ℓ. More precisely, if $V(Z_\ell) = \{1, \dots, 3\ell\}$ and $V_1, \dots, V_{3\ell}$ are the corresponding clusters in G, then

- $V(\mathcal{C}) = V_1 \cup \dots \cup V_{3\ell}$;
- $\mathcal{C}[V_i, V_j]$ is ε-regular whenever $ij \in E(Z_\ell)$;
- If ij is an edge in a triangle $T \in T_\ell$ then $\mathcal{C}[V_i, V_j]$ is ε-superregular.

We call $\mathcal{C}$ a 'cycle structure' (see Section 8.1 for the formal definition). The initial structure of $\mathcal{C}$ is such that it contains a spanning square cycle. However, since $\mathcal{C}$ is not necessarily spanning in G, this does not correspond to the square of a Hamilton cycle in G. We thus need to incorporate the 'exceptional vertices' of G into this cycle structure $\mathcal{C}$ in a balanced way so that at the end $\mathcal{C}$ (and hence G) contains the square of a Hamilton cycle. The rich structure of Z_ℓ and thus $\mathcal{C}$ is vital for this. Again particular care is needed when incorporating exceptional vertices of small degree into our cycle structure. This is achieved in Section 8. This part of the proof builds on ideas used in [8, 9].

3. Extremal examples for Theorem 1.3

In this section we describe examples which show that Theorem 1.3 is asymptotically best possible.

Given a fixed graph H, an *H-packing* in a graph G is a collection of vertex-disjoint copies of H in G. We say that an H-packing is *perfect* if it contains $\lfloor |G|/|H| \rfloor$ copies of H in G, i.e. the maximum number. Observe that the square of a Hamilton cycle contains a perfect K_3-packing. The following proposition is a special case of Proposition 17 in [6]. It implies that one cannot replace ηn with -1 in Theorem 1.3.

Proposition 3.1. *Suppose that $n \in 3\mathbb{N}$, $k \in \mathbb{N}$ and $1 \leq k < n/3$. Then there exists a graph G on n vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

$$d_i = \begin{cases} n/3 + k - 1 & \text{if} \quad 1 \leq i \leq k \\ 2n/3 & \text{if} \quad k+1 \leq i \leq n/3 + k \\ n - k - 1 & \text{if} \quad n/3 + k + 1 \leq i \leq n - k + 1 \\ n - 1 & \text{if} \quad n - k + 2 \leq i \leq n, \end{cases}$$

but such that G does not contain a perfect K_3-packing.

Proof. Construct G as follows. The vertex set of G is the union of disjoint sets V_1, V_2, A, B of sizes $n/3$, $2n/3 - 2k + 1$, $k - 1$, k respectively. Add all edges from $B \cup V_2 \cup A$ to V_1. Further, add all edges with both endpoints in $V_2 \cup A$. Add all possible edges between A and B.

Consider an arbitrary copy T of K_3 in G which contains $b \in B$. Since B is an independent set in G and there are no edges between B and V_2, we have that $V(T) \setminus \{b\} \subseteq A \cup V_1$. But V_1 is an independent set in G, so T contains at most one vertex in V_1 and hence at least one vertex in A. But since $|B| > |A|$ this implies that G does not contain a perfect K_3-packing. Furthermore, it is easy to check that G has our desired degree sequence. □

Note that Proposition 3.1 shows that, if true, Conjecture 1.4 is close to best possible in the following sense: Given any $1 \leq k < n/3$, there is a graph G on n vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that (i) G does not contain the square of a Hamilton cycle and (ii) G satisfies the degree sequence condition in Conjecture 1.4 except for the terms $d_{k-\eta n}, \dots, d_k$ which only 'just' fail to satisfy the desired condition.

At first sight, one might think that the ηn term in Theorem 1.3 is an artifact of our proof, but in fact it is a feature of the problem: indeed, it cannot be replaced by $o(\sqrt{n})$. This is shown by an example in Proposition 22 in the arXiv version of [6].

4. Preliminaries

4.1. **Notation.** We write $|G|$ for the order of a graph G and $\delta(G)$ and $\Delta(G)$ for its minimum and maximum degrees respectively. The degree of a vertex $x \in V(G)$ is denoted by $d_G(x)$ and its neighbourhood by $N_G(x)$. Given $A \subseteq V(G)$, we write $N_G(A) := \bigcup_{a \in A} N_G(a)$. We will write $N(A)$, for example, if this is unambiguous. For $x \in V(G)$ and $A \subseteq V(G)$ we write $d_G(x, A)$ for the number of edges xy in G with $y \in A$. Given (not necessarily disjoint) $X, Y \subseteq V(G)$, we write $E(G[X, Y])$ for the collection of edges with one endpoint in X and the other endpoint in Y. Define $e_G(X, Y) := |E(G[X, Y])|$. For each $k \in \mathbb{N}$, we let K_k denote the complete graph on k vertices.

Given a graph G, $X \subseteq V(G)$ and an integer $k \leq |X|$, we define the *k-neighbourhood of X in G* by

$$N_G^k(X) := \bigcup_{\substack{X' \subseteq X \\ |X'|=k}} \bigcap_{x \in X'} N_G(x),$$

that is, the set of all vertices in G adjacent to at least k members of X. When $X = \{x_1, \ldots, x_\ell\}$, we will also write $N_G^k(x_1, \ldots, x_\ell) := N_G^k(X)$. Observe that, if $X, Y \subseteq V(G)$ are disjoint, then

$$N_G^{|X|}(X) \cap N_G^{|Y|}(Y) = N_G^{|X|+|Y|}(X \cup Y). \tag{4.1}$$

If $H \subseteq G$ we set $N_G^k(H) := N_G^k(V(H))$. For $A \subseteq V(G)$ we define $N_A^k(X) := N_G^k(X) \cap A$, and (4.1) holds with G replaced by A.

Given a graph G and a subset $X \subseteq V(G)$, we write $G[X]$ for the subgraph of G induced by X. We write $G \setminus X$ for the subgraph of G induced by $V(G) \setminus X$. Given disjoint $X, Y \subseteq V(G)$ we let $G[X, Y]$ denote the graph with vertex set $X \cup Y$ whose edge set consists of all those edges $xy \in E(G)$ with $x \in X$ and $y \in Y$.

Given a function $f : D \to C$ and $D' \subseteq D$, we write $f(D') := \{f(d) : d \in D'\} \subseteq C$.

Given a graph H, the *square* of H is obtained from H by adding an edge between every pair of vertices of distance two in H. In particular, we say that $P = v_1 \ldots v_k$ is a *square path* if $V(P) = \{v_1, \ldots, v_k\}$ and

$$E(P) = \{v_i v_{i+1} : 1 \leq i \leq k-1\} \cup \{v_i v_{i+2} : 1 \leq i \leq k-2\}.$$

So we always implicitly assume that a square path P is equipped with an ordering. We write $P^* = v_k \ldots v_1$ for P 'ordered backwards'; so $P \neq P^*$. Given vertices $x_1, \ldots, x_\ell \in V(G)$ such that $v_1 \ldots v_k x_1 \ldots x_\ell$ is a square path, we sometimes write $Px_1 \ldots x_k := v_1 \ldots v_k x_1 \ldots x_\ell$. The square path $x_1 \ldots x_k P$ is defined similarly. Given sets $X_1, \ldots, X_k$, we write $P \in X_1 \times \ldots \times X_k$ if $v_i \in X_i$ for all $1 \leq i \leq k$.

Given a square path P and a positive integer $\ell \leq |P|$, we say that $[P]_\ell$ is an *ℓ-segment* if it is an ordered set whose members are ℓ consecutive vertices of P, endowed with the ordering of P. We usually write $a_1 \ldots a_\ell$ for the ℓ-segment $(a_1, \ldots, a_\ell)$. We define the *final ℓ-segment* $[P]_\ell^+$ of P to be the ordered set of the final ℓ vertices in P, whose order is inherited from P. The *initial ℓ-segment* $[P]_\ell^-$ is defined analogously. We write $(P)_\ell^+, (P)_\ell^-$ for the unordered versions. By a slight abuse of notation, we also write $[P]_\ell$ for the square path $P[(P)_\ell]$ and similarly for $[P]_\ell^\pm$.

Throughout we will omit floors and ceilings where the argument is unaffected. The constants in the hierarchies used to state our results are chosen from right to left. For example, if we claim that a result holds whenever $0 < 1/n \ll a \ll b \ll c \leq 1$ (where n is the order of the graph), then there are non-decreasing functions $f : (0,1] \to (0,1]$, $g : (0,1] \to (0,1]$ and $h : (0,1] \to (0,1]$ such that the result holds for all $0 < a, b, c \leq 1$ and all $n \in \mathbb{N}$ with $b \leq f(c)$, $a \leq g(b)$ and $1/n \leq h(a)$. Hierarchies with more constants are defined in a similar way. Note that $a \ll b$ implies that we may assume in the proof that e.g. $a < b$ or $a < b^2$. Given $n, n' \in \mathbb{N}$ with $n \leq n'$, we write $[n, n'] := \{n, \ldots, n'\}$ and $[n] := [1, n]$. We write $a\mathbb{N} := \{an : n \in \mathbb{N}\}$. We also write $a = b \pm \varepsilon$ for $a \in [b - \varepsilon, b + \varepsilon]$.

We will need the following simple consequence of the inclusion-exclusion principle.

Proposition 4.1. *Let G be a graph on n vertices and let $w, x, y \in V(G)$ be distinct. Then*

(i) $|N_G^2(x, y)| \geq d_G(x) + d_G(y) - n$;

(ii) $|N_G^2(w, x, y)| + |N_G^3(w, x, y)| \geq d_G(w) + d_G(x) + d_G(y) - n$.

Proof. We will only prove (ii). Observe that

$$n \geq |N_G(w) \cup N_G(x) \cup N_G(y)| = d_G(w) + d_G(x) + d_G(y) - |N_G^2(w, x, y)| - |N_G^3(w, x, y)|,$$

as required. □

4.2. **The Regularity and Blow-up lemmas.** In the proof of Theorem 1.3 we apply Szemerédi's Regularity lemma [37]. To state it we need some more definitions. We write $d_G(A,B)$ for the *density* $\frac{e_G(A,B)}{|A||B|}$ of a bipartite graph G with vertex classes A and B. Given $\varepsilon > 0$ we say that G is *ε-regular* if every $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ satisfy $|d(A,B) - d(X,Y)| \leq \varepsilon$. Given $\varepsilon, d \in (0,1)$ we say that G is *(ε,d)-regular* if G is ε-regular and $d_G(A,B) \geq d$. Observe the following:

($\star$) Given an (ε,d)-regular bipartite graph $G[A,B]$ and $X \subseteq A$ with $|X| \geq \varepsilon|A|$, there are less than $\varepsilon|B|$ vertices in B which have less than $(d-\varepsilon)|X|$ neighbours in X.

We say that G is *(ε,d)-superregular* if both of the following hold:

- G is (ε,d)-regular;
- $d_G(a) \geq d|B|$ and $d_G(b) \geq d|A|$ for all $a \in A$, $b \in B$.

We will use the degree form of the Regularity lemma, which can be easily derived from the standard version [37].

Lemma 4.2. (Degree form of the Regularity lemma) *For every $\varepsilon \in (0,1)$ and every $M' \in \mathbb{N}$ there exist $M, n_0 \in \mathbb{N}$ such that if G is a graph on $n \geq n_0$ vertices and $d \in [0,1]$ is any real number, then there is a partition of the vertex set of G into $V_0, V_1, \dots, V_L$ and a spanning subgraph G' of G such that the following holds:*

(i) *$M' \leq L \leq M$;*
(ii) *$|V_0| \leq \varepsilon n$;*
(iii) *$|V_1| = \ldots = |V_L| =: m$;*
(iv) *$d_{G'}(x) > d_G(x) - (d+\varepsilon)n$ for all $x \in V(G)$;*
(v) *for all $1 \leq i \leq L$ the graph $G'[V_i]$ is empty;*
(vi) *for all $1 \leq i < j \leq L$, $G'[V_i, V_j]$ is ε-regular and has density either 0 or at least d.*

We call $V_1, \dots, V_L$ *clusters*, V_0 the *exceptional set* and the vertices in V_0 *exceptional vertices*. We refer to G' as the *pure graph*. The last condition of the lemma says that all pairs of clusters are ε-regular (but possibly with different densities). The *reduced graph* R of G with parameters ε, d and M' is the graph whose vertices are $1, \dots, L$ and in which ij is an edge precisely when $G'[V_i, V_j]$ is ε-regular and has density at least d.

The following simple observation is well known.

Proposition 4.3. *Suppose that $0 < \varepsilon \leq d' \leq d \leq 1$ and $d' \leq d/2, 1/6$. Let G be a bipartite graph with vertex classes A and B of size $(1 \pm \varepsilon)m$. Suppose that G' is obtained from G by removing at most $d'm$ vertices from each vertex class.*

(i) *If G is (ε,d)-regular then G' is $(2d', d-\varepsilon)$-regular.*
(ii) *If G is (ε,d)-superregular then G' is $(2d', d-2d')$-superregular.*

Proof. To prove (i), let $A' \subseteq A$ and $B' \subseteq B$ denote the vertex classes of G'. Since $|A'|, |B'| \geq \varepsilon|A|, \varepsilon|B|$ and G is (ε,d)-regular we have that

$$d_{G'}(A',B') = d_G(A',B') = d_G(A,B) \pm \varepsilon \tag{4.2}$$

and

$$d_{G'}(A',B') \geq d - \varepsilon. \tag{4.3}$$

Suppose that $X \subseteq A'$, $Y \subseteq B'$ are such that $|X| \geq 2d'|A'| \geq \varepsilon|A|$ and $|Y| \geq 2d'|B'| \geq \varepsilon|B|$. Then as G is (ε,d)-regular $d_{G'}(X,Y) = d_G(A,B) \pm \varepsilon$. Together with (4.2) this implies that $|d_{G'}(A',B') - d_{G'}(X,Y)| \leq 2\varepsilon \leq 2d'$, and so by (4.3) G' is $(2d', d-\varepsilon)$-regular. To see (ii), note further that, for any $a \in A'$ we have

$$d_{G'}(a, B') \geq d|B| - d'm \geq \left(d - \frac{d'}{1-\varepsilon}\right)|B| \geq (d - 2d')|B'|,$$

and similarly $d_{G'}(b, A') \geq (d - 2d')|A'|$ for all $b \in B'$. □

The next proposition appears as Proposition 8 in [8], and is a slight variant of Proposition 4.3.

Proposition 4.4. *Let G be a graph with $A, B \subseteq V(G)$ disjoint. Suppose that $G[A, B]$ is (ε, d)-regular and let $A', B' \subseteq V(G)$ be such that $|A \triangle A'| \leq \alpha|A'|$ and $|B \triangle B'| \leq \alpha|B'|$ for some $0 \leq \alpha < 1$. Then $G[A', B']$ is (ε', d')-regular, with*

$$\varepsilon' := \varepsilon + 6\sqrt{\alpha} \quad \text{and} \quad d' := d - 4\alpha.$$

If, moreover, $G[A, B]$ is (ε, d)-superregular and each vertex $x \in A'$ has at least $d'|B'|$ neighbours in B' and each vertex $x \in B'$ has at least $d'|A'|$ neighbours in A', then $G[A', B']$ is (ε', d')-superregular with ε' and d' as above.

The following lemma is well known in several variations. The version here is almost identical to Proposition 8 in [30].

Lemma 4.5. *Let $L \in \mathbb{N}$ and suppose that $0 < 1/m \ll 1/L \ll \varepsilon \ll d, 1/\Delta \leq 1$. Let R be a graph with $V(R) = [L]$. Let G be a graph with vertex partition $V_1, \ldots, V_L$ such that $|V_i| = (1 \pm \varepsilon)m$ for all $1 \leq i \leq L$, and in which $G[V_i, V_j]$ is (ε, d)-regular whenever $ij \in E(R)$. Let H be a subgraph of R with $\Delta(H) \leq \Delta$. Then for each $i \in V(H)$, V_i contains a subset V_i' of size $(1 - \sqrt{\varepsilon})m$ such that for every edge ij of H, the graph $G[V_i', V_j']$ is $(4\sqrt{\varepsilon}, d/2)$-superregular.*

Proof. Consider an edge $ij \in E(H)$. By $(\star)$, there are less than $\varepsilon|V_i|$ vertices in V_i which have less than $(d - \varepsilon)(1 - \varepsilon)m \geq (d - 2\varepsilon)m$ neighbours in V_j. So for every vertex i of H we can choose a set $V_i' \subseteq V_i$ of size $(1 - \sqrt{\varepsilon})m \leq (1 - \varepsilon\Delta)|V_i|$ such that for each neighbour j of i in H, all vertices $x \in V_i'$ have at least $(d - 2\varepsilon)m$ neighbours in V_j. Proposition 4.3(i) with $2\sqrt{\varepsilon}$ playing the role of d' implies that, for each edge $ij \in E(H)$, $G[V_i', V_j']$ is $(4\sqrt{\varepsilon}, d - \varepsilon)$-regular, and hence $(4\sqrt{\varepsilon}, d/2)$-regular. Moreover, for each $x \in V_i'$, $d_G(x, V_j') \geq (d - 2\varepsilon)m - 2\sqrt{\varepsilon}m \geq d(1 - \sqrt{\varepsilon})m/2$. Therefore $G[V_i', V_j']$ is $(4\sqrt{\varepsilon}, d/2)$-superregular. □

The following proposition is an easy consequence of (ε, d)-regularity, so we only sketch the proof.

Proposition 4.6. *Let $0 < 1/m \ll \varepsilon \ll c, d < 1$. Let G be a graph with vertex partition X_1, X_2, X_3 where $|X_i| = (1 \pm \varepsilon)m$ for all $1 \leq i \leq 3$ and such that $G[X_i, X_j]$ is (ε, d)-regular for all $1 \leq i < j \leq 3$. For each $i = 1, 2$, let $A_i, B_i \subseteq X_i$, where $|A_i|, |B_i| \geq cm$. Let $W \subseteq V(G)$ be such that $|W \cap X_i| \leq \varepsilon m/2$ for all $1 \leq i \leq 3$. Then there exists a square path $P \in A_1 \times A_2 \times X_3 \times B_1 \times B_2$ with $V(P) \cap W = \emptyset$.*

Sketch proof. For $j = 1, 2$, let $A_j' \subseteq A_j$, $B_j' \subseteq B_j$, $X_3' \subseteq X_3$ be such that A_j', B_j', W are pairwise disjoint, $X_3' \cap W = \emptyset$, and $|A_j'|, |B_j'|, |X_3'| \geq cm/4$. So $\mathcal{X} := \{A_1', A_2', B_1', B_2', X_3'\}$ is a collection of vertex-disjoint susbets of $V(G)$, and $W \cap Y = \emptyset$ for all $Y \in \mathcal{X}$. Let R be the graph whose vertices are elements of $\mathcal{X}$, in which, for all $Y, Z \in \mathcal{X}$, we have $YZ \in E(R)$ whenever $G[Y, Z]$ is $(\sqrt{\varepsilon}, d/2)$-regular. Then R is a complete tripartite graph with vertex classes $\{A_1', B_1'\}, \{A_2', B_2'\}, \{X_3'\}$. So R contains the square path $P' := A_1'A_2'X_3'B_1'B_2'$. It is a simple consequence of regularity that therefore G contains a square path $P \in A_1' \times A_2' \times X_3' \times B_1' \times B_2'$. Note that P has the required properties. □

Given two graphs H, G, we say that a function $\phi : V(H) \to V(G)$ is a *graph homomorphism* if, for all edges $uv \in E(H)$, we have that $\phi(u)\phi(v) \in E(G)$. If ϕ is injective, then we call it an *embedding*, in which case $H \subseteq G$.

We need the following result from [8] which, given a homomorphism from a graph H into the reduced graph R, allows us to embed H into G. Furthermore, under certain conditions we can guarantee that a small fraction of the vertices of H are mapped into specific sets. A similar result was first obtained by Chvátal, Rödl, Szemerédi and Trotter [14].

Lemma 4.7. *(Partial embedding lemma) Suppose that $L \in \mathbb{N}$ and $0 < 1/m \ll 1/L \ll \varepsilon \ll c \ll d, 1/\Delta < 1$. Let R be a graph with $V(R) = [L]$. Let G be a graph with vertex partition $V_1, \ldots, V_L$ such that $|V_i| = (1 \pm \varepsilon)m$ for all $1 \leq i \leq L$, and in which $G[V_i, V_j]$ is (ε, d)-regular whenever $ij \in E(R)$.*

Let H be a graph with vertex partition X, Y and let $f : V(H) \to V(R)$ be a graph homomorphism (so $f(h)f(h') \in E(R)$ whenever $hh' \in E(H)$).

Then, if $|H| \leq \varepsilon m$ and $\Delta(H) \leq \Delta$, there exists an injective mapping $\tau : X \to V(G)$ with $\tau(x) \in V_{f(x)}$ for all $x \in X$, such that for all $y \in Y$ there exist sets $C_y \subseteq V_{f(y)} \setminus \tau(X)$ such that the following hold:

(i) *if $x, x' \in X$ and $xx' \in E(H)$, then $\tau(x)\tau(x') \in E(G)$;*
(ii) *for all $y \in Y$ we have that $C_y \subseteq N_G(\tau(x))$ for all $x \in N_H(y) \cap X$;*
(iii) *$|C_y| \geq c|V_{f(y)}|$ for all $y \in Y$.*

In its simplest form, the Blow-up lemma of Komlós, Sárközy and Szemerédi [26] states that for the purposes of embedding a spanning bipartite graph of bounded degree, a superregular pair behaves like a complete bipartite graph.

Theorem 4.8. *(Blow-up lemma [26]) For every $d, \Delta, c > 0$ and $k \in \mathbb{N}$ there exist constants ε_0 and α such that the following holds. Let $n_1, \ldots, n_k$ be positive integers, $0 < \varepsilon < \varepsilon_0$, and G be a k-partite graph with vertex classes $V_1, \ldots, V_k$ where $|V_i| = n_i$ for $i \in [k]$. Let J be a graph on vertex set $[k]$ such that $G[V_i, V_j]$ is (ε, d)-superregular whenever $ij \in E(J)$. Suppose that H is a k-partite graph with vertex classes $W_1, \ldots, W_k$ of size at most $n_1, \ldots, n_k$ respectively with $\Delta(H) \leq \Delta$. Suppose further that there exists a graph homomorphism $\phi : V(H) \to V(J)$ such that $|\phi^{-1}(i)| \leq n_i$ for every $i \in [k]$. Moreover, suppose that in each class W_i there is a set of at most αn_i special vertices y, each of them equipped with a set $S_y \subseteq V_i$ with $|S_y| \geq c n_i$. Then there is an embedding of H into G such that every special vertex y is mapped to a vertex in S_y.*

4.3. **η-good degree sequences.** We will often think of the collection of degrees of the vertices of a graph G as a function $d_G : V(G) \to \{0, 1, \ldots, n-1\}$. The notation d_G will always be used in this way. Later we will define a different notion of degree, a function whose image is not necessarily a subset of $\mathbb{N} \cup \{0\}$.

Definition 4.9. (η-goodness) *Given $\eta > 0$, $n \in \mathbb{N}$, a finite set V, and a function $d : V \to \mathbb{R}$, let $v_1, \ldots, v_{|V|}$ be an ordering of the elements of V such that $d(v_i) \leq d(v_j)$ whenever $1 \leq i \leq j \leq |V|$. We say that d is* (η, n)-good *if $d(v_i) \geq (1/3 + \eta)n + i + 1$ for all $1 \leq i \leq |V|/3$. If V is the vertex set of a graph G, and $d(v)$ is the degree of $v \in V$ in G, we say that G is* (η, n)-good. *If $|V| = n$ we say that G is* η-good.

The next simple proposition is very useful. Its proof follows immediately from the definition of (η, n)-good, so we omit it.

Proposition 4.10. *Let $\eta > 0$ and $n \in \mathbb{N}$. Let G be a graph on n vertices and let $d : V(G) \to \mathbb{R}$ be an (η, n)-good function. Then the following hold:*

(i) *for all $X \subseteq V(G)$ with $|X| \geq n/3$, there exist at least $|X| - n/3$ vertices $x \in X$ with $d(x) \geq (2/3 + \eta)n$;*
(ii) *for all $X \subseteq V(G)$ with $k \leq |X| \leq n/3$, there exist at least k vertices $x \in X$ with $d(x) \geq (1/3 + \eta)n + |X| - k + 2$.*

Given a graph G on n vertices and a set $X \subseteq V(G)$, we write

$$X_\eta := \{x \in X : d_G(x) \geq (2/3 + \eta)n\}.$$

Observe that, if G is η-good, then

$$|V(G)_\eta| \geq 2n/3. \tag{4.4}$$

The following proposition collects together some useful facts about η-good graphs.

Proposition 4.11. *Let $n, k \in \mathbb{N}$ and $\eta > 0$ such that $0 \leq 1/n \leq 1/k, \eta \leq 1$. Let G be an η-good graph on n vertices and let $X, Y \subseteq V(G)$. Then the following hold:*

(i) *if $X_\eta = \emptyset$, then $|X| < n/3$;*
(ii) *if $|X_\eta| \geq (1/3 - \eta/2)n$, then there are no isolated vertices in $G[X_\eta]$;*
(iii) *if $|X| \geq n/3 + k$, then $e_G(X) > k^2/2$;*
(iv) *if $X, Y \neq \emptyset$ and $E(G[X, Y]) = \emptyset$, then $|X| + |Y| < (2/3 - \eta)n$.*

Proof. First note that (i) and (ii) follow immediately from the definition of X_η and η-goodness.

We now prove (iii). By (i), $|X_\eta| \geq k$. For each $x \in X_\eta$ we have

$$d_G(x, X) \geq d_G(x) - (n - |X|) \geq (2/3 + \eta)n - (2n/3 - k) > k.$$

So $e(G[X]) \geq \frac{1}{2}\sum_{x \in X} d_G(x, X) > k^2/2$, as required.

To prove (iv), suppose, without loss of generality, that $|X| \leq |Y|$. Note that $|X| \leq n/3$ otherwise (i) implies that $X_\eta \neq \emptyset$ and then since $|Y| \geq |X| \geq n/3$ we have that $e_G(X, Y) > 0$, a contradiction. Let $x_0 \in X$ be such that $\max_{x \in X}\{d_G(x)\} = d_G(x_0)$. Proposition 4.10(ii) applied with $k := 1$ implies that

$$(1/3 + \eta)n + |X| + 1 \leq d_G(x_0) \leq n - |Y|,$$

and so $|X| + |Y| < (2/3 - \eta)n$, as desired. □

We now define what it means for a square path to be head- or tail-heavy. We will show in Section 6 that if P is a tail-heavy square path and Q is a head-heavy square path, then we can 'connect' them in an appropriate manner.

Definition 4.12. (η-heaviness) *Let $n \in \mathbb{N}$ and $\eta > 0$. Let G be an η-good graph on n vertices containing a square path P. We say that P is* η-tail-heavy *if $[P]_2^+ \in V(G)_\eta \times V(G)_\eta$. We say that P is* η-head-heavy *if $[P]_2^- \in V(G)_\eta \times V(G)_\eta$. If P is both η-head- and η-tail-heavy, we say that it is* η-heavy. *We omit the prefix η- if it is clear from the context.*

Equivalently, P is η-tail-heavy if $d_G(x) \geq (2/3 + \eta)n$ for all $x \in (P)_2^+$, and analogously for head-heavy. Note that P is η-tail-heavy if and only if P^* is η-head-heavy.

4.4. **Core degree.** Suppose that R is the reduced graph (with parameters ε, d and M') of a graph G. If G is η-good then we will show that R 'inherits' the degree sequence of G (see Lemma 4.13(ii)). Note though that the degree of a vertex $i \in V(R)$ does not provide precise information about the degrees of the vertices $x \in V_i$ in G. In particular, if d is small it is possible for i to have 'large' degree in R but for *every* vertex $x \in V_i$ to have 'small' degree in G. In the proof of Theorem 1.3 it will be important to ensure that certain clusters contain a 'significant' number of vertices of 'large' degree in G. For this, we introduce the notion of the 'core degree' of a cluster in R.

Given $0 < \alpha \leq 1$, a graph G on n vertices and a collection $\mathcal{R}$ of disjoint subsets of $V(G)$, we define the *α-core degree $d^\alpha_{\mathcal{R},G}(X)$ of $X \in \mathcal{R}$ (with respect to G)* as follows. Let $d_1 \leq \ldots \leq d_{|X|}$ be the vertex degrees in G of the vertices in X. Then we let

$$d^\alpha_{\mathcal{R},G}(X) := d_{\lfloor (1-\alpha)|X| \rfloor + 1}|\mathcal{R}|/n.$$

So $d^\alpha_{\mathcal{R},G}(X) \geq k|\mathcal{R}|$ if and only if there are at least $\alpha|X|$ vertices $x \in X$ with $d_G(x) \geq kn$. Note that whenever $\alpha' \leq \alpha$ we have that $d^{\alpha'}_{\mathcal{R},G}(X) \geq d^\alpha_{\mathcal{R},G}(X)$ for all $X \in \mathcal{R}$.

Suppose that $\mathcal{R} := \{V_1, \dots, V_k\}$. If R is a graph such that each $j \in V(R)$ corresponds to the set $V_j \in \mathcal{R}$, we define

$$d^{\alpha}_{R,G}(j) := d^{\alpha}_{\mathcal{R},G}(V_j).$$

(Typically R will be a reduced graph and so its vertex set $\{1, \dots, k\}$ naturally corresponds to clusters $V_1, \dots, V_k$ in G.) In this case, we often think of $d^{\alpha}_{R,G}$ as a function which maps each vertex of R to some rational less than $|R|$, and call this function the *α-core degree function of R (with respect to G)*.

The next lemma shows that the reduced graph R and the function $d^{\alpha}_{R,G}$ 'inherit' the degree sequence of G.

Lemma 4.13. *Let $0 < 1/n \ll 1/M' \ll \varepsilon \ll d, \alpha \ll \eta < 1$ and let G be a graph of order n which is η-good. Apply Lemma 4.2 with parameters ε, d and M' to obtain a pure graph G' and a reduced graph R of G. Then*

(i) *$d_R(j) \geq (1 - 6d) d^{\alpha}_{R,G}(j)$ for all $j \in V(R)$;*
(ii) *$d^{\alpha}_{R,G}$ and R are both $(\eta/2, |R|)$-good.*

Proof. Let $L := |R|$ and $\mathcal{R} := \{V_1, \dots, V_L\}$ be the clusters of G such that V_j is associated with $j \in V(R)$. Set $m := |V_1| = \ldots = |V_L|$. Lemma 4.2(ii) implies that

$$mL \leq n = mL + |V_0| \leq mL + \varepsilon n. \tag{4.5}$$

To prove (i), fix $j \in V(R)$ and let $D := d^{\alpha}_{R,G}(j)$. Note first that $D \geq \delta(G)L/n \geq (1/3+\eta)L$ since G is η-good. By the definition of $d^{\alpha}_{R,G}$, there is a set $X_j \subseteq V_j$ such that $|X_j| \geq \alpha m$ and $d_G(x) \geq Dn/L$ for all $x \in X_j$. So by Lemma 4.2(iv), $d_{G'}(x) > Dn/L - (d+\varepsilon)n$. Given any vertex $x \in X_j$, the number of clusters $V_i \in \mathcal{R}$ containing a neighbour of x in G' is at least

$$\frac{Dn/L - (d+2\varepsilon)n}{m} \overset{(4.5)}{\geq} D - \frac{(d+2\varepsilon)n}{m} \overset{(4.5)}{\geq} D - 2dL \geq D(1-6d).$$

Lemma 4.2(vi) implies that j is adjacent to each of the vertices corresponding to these clusters in R. So $d_R(j) \geq D(1-6d)$, proving (i).

To prove (ii), fix $1 \leq i \leq L/3$ and $\mathcal{X} \subseteq V(R)$ with $|\mathcal{X}| = i$. Let $X' := \bigcup_{j \in \mathcal{X}} V_j \subseteq V(G)$. Then $|X'| = im \leq Lm/3 \leq n/3$ by (4.5). Since G is η-good, Proposition 4.10(ii) implies that there is a subset Y of X' with $|Y| \geq \alpha im$ such that $\min_{y \in Y}\{d_G(y)\} \geq (1/3+\eta)n + (1-\alpha)im + 2$. Observe further that there exists some $j \in \mathcal{X}$ such that $|Y \cap V_j| \geq \alpha m$. Thus

$$\begin{aligned} d^{\alpha}_{R,G}(j) \quad &\geq \quad \min_{y \in Y}\{d_G(y)\}\frac{L}{n} \geq \left(\frac{1}{3}+\eta\right)L + \frac{(1-\alpha)imL}{n} \\ &\overset{(4.5)}{\geq} \quad \left(\frac{1}{3}+\eta\right)L + (1-\alpha)(1-\varepsilon)i \geq \left(\frac{1}{3}+\frac{2\eta}{3}\right)L + i + 1. \end{aligned}$$

Since $\mathcal{X}$ was arbitrary, this proves that $d^{\alpha}_{R,G}$ is $(2\eta/3, L)$-good and hence $(\eta/2, L)$-good.

Let $1 \leq i \leq L/3$. Now, by (i), the vertex j_i of R with ith smallest degree satisfies

$$d_R(j_i) \geq (1-6d)\left(\left(\frac{1}{3}+\frac{2\eta}{3}\right)L + i + 1\right) \geq \left(\frac{1}{3}+\frac{\eta}{2}\right)L + i + 1.$$

So R is $(\eta/2, L)$-good, completing the proof of (ii). □

The next proposition shows that, given an (η, n)-good function d, after slightly shrinking the domain of d or by slightly reducing each of the values that d takes, the function that remains is $(\eta/2, n)$-good.

Proposition 4.14. *Let $n \in \mathbb{N}$ and $\eta > 0$ such that $0 < 1/n \ll \eta < 1$. Let V be a set of order n and let $d : V \to \mathbb{R}$ be (η, n)-good. Let $V' \subseteq V$ with $|V'| \geq (1 - \eta/4)n$ and let $d' : V' \to \mathbb{R}$ be such that $d'(v) \geq d(v) - \eta n/4$ for all $v \in V'$. Then d' is $(\eta/2, n)$-good. In particular, any graph obtained from an η-good graph G on n vertices by removing at most $\eta n/4$ vertices and $\eta n/4$ edges from each vertex is $(\eta/2, n)$-good.*

Proof. Let $v_1, \ldots, v_n$ be an ordering of V such that $d(v_i) \leq d(v_j)$ whenever $1 \leq i \leq j \leq n$. Then $d(v_i) \geq (1/3 + \eta)n + i + 1$ for all $1 \leq i \leq n/3$.

Let $i_1, \ldots, i_k$ be the subsequence of $1, \ldots, n$ corresponding to the vertices in V'. So $k := |V'| \geq (1 - \eta/4)n$. Let $1 \leq j \leq k/3$ be arbitrary. Since $j \leq k/3 \leq n/3$ and $i_j \geq j$ we have

$$\begin{aligned} d'(v_{i_j}) &\geq d(v_{i_j}) - \eta n/4 \geq d(v_j) - \eta n/4 \geq (1/3 + \eta)n + (j + 1) - \eta n/4 \\ &\geq (1/3 + \eta/2)n + j + 1. \end{aligned}$$

This implies that d' is $(\eta/2, n)$-good. The final assertion follows by taking $d := d_G$. □

5. An almost perfect packing of heavy square paths

The aim of this section is to prove the following lemma, which ensures that every sufficiently large η-good graph G on n vertices contains an almost perfect packing of square paths, and the number of these paths is bounded. As mentioned in Section 2, a relatively simple application of Lemma 4.2 and Theorems 4.8 and 5.2 can achieve this. However, we also require that the first and last two vertices of each of these paths have degree at least $(2/3 + \eta)n$ in G, for which considerably more work is needed. This property is crucial when, in Section 6, we connect these paths to obtain an almost spanning square cycle.

Lemma 5.1. *Let $0 < \varepsilon, \eta \ll 1$. Then there exist $n_0, M \in \mathbb{N}$ such that the following holds. For every η-good graph G on $n \geq n_0$ vertices, G contains a collection $\mathcal{P}$ of at most M vertex-disjoint η-heavy square paths such that $\sum_{P \in \mathcal{P}} |P| \geq (1 - \varepsilon)n$.*

To prove Lemma 5.1, we will use the following result of the second author [38] which guarantees a perfect triangle packing in a sufficiently large η-good graph.

Theorem 5.2. [38] *For every $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that every η-good graph G on $n \geq n_0$ vertices contains a perfect K_3-packing.*

Theorem 5.2 is a special case of a more general result from [38] on a degree sequence condition that forces a graph to contain a perfect H-packing for arbitrary H.

To find a bounded number of vertex-disjoint square paths which together cover almost every vertex of G, we apply Szemerédi's Regularity lemma to G and then apply Theorem 5.2 to the reduced graph R of G to find a perfect triangle packing $(T_j)_j$ in R. Then we use the Blow-up lemma to find a square path in G for each triangle T_j that covers almost all of the vertices in the clusters of T_j.

However, to guarantee that our paths are η-heavy, more work is needed. We extend each triangle T_j in R into two 'folded paths' F_j, F'_j – a folded path is essentially a sequence of triangles such that the ith triangle shares exactly two vertices with the $(i-1)$th triangle. A folded path is therefore a generalisation of a square path. We choose both F_j and F'_j so that their final two clusters each contain many vertices of degree at least $(2/3 + \eta)n$. Further, the initial triangle of both F_j and F'_j is T_j. These properties will allow us to find a square path Q_j in G so that:

(i) Q_j only contains vertices from the clusters in F_j and F'_j;
(ii) Q_j contains most of the vertices from the clusters in T_j;
(iii) Q_j is η-heavy.

Note that for distinct T_j, $T_{j'}$ in R, the folded paths F_j, F'_j, $F_{j'}$ and $F'_{j'}$ may intersect. Thus care is needed to ensure the square paths $Q_j, Q_{j'}$ constructed are vertex-disjoint.

5.1. **Folded paths.** Here we define a structure – a 'folded path' – which will be useful when embedding square paths. Indeed, if the reduced graph of G contains a short folded path F, we can embed a short square path into G using only vertices lying in the clusters which form the vertex set of F.

Definition 5.3. (Folded paths) *We say a graph F is a* folded path *if there exists an ordered sequence $v_1, \dots, v_n$ of distinct vertices and integers $k_3, \dots, k_n$ such that*

- $V(F) = \{v_1, \dots, v_n\}$;
- $k_3 := 1$ *and* $k_i \in \{i-2, k_{i-1}\}$ *for* $4 \leq i \leq n$;
- $E(F) = \{v_1v_2\} \cup \{v_iv_{k_i}, v_iv_{i-1} : 3 \leq i \leq n\}$.

We implicitly assume that a folded path is equipped with ordered sequences $v_1, \dots, v_n$ and $k_3, \dots, k_n$. We will sometimes write $F = v_1 \dots v_n$, and say that $k_3, \dots, k_n$ is the *ordering* of F. Observe that $k_3, \dots, k_n$ is a non-decreasing sequence and $k_i \leq i - 2$ for all $i \geq 3$.

A folded path F is a generalisation of a square path. Indeed, the special case when the ordering of F is $1, \dots, n-2$ (i.e. $k_i = i-2$ for all $i \geq 3$) corresponds to the square path on n vertices. When $k_i \neq i-2$, one can think of v_{k_i} as a 'pivot', at which the triangles that form the structure 'change direction'. The top of Figure 1 shows a folded path, with arrows drawn from a pivot v_{k_i} to v_i.

Another way to view a folded path is as a sequence of square paths which are disjoint apart from initial and final triangles, which are shared by consecutive paths. Figure 1 gives an example of a homomorphism from a square path to a folded path. One can visualise folding a square path so that triangles map onto triangles, using the pivots as directions for where to fold. In Lemma 5.5, we show that given any folded path F, there is a homomorphism from some square path P to F where P 'stretches along the length' of F and where $|P|$ is not significantly greater than $|F|$.

In the next proposition, we prove that, in a folded path, every edge lies in a triangle.

Proposition 5.4. *Let $F := v_1 \dots v_n$ be a folded path with ordering $k_3, \dots, k_n$. Then, for all $xy \in E(F)$, we have $N^2_F(x, y) \neq \emptyset$.*

Proof. Write $x =: v_j$ and $y =: v_\ell$ where $j < \ell$. Then $j \in \{\ell - 1, k_\ell\}$. Recall that $k_\ell \in \{\ell - 2, k_{\ell-1}\}$. For each of the four possible values of (j, k_ℓ), we will exhibit a vertex $z \in N^2_F(v_j, v_\ell)$. Suppose first that $k_\ell = \ell - 2$. If $j = \ell - 1$, then we set $z := v_{\ell-2}$. If $j = k_\ell$, then we set $z := v_{\ell-1}$. Suppose instead that $k_\ell = k_{\ell-1}$. If $j = \ell - 1$, then we set $z := v_{k_{\ell-1}}$. If $j = k_\ell$, then we set $z := v_{\ell-1}$. □

5.2. **Embedding a square path into a folded path.** The next lemma guarantees a homomorphism from a square path P into a given folded path F, with some special properties. Later (in the proof of Lemma 5.7), we will use this lemma in combination with Lemma 4.7 to embed a square path P into a graph G whose reduced graph contains a copy of some folded path F.

Lemma 5.5. *Let $n \geq 3$ be an integer and let $F = v_1 \dots v_n$ be a folded path. Then there exists a square path $P := x_1 \dots x_p$ with $n \leq p \leq 2n + 1$ and a mapping $g : [p] \to [n]$ such that*

(i) $v_{g(i)}v_{g(j)} \in E(F)$ *whenever* $x_ix_j \in E(P)$;
(ii) $g(1) = 1$, $g(2) = 2$, $g(3) = 3$, $g(\{p - 1, p\}) = \{n - 1, n\}$.

Proof. Let $k_3, \dots, k_n$ be the ordering of F. We will begin by finding a square path $P' := x_1 \dots x_{p'}$ with $n \leq p' \leq 2n$ and a function $g : [p'] \to [n]$ such that $v_{g(i)}v_{g(j)} \in E(F)$ whenever $x_ix_j \in E(P')$; $g(1) = 1$, $g(2) = 2$, $g(3) = 3$ and $g(p') = n$. We prove this iteratively. Suppose that for some $3 \leq i \leq n - 1$ there exists $p_i \in \mathbb{N}$ such that $i \leq p_i \leq 2i$ and a function $g : [p_i] \to [n]$ such that

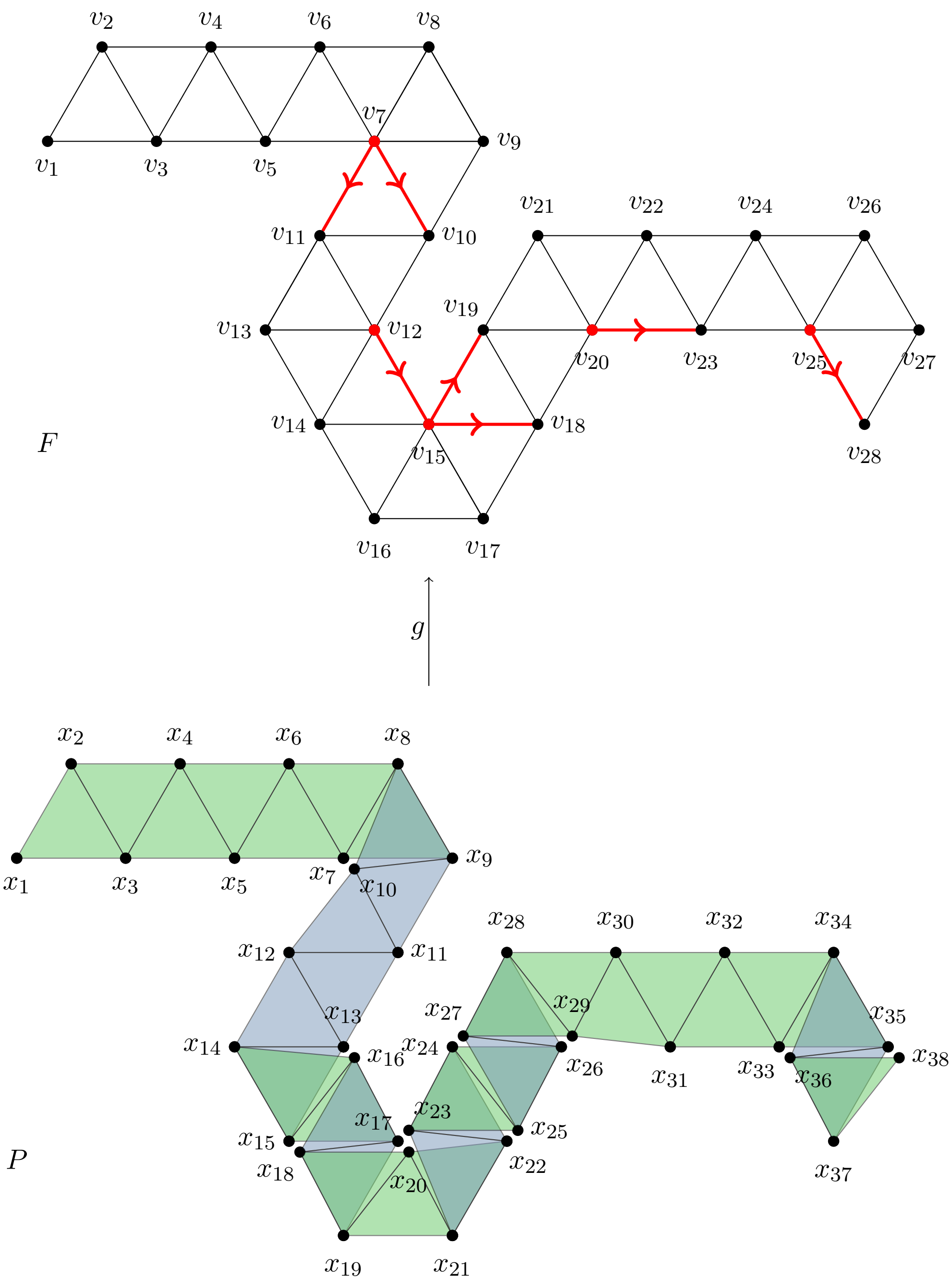

Figure 1: A folded path F with $|F| = 28$, and a square path P with $|P| = 38$, drawn to show a homomorphism $g : [|P|] \to [|F|]$ (so, for example, $g(\{17, 20, 23\}) = \{15\}$), as described in Lemma 5.5. The final two vertices map to each other: $g(37) = 28$ and $g(38) = 27$.

(A_i) $g(1) = 1$, $g(2) = 2$, $g(3) = 3$, $g(p_i) = i$, $g([p_i]) = [i]$, and $g(j) < i$ for all $j < p_i$;

(B_i) $v_{g(j)}v_{g(j')} \in E(F)$ whenever $1 \leq j < j' \leq p_i$ and $j' - j \in \{1, 2\}$.

Observe that (B_i) is equivalent to the statement that $v_{g(j)}v_{g(j')} \in E(F)$ whenever $x_j x_{j'} \in E(P_i)$, where $P_i := x_1 \ldots x_{p_i}$ is a square path. So our aim is to find $p' := p_n \in \mathbb{N}$ and $g : [p'] \to [n]$ such that (A_n) and (B_n) hold. Certainly (A_3) and (B_3) hold. Assume that (A_i) and (B_i) hold for some fixed

$3 \le i \le n-1$. We will extend the domain of g by defining $p_{i+1} \ge p_i + 1$ and $g(p_i+1), \dots, g(p_{i+1})$ so that (A_{i+1}) and (B_{i+1}) hold.

We first give some motivation. Ideally, we would like to set $p_{i+1} := p_i + 1$, so $g(p_i+1) = i+1$. But we can only do so if $v_{g(p_i-1)}, v_{g(p_i)}$ are both neighbours of v_{i+1} in F such that $g(p_i-1), g(p_i) \le i$, and by definition of F, this only occurs if $g(\{p_i-1, p_i\}) = \{k_{i+1}, i\}$. By (A_i), $g(p_i) = i$, so we need that $g(p_i - 1) = k_{i+1}$. If this is not the case, we cannot set $p_{i+1} = p_i + 1$, but it turns out that we can take $p_{i+1} = p_i + 2$ (by taking a single intermediate step via k_{i+1}).

Indeed, set

$$p_{i+1} := p_i + 1 \quad \text{and} \quad g(p_i+1) := i+1 \qquad \text{if} \quad k_{i+1} = g(p_i - 1); \tag{5.1}$$

$$p_{i+1} := p_i + 2 \quad \text{and} \quad g(p_i+1) := k_{i+1},\ g(p_i+2) := i+1 \quad \text{if} \quad k_{i+1} \ne g(p_i - 1). \tag{5.2}$$

First we check that (A_{i+1}) holds. We have $g(1) = 1$, $g(2) = 2$, $g(3) = 3$ and $p_{i+1} \ge p_i + 1 \ge i + 1$. Moreover, $p_{i+1} \le p_i + 2 \le 2(i+1)$ and $g(p_{i+1}) = i+1$. Furthermore,

$$g([p_{i+1}]) = g([p_i]) \cup g([p_i+1, p_{i+1}]) \in \{[i] \cup \{i+1\}, [i] \cup \{k_{i+1}, i+1\}\} = [i+1],$$

and $g(j) < i+1$ for all $j < p_{i+1}$. (Here we used that $k_{i+1} \le i-1$.) So (A_{i+1}) holds.

Now we show that (B_{i+1}) holds. Let $E'(F)$ be the set of all triples $\{j, j', j''\}$ such that $v_j v_{j'} v_{j''}$ is a triangle in F. It suffices to prove that $g(\{j-2, j-1, j\}) \in E'(F)$ for all $p_i + 1 \le j \le p_{i+1}$. We claim that

$$\{k_t, t-1, t\} \in E'(F) \tag{5.3}$$

for all $3 \le t \le n$. To see this, note that $\{v_{k_{t-1}} v_{t-1}, v_{t-2} v_{t-1}, v_{k_t} v_t, v_{t-1} v_t\} \subseteq E(F)$ by definition and $k_t \in \{k_{t-1}, t-2\}$, proving the claim.

By the definition of a folded path,

$$N_F(v_i) \cap \{v_{g(1)}, \dots, v_{g(p_i)}\} \overset{(A_i)}{=} N_F(v_i) \cap \{v_1, \dots, v_i\} = \{v_{k_i}, v_{i-1}\}. \tag{5.4}$$

(B_i) implies that $v_{g(p_i-1)} v_{g(p_i)} \in E(F)$, i.e. that $v_{g(p_i-1)} \in N_F(v_i)$. So (5.4) implies that

$$g(p_i - 1) \in \{k_i, i-1\}, \quad \text{and also} \quad k_{i+1} \in \{k_i, i-1\}. \tag{5.5}$$

There are now two cases to consider depending on the value of $g(p_i - 1)$.

Suppose first that $k_{i+1} = g(p_i - 1)$. Then $p_{i+1} = p_i + 1$ by (5.1). We have that

$$g(\{p_i - 1, p_i, p_i + 1\}) \overset{(5.1)}{=} \{k_{i+1}, i, i+1\} \overset{(5.3)}{\subseteq} E'(F),$$

proving (B_{i+1}) in this case.

Suppose instead that $k_{i+1} \ne g(p_i - 1)$. Then $p_{i+1} = p_i + 2$ by (5.2). Now (5.5) gives that

$$\{k_{i+1}, g(p_i - 1)\} = \{k_i, i-1\}. \tag{5.6}$$

Therefore

$$\begin{aligned} g(\{\{p_i - 1, p_i, p_i + 1\}, \{p_i, p_i + 1, p_i + 2\}\}) &\overset{(5.2)}{=} \{\{g(p_i - 1), i, k_{i+1}\}, \{i, k_{i+1}, i+1\}\} \\ &\overset{(5.6)}{=} \{\{k_i, i-1, i\}, \{k_{i+1}, i, i+1\}\} \\ &\overset{(5.3)}{\subseteq} E'(F), \end{aligned}$$

proving (B_{i+1}) here.

We have proved that we can obtain $p' := p_n$ where $n \le p' \le 2n$ and $g : [p'] \to [n]$ such that (A_n) and (B_n) hold. Now $g(p') = n$ by (A_n). Therefore if $g(p' - 1) = n - 1$, we are done. So suppose not (see e.g. Figure 1 with $p' = 37$). The definition of F implies that $N_F(v_n) = \{v_{k_n}, v_{n-1}\}$. So v_n has exactly one neighbour v_{k_n} in F which is not v_{n-1}. Then (B_n) implies that $g(p' - 1) = k_n$ and

$g(p'-2)=n-1$. Now (5.3) implies that $\{k_n, n-1, n\} \in E'(F)$, i.e. $v_{k_n}v_{n-1}v_n$ is a triangle in F. We are done by setting $p := p'+1$ and $g(p) := n-1$. □

5.3. **Finding a folded path in an η-good graph.** Recall that we can use Theorem 5.2 to find a triangle packing $(T_i)_i$ in the reduced graph R of G, and then apply the Blow-up lemma (Theorem 4.8) so that for each i we find a square path P_i in G that almost spans the vertex set of G corresponding to T_i. So $(P_i)_i$ covers almost all the vertices of G. However, to prove Lemma 5.1, we require that each P_i is both head- and tail-heavy. We will extend each P_i both forwards and backwards by finding square paths R_i, R_i' such that $R_iP_iR_i'$ is a top- and tail-heavy square path, and $|R_i|, |R_i'|$ are small. To do so, we will find folded paths F_i and F_i' in R which will form the 'framework' for R_i and R_i' respectively.

This is achieved in Lemma 5.6, whose proof is the aim of this subsection. Given a triangle T_i, in order to find two 'types' of paths R_i, R_i', Lemma 5.6 'produces' two folded paths such that the first three clusters in both of these folded paths are the clusters from T_i, but the order of these clusters differs. Further, in both folded paths the last two vertices correspond to clusters containing many high-degree vertices in G.

We use standard cycle notation for permutations, so, for example, (132) maps 1 to 3, 3 to 2 and 2 to 1.

Lemma 5.6. *Let $\gamma, \eta > 0$ and $n \in \mathbb{N}$ where $0 < 1/n \ll \gamma \ll \eta \ll 1$. Suppose that G is a graph on n vertices. Let $d'_G : V(G) \to \mathbb{R}$ be an (η, n)-good function such that*

$$d_G(x) \geq (1-\gamma)d'_G(x) \tag{5.7}$$

for all $x \in V(G)$. Let T be the vertex set of a triangle in G. Then there exists $8 \leq t \leq 5/\eta$ and an ordering v_1, v_2, v_3 of T such that G contains a folded path $F = v_1v_2v_3 \dots v_t$ such that $d'_G(v_{t-1}), d'_G(v_t) \geq (2/3+\eta)n$. Moreover, there exists $\sigma \in \{(132), (213)\}$ and $8 \leq t' \leq 5/\eta$ such that G contains a folded path $H = v_{\sigma(1)}v_{\sigma(2)}v_{\sigma(3)}v_4' \dots v_{t'}'$ such that $d'_G(v'_{t'-1}), d'_G(v'_{t'}) \geq (2/3+\eta)n$.

Proof. We split the proof into three steps. In the first step, we find a short folded path F' whose final vertex v_s has $d'_G(v_s) \geq (2/3+\eta)n$. In the second step, we extend F' into F'' so that the final vertex v_r of F'' is a neighbour of v_s in F'', and $d'_G(v_r) \geq (2/3+\eta)n$. Finally, in the third step, we extend F'' into F by appending four additional vertices. Simultaneously we will construct H (using the same process used to construct F).

Step 1. Obtain an ordering v_1, v_2, v_3 of T such that there exists a folded path $F' = v_1v_2v_3 \dots v_s$ where $d'_G(v_s) \geq (2/3+\eta)n$ and $3 \leq s \leq 4/\eta$. Obtain $\sigma \in \{(132), (213)\}$ such that there exists a folded path $H' = v_{\sigma(1)}v_{\sigma(2)}v_{\sigma(3)}v_4' \dots v_{s'}'$ where $3 \leq s' \leq 4/\eta$ and $d'_G(v) \geq (2/3+\eta)n$ for the final vertex v on H'.

Consider any $S \subseteq V(G)$ with $|S| = 3$ and write $S = \{x_1, x_2, x_3\}$ where $d'_G(x_1) \leq d'_G(x_2) \leq d'_G(x_3)$. We define $\mathcal{C}(S) := (\alpha_1, \alpha_2, \alpha_3)$ where $d'_G(x_i) = (1/3 + \alpha_i\eta)n$ for all $1 \leq i \leq 3$. So $1 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq 2/3\eta$. We also write $c(S) := \alpha_1 + \alpha_2 + \alpha_3$.

Suppose that there exists $z \in T$ with $d'_G(z) \geq (2/3+\eta)n$. Then, writing $T := \{v_1, v_2, v_3 := z\}$, we are done by setting $F' := v_1v_2v_3$ and $H' := v_2v_1v_3$ (so $\sigma = (213)$).

Therefore we may assume that $d'_G(z) < (2/3+\eta)n$ for all $z \in T$. Note that $c(T) \geq 3$ since d'_G is (η, n)-good. Further, the minimum degree of G implies that there are two vertices v_2, v_3 in T with common neighbour $v_4 \in N^2_G(T) \setminus T$. Let v_1 be the vertex in $T \setminus \{v_2, v_3\}$. Set $F_1' := v_1v_2v_3$ and $F_2' := v_1v_2v_3v_4$. Then F_2' is a folded path with $\{v_1, v_2, v_3\} = T$; and k_3, k_4 is the ordering of F_2' where $k_3 := 1$ and $k_4 := 2$. Since d'_G is (η, n)-good, we have $c(\{v_2, v_3, v_4\}) \geq 3$. Note that the ordering v_2, v_3 was arbitrary.

To achieve Step 1, we will now concentrate on achieving Step 1′.

Step 1′. Obtain a folded path $F' = v_1v_2v_3 \dots v_s$ where $d'_G(v_s) \geq (2/3+\eta)n$ and $4 \leq s \leq 4/\eta$.

Suppose that for some $2 \le i \le 3/\eta$, we have defined a folded path F'_i such that the following hold.

(A_i) $F'_i := v_1 \dots v_{m_i}$ for some $4 \le m_i \le i+2$;

(B_i) $T_i := \{v_{m_i}, v_{m_i-1}, v_{k_{m_i}}\}$ is such that $c(T_i) \ge i/2$.

Note that $T_i = \{v_{m_i}\} \cup N_{F'_i}(v_{m_i})$. We have shown that (A_2) and (B_2) hold.

If $d'_G(v_{m_i}) \ge (2/3+\eta)n$, set $s := m_i$ and $F' := F'_i$. Otherwise, we will obtain F'_{i+1} from F'_i so that F'_{i+1} satisfies (A_{i+1}) and (B_{i+1}). Write $m := m_i$ and let $k_3, \dots, k_m$ be the ordering of F'_i. Note that if there exist

$$k_{m+1} \in \{m-1, k_m\} \quad \text{and} \quad v_{m+1} \in N^2_G(v_m, v_{k_{m+1}}) \setminus V(F'_i), \tag{5.8}$$

then $v_1 \dots v_{m+1}$ is a folded path in G with ordering $k_3, \dots, k_{m+1}$.

Proposition 4.1(ii) and (5.7) imply that

$$|N^2_G(T_i)| + |N^3_G(T_i)| \quad \ge \quad (1-\gamma)\sum_{x \in T_i} d'_G(x) - n = (1-\gamma)c(T_i)\eta n - \gamma n. \tag{5.9}$$

Write

$$\mathcal{C}(T_i) =: (\alpha_1, \alpha_2, \alpha_3) \quad \text{so} \quad \alpha_1 + \alpha_2 + \alpha_3 \ge i/2. \tag{5.10}$$

There are two cases to consider, depending on the sizes of $N^2_G(T_i)$ and $N^3_G(T_i)$.

Case 1. $|N^3_G(T_i)| \ge (1-\gamma)\alpha_1\eta n - \gamma n/2$.

Suppose that there is no vertex $v_{m+1} \in N^3_G(T_i) \setminus V(F'_i)$ with $d'_G(v_{m+1}) \ge (2/3+\eta)n$. Then

$$(1-\gamma)\alpha_1\eta n - \gamma n/2 - |F'_i| \le |N^3_G(T_i) \setminus V(F'_i)| \le n/3$$

by Proposition 4.10(i). So Proposition 4.10(ii) implies that we can choose $v_{m+1} \in N^3_G(T_i) \setminus V(F'_i)$ with

$$\begin{aligned} d'_G(v_{m+1}) \quad &\ge \quad \left(\frac{1}{3}+\eta\right)n + (1-\gamma)\alpha_1\eta n - \gamma n/2 - |F'_i| \\ &\overset{(A_i)}{\ge} \quad \left(\frac{1}{3} + (\alpha_1+1)\,\eta\right)n - 2\gamma n - \left(\frac{3}{\eta}+2\right) \\ &\ge \quad \left(\frac{1}{3} + (\alpha_1+1/2)\,\eta\right)n. \end{aligned} \tag{5.11}$$

Let $v_j, v_\ell \in T_i$ be such that $d'_G(v_j) = (1/3+\alpha_2\eta)n$ and $d'_G(v_\ell) = (1/3+\alpha_3\eta)n$. Observe that $v_{m+1} \in N^2_G(v_j, v_\ell)$. Set $T_{i+1} := \{v_{m+1}, v_j, v_\ell\}$. Then, by (5.11),

$$c(T_{i+1}) \ge (\alpha_1 + 1/2) + \alpha_2 + \alpha_3 \quad \text{or} \quad d'_G(v_{m+1}) \ge (2/3+\eta)n. \tag{5.12}$$

Case 2. $|N^2_G(T_i)| \ge (1-\gamma)(\alpha_2+\alpha_3)\eta n - \gamma n/2$.

Note that this is indeed the only other case by (5.9) and (5.10). Then, similarly as above, Proposition 4.10(i) and (ii) imply that we can choose $v_{m+1} \in N^2_G(T_i) \setminus V(F'_i)$ with

$$d'_G(v_{m+1}) \ge \min\left\{\left(\frac{2}{3}+\eta\right)n, \left(\frac{1}{3} + (\alpha_2+\alpha_3+1/2)\,\eta\right)n\right\}. \tag{5.13}$$

Let v_j, v_ℓ be two neighbours of v_{m+1} in T_i, where $d'_G(v_j) \le d'_G(v_\ell)$. So $d'_G(v_j) \ge (1/3+\alpha_1\eta)n$ and $d'_G(v_\ell) \ge (1/3+\alpha_2\eta)n$. In this case we set $T_{i+1} := \{v_{m+1}, v_j, v_\ell\}$. Then

$$c(T_{i+1}) \ge \alpha_1 + \alpha_2 + (\alpha_2+\alpha_3+1/2) \quad \text{or} \quad d'_G(v_{m+1}) \ge (2/3+\eta)n. \tag{5.14}$$

In both Cases 1 and 2, $v_{m+1} \notin V(F'_i)$ and $\{j,\ell\} \subseteq \{k_m, m-1, m\}$. If $\{j,\ell\} = \{k_m, m-1\}$ then we obtain F'_{i+1} from F'_i by replacing v_m with v_{m+1} (and k_m is unchanged). Then F'_{i+1} is certainly a folded path in G. Otherwise, one of j,ℓ equals m, and we choose k_{m+1} so that $\{m, k_{m+1}\} = \{j,\ell\}$.

Note that $k_{m+1} \in \{k_m, m-1\}$. So $F'_{i+1} := v_1 \dots v_m v_{m+1}$ is a folded path in G by (5.8). In both cases, F'_{i+1} is a folded path with

$$4 \leq m \leq |F'_{i+1}| \leq m+1 \leq i+3.$$

Moreover, since $m = m_i \geq 4$, the first three vertices of F'_{i+1} are v_1, v_2, v_3, as required. So (A_{i+1}) holds.

If $d'_G(v_{m+1}) \geq (2/3+\eta)n$ then we set $F' := F'_{i+1}$ and Step $1'$ is complete. Otherwise, (5.10), (5.12), (5.14) and (B_i) imply that $c(T_{i+1}) \geq (i+1)/2$. So (B_{i+1}) holds. We have thus defined F'_{i+1} so that (A_{i+1}) and (B_{i+1}) both hold.

Therefore after repeating this process at most $S := 3/\eta$ times either we obtain a folded path F' as desired in Step $1'$ or we obtain a folded path $F'_S = v_1 \dots v_s$ (where $s := m_S$) that satisfies (A_S) and (B_S) and so $c(T_S) \geq 3/2\eta$. In the latter case, we may assume that $d'_G(v_j) < (2/3+\eta)n$ for all $4 \leq j \leq s-1$ (otherwise setting $F' := v_1 \dots v_j$ yields our desired folded path). Note that

$$\frac{1}{3}\sum_{x \in T_S} d'_G(x) = \frac{1}{3}(1 + c(T_S)\eta)n \overset{(B_S)}{\geq} \frac{1}{3}(1+3/2)n = 5n/6 \geq (2/3+\eta)n$$

and so

$$d'_G(v_s) \geq (2/3+\eta)n. \tag{5.15}$$

Observe that

$$s \overset{(A_S)}{\leq} S+2 \leq 3/\eta + 2 \leq 4/\eta. \tag{5.16}$$

Set $F' := F'_S$. This completes the proof of Step $1'$. Since the choice of the ordering v_2, v_3 was arbitrary, we can argue precisely as in Step $1'$, now with F'_2 replaced with $H'_2 := v_1 v_3 v_2 v_4$ to obtain a folded path H' as desired in Step 1 (so $\sigma = (132)$). This completes Step 1. From now on we only extend F' to F since the process of extending H' to H is identical.

Step 2. Obtain a folded path $F'' := v_1 v_2 v_3 \dots v_r$ where $4 \leq s+1 \leq r \leq 49/10\eta$, where $k_r = s$ and $d'_G(v_s), d'_G(v_r) \geq (2/3+\eta)n$.

Let $F_0 := F' = v_1 \dots v_s$. Suppose that for some $0 \leq i \leq 1/3\eta$, we have defined folded paths $F_0, \dots, F_i$ such that $F_i := v_1 \dots v_{s+i}$ where for all $1 \leq j \leq i$ we have $v_{s+j} \in N_{F_i}(v_s)$ and $d'_G(v_{s+i}) \geq (1/3 + i\eta)n$. So $k_{s+j} = s$ for all $2 \leq j \leq i$.

By choosing an arbitrary $v_{s+1} \in N^2_G(v_{s-1}, v_s) \setminus V(F')$ we can find our desired folded path $F_1 = v_1 \dots v_s v_{s+1}$. (Such a vertex v_{s+1} exists by (5.7) and (5.15) and since d'_G is (η, n)-good.) Thus, we may assume that $i \geq 1$.

If there exists $1 \leq j \leq i$ with $d'_G(v_{s+j}) \geq (2/3+\eta)n$, we are done by setting $r := s+j \geq 1$ and $F'' := F_j$. Otherwise, Proposition 4.1(i) implies that

$$\begin{aligned} |N^2_G(v_s, v_{s+i}) \setminus V(F_i)| &\overset{(5.16)}{\geq} (d_G(v_s) + d_G(v_{s+i})) - n - (4/\eta + i) \\ &\overset{(5.7)}{\geq} (1-\gamma)(d'_G(v_s) + d'_G(v_{s+i})) - n - 5/\eta \\ &\overset{(5.15)}{\geq} (1-\gamma)(i+1)\eta n - \gamma n - 5/\eta \geq i\eta n. \end{aligned}$$

Since $i \leq 1/3\eta$, Proposition 4.10(ii) implies that $N^2_G(v_s, v_{s+i}) \setminus V(F_i)$ contains a vertex v_{s+i+1} with

$$d'_G(v_{s+i+1}) \geq (1/3 + (i+1)\eta)n.$$

Therefore (5.8) implies that $F_{i+1} := v_1 \dots v_{s+i+1}$ is a folded path with $k_{s+i+1} = s$.

After $r - s \leq 1/3\eta + 1$ steps we obtain $F'' := F_{r-s} = v_1 \dots v_r$, where $v_r \in N_{F''}(v_s)$ and

$$d'_G(v_r) \geq (1/3 + (1/3\eta + 1)\eta)n = (2/3+\eta)n. \tag{5.17}$$

Now (5.16) implies that $4 \le r \le 4/\eta + 1/3\eta + 1 \le 49/10\eta$. This completes the proof of Step 2.

Step 3. Obtain F.

Let $a, b \in V(G)_{\eta/2}$ be arbitrary. By Proposition 4.1(i), we have that

$$|N_G^2(a,b) \setminus V(F'')| \ge \left(\frac{1}{3} + \eta\right) n - \left(\frac{4}{\eta} + \frac{1}{3\eta} + 1\right) \ge \left(\frac{1}{3} + \frac{\eta}{2}\right) n.$$

Proposition 4.10(i) implies that there exists a set $K(a,b) \subseteq N_G^2(a,b) \setminus V(F'')$ with $|K(a,b)| \ge \eta n/2$, such that for each $x \in K(a,b)$ we have $d'_G(x) \ge (2/3 + \eta)n$. Furthermore, (5.7) implies that $K(a,b) \subseteq V(G)_{\eta/2}$. Observe that $\{v_s, v_r\} \subseteq V(G)_{\eta/2}$. So, for each $1 \le j \le 4$, we can find a distinct vertex v_{r+j} so that $v_{r+1} \in K(v_s, v_r)$, and $v_{r+j} \in K(v_{r+j-2}, v_{r+j-1})$ for all $2 \le j \le 4$. Let $k_1, \dots, k_r$ be the ordering of F''. Then $F := v_1 \dots v_{r+4}$ is a folded path with ordering $k_1, \dots, k_{r+4}$, where

$$k_{r+1} := s, \quad k_{r+2} := r, \quad k_{r+3} := r+1, \quad k_{r+4} := r+2.$$

To see this, observe that $k_i \in \{i-2, k_{i-1}\}$ for all $r+1 \le i \le r+4$ since $k_r = s < r$.

Let $t := r + 4$. Then Step 2 implies that $8 \le t \le 49/10\eta + 4 \le 5/\eta$, as required. Finally, $d'_G(v_{t-1}), d'_G(v_t) \ge (2/3 + \eta)n$, as required. □

Note that in Step 3 of the proof of Lemma 5.6 we add 4 vertices only to ensure that the folded path F has length at least 8 (this property will be useful later on). In particular, we could have guaranteed that the last two vertices of F have 'large' degree by only adding a single vertex in this final step.

5.4. **The proof of Lemma 5.1.** The next lemma shows that, given a suitable framework in the reduced graph R of G, we can find a tail-heavy square path P such that, in two (or three) given clusters, there are many pairs (or triples) of vertices that can be added to the start of P to extend the square path. The necessary framework is a folded path whose first three vertices correspond to these given clusters, and whose final two vertices have large core degree. (Recall that the α-core degree $d^\alpha_{R,G}$ is defined in Section 4.4.) The proof is essentially just an application of Lemmas 4.7 and 5.5; its length is due to technical issues.

Lemma 5.7. *Let $n, L \in \mathbb{N}$ and suppose that $0 < 1/n \ll 1/L \ll \varepsilon \ll c \ll d \ll \eta \ll 1$. Let R be a graph with $V(R) = [L]$. Let G be a graph on n vertices with vertex partition $V_0, V_1, \dots, V_L$ such that $|V_0| \le \varepsilon n$ and so that there exists $m \in \mathbb{N}$ with $|V_i| = (1 \pm \varepsilon)m$ for all $1 \le i \le L$. Further, suppose that $G[V_i, V_j]$ is (ε, d)-regular whenever $ij \in E(R)$. Let $F = i_1 \dots i_t$ be a folded path in R with $8 \le t \le 5/\eta$ such that $d^{2c}_{R,G}(i_{t-1}), d^{2c}_{R,G}(i_t) \ge (2/3 + \eta)L$. Then*

(1) *G contains an η-tail-heavy square path Q with $|Q| \le 11/\eta$, and sets $A_k \subseteq V_{i_k} \setminus V(Q)$ for $k = 1, 2$ with $|A_k| \ge cm$, such that for any $z_k \in A_k$ where $z_1 z_2 \in E(G)$, we have that $z_1 z_2 Q$ is a square path;*
(2) *G contains an η-tail-heavy square path P with $|P| \le 11/\eta$, and sets $B_k \subseteq V_{i_k} \setminus V(P)$ for $k = 1, 2, 3$ with $|B_k| \ge cm$, such that for any $z_k \in B_k$ where $z_1 z_2 z_3$ is a triangle in G, we have that $z_1 z_2 z_3 P$ is a square path. Further, for any $z_2 \in B_2$, $z_3 \in B_3$ such that $z_2 z_3 \in E(G)$ we have that $z_2 z_3 P$ is a square path.*

Moreover, neither P nor Q contain any vertices from V_0.

Proof. We will only prove (2) since the proof of (1) is very similar. Apply Lemma 5.5 with t playing the role of n to obtain a square path $P' := x_1 \dots x_p$ where p satisfies

$$8 \le t \le p \le 2t + 1 \le 10/\eta + 1 \tag{5.18}$$

and a mapping $g : [p] \to [t]$ such that $i_{g(j)} i_{g(k)} \in E(F)$ whenever $x_j x_k \in E(P')$; and $g(1) = 1$, $g(2) = 2$, $g(3) = 3$ and $g(\{p-1, p\}) = \{t-1, t\}$. Let $f : V(P') \to V(F)$ be such that $f(x_j) = i_{g(j)}$ for all $1 \leq j \leq p$. So $f(x)f(y) \in E(F)$ whenever $xy \in E(P')$. Moreover,

$$f(x_1) = i_1, \quad f(x_2) = i_2, \quad f(x_3) = i_3 \quad \text{and} \quad f(\{x_{p-1}, x_p\}) = \{i_{t-1}, i_t\}. \tag{5.19}$$

Let

$$Y := (P')_3^- \cup (P')_2^+ = \{x_1, x_2, x_3, x_{p-1}, x_p\} \quad \text{and let} \quad X := V(P') \setminus Y. \tag{5.20}$$

Observe that $X \neq \emptyset$ by (5.18). Then Lemma 4.7 with $G \setminus V_0, R, P', X, Y, 2c, f$ playing the roles of G, R, H, X, Y, c, f implies that there exists an injective mapping $\tau : X \to V(G)$ with $\tau(x_j) \in V_{f(x_j)}$ for all $4 \leq j \leq p-2$, such that there exist sets

$$C_k \subseteq V_{f(x_k)} \setminus \tau(X) \quad \text{for all} \quad k \in \{1, 2, 3, p-1, p\} \tag{5.21}$$

such that

(i) if $x, x' \in X$ and $xx' \in E(P')$ then $\tau(x)\tau(x') \in E(G)$;
(ii) for all $k \in \{1, 2, 3, p-1, p\}$ we have that $C_k \subseteq N_G(\tau(x))$ for all $x \in N_{P'}(x_k) \cap X$;
(iii) $|C_k| \geq 2c(1-\varepsilon)m$ for all $k \in \{1, 2, 3, p-1, p\}$.

Property (i) implies that $\tau(X)$ spans a square path $P_1 := \tau(x_4) \dots \tau(x_{p-2})$ in G (which contains at least three vertices by (5.18)). We would like to use P_1 to find an η-tail-heavy path. To do this, we will find a short square path whose first two vertices lie in C_{p-1}, C_p respectively, and whose last two vertices both lie in $V(G)_\eta$. Since $d^{2c}_{R,G}(i_t), d^{2c}_{R,G}(i_{t-1}) \geq (2/3 + \eta)L$, there exist sets $B^{t-1} \subseteq V_{i_{t-1}}$ and $B^t \subseteq V_{i_t}$ such that $|B^{t-1}|, |B^t| \geq 2c(1-\varepsilon)m$ and $B^{t-1} \cup B^t \subseteq V(G)_\eta$. Let $C^{t-1} := B^{t-1} \setminus \tau(X)$ and define C^t similarly. Then

$$|C^{t-1}|, |C^t| \geq 2c(1-\varepsilon)m - |\tau(X)| \geq 2c(1-\varepsilon)m - p + 5 \overset{(5.18)}{\geq} 2c(1-\varepsilon)m - 10/\eta + 4 \geq cm.$$

Recall that $i_{t-1} i_t \in E(F)$ by the definition of a folded path. Proposition 5.4 implies that there exists $i_s \in N^2_F(i_{t-1}, i_t)$, i.e. $i_s i_{t-1} i_t$ is a triangle in F.

We will assume that $f(x_{p-1}) = i_{t-1}$ and so $f(x_p) = i_t$ by (5.19) (the other case, when $f(x_{p-1}) = i_t$ and $f(x_p) = i_{t-1}$, is almost identical). Then (5.21) implies that $C_{p-1} \subseteq V_{f(x_{p-1})} \setminus \tau(X) = V_{i_{t-1}} \setminus \tau(X)$. Similarly $C_p \subseteq V_{i_t} \setminus \tau(X)$. Proposition 4.6 applied with $V_{i_{t-1}}, V_{i_t}, V_{i_s}, C_{p-1}, C_p, C^{t-1}, C^t, \tau(X)$ playing the roles of $X_1, X_2, X_3, A_1, A_2, B_1, B_2, W$ implies that G contains a square path $P_2 \in C_{p-1} \times C_p \times V_{i_s} \times C^{t-1} \times C^t$. Write $P_2 := y_1 y_2 y_3 y_4 y_5$. Observe that, by construction, $V(P_2) \cap \tau(X) = \emptyset$, and P_2 is η-tail-heavy. We claim that

$$P := P_1 P_2 = \tau(x_4) \dots \tau(x_{p-2}) y_1 y_2 y_3 y_4 y_5$$

is an η-tail-heavy square path. Since P_1 and P_2 are vertex-disjoint square paths each containing at least two vertices (by (5.18)), it suffices to show that the necessary edges between P_1 and P_2 are present, i.e. that the necessary edges between $\tau(x_{p-3}), \tau(x_{p-2})$ and y_1, y_2 are present. Observe that $N_{P'}(x_{p-1}) \cap X = \{x_{p-3}, x_{p-2}\}$. Then (ii) implies that $y_1 \in C_{p-1} \subseteq N_G(\tau(x_{p-3})) \cap N_G((\tau(x_{p-2}))$, as required. Similarly $y_2 \in C_p \subseteq N_G((\tau(x_{p-2}))$, as required. So P is a square path. Further, by construction, P is disjoint from V_0.

Note further that

$$|P| = |P_1| + |P_2| = p \overset{(5.18)}{\leq} 11/\eta.$$

Let $B_k := C_k \setminus V(P_2)$ for $k = 1, 2, 3$. Then (5.21) implies that

$$B_k \subseteq V_{f(x_k)} \setminus (\tau(X) \cup V(P_2)) \overset{(5.19)}{=} V_{i_k} \setminus V(P).$$

Moreover, (iii) implies that, for $k = 1, 2, 3$, we have $|B_k| \geq 2c(1-\varepsilon)m - |P_2| = 2c(1-\varepsilon)m - 5 \geq cm$. Let $z_k \in B_k$ for $k = 1, 2, 3$ such that $z_1 z_2 z_3$ is a triangle in G. We must show that $z_1 z_2 z_3 P$ is a square path. That is, we need to show that $z_2 \in N_G(\tau(x_4))$ and $z_3 \in N_G(\tau(x_4)) \cap N_G(\tau(x_5))$. But, since $z_k \in C_k$ for all $k = 1, 2, 3$, this is implied by (ii). Similarly, given any $z_2 \in B_2$, $z_3 \in B_3$ such that $z_2 z_3 \in E(G)$, (ii) implies that $z_2 z_3 P$ is a square path. $\square$

In the next lemma, given a small collection of folded paths, we obtain a small collection of short square paths, with certain useful properties. We find a pair of square paths in G corresponding to each of the ℓ triangles $T_i = (i, 1)(i, 2)(i, 3)$ in the reduced graph R of G. The first, P_i, is tail-heavy, and there are many pairs of vertices in $(i, 1) \times (i, 2)$ which can precede P_i. The second, $P_{\ell+i}$, is head-heavy, and there are many pairs of vertices in $(i, 3) \times (i, 1)$ which can succeed $P_{\ell+i}$. The proof is by repeated application of Lemma 5.7.

In the proof of Lemma 5.1, we will find a square path Q_i containing most of the vertices in T_i which will be sandwiched between $P_{\ell+i}$ and P_i. In order to connect Q_i with P_i and $P_{\ell+i}$ we need many pairs of possible start- and endpoints.

Lemma 5.8. *Suppose that $0 < 1/n \ll 1/\ell \ll \varepsilon \ll c \ll d \ll \eta \ll 1$. Let R be a graph with $V(R) = [\ell] \times [3]$. Suppose that G is a graph on n vertices with vertex partition $\{V_0\} \cup \{V_{i,j} : (i, j) \in [\ell] \times [3]\}$ such that $|V_0| \leq \varepsilon n$ and $|V_{i,j}| =: m$ for all $(i, j) \in [\ell] \times [3]$, and $G[V_{i,j}, V_{i',j'}]$ is (ε, d)-regular whenever $(i, j)(i', j') \in E(R)$. Define $\mathcal{V} := \{(i, j) : d^{2c}_{R,G}(V_{i,j}) \geq (2/3 + \eta)3\ell\}$. Let $F_1, \ldots, F_\ell, F'_1, \ldots, F'_\ell$ be a collection of folded paths in R such that, for all $1 \leq i \leq \ell$ we have*

- (F1) *$F_i := v_{i,1} \ldots v_{i,t_i}$ and $F'_i := u_{i,1} \ldots u_{i,s_i}$ where $8 \leq s_i, t_i \leq 5/\eta$;*
- (F2) *$\{v_{i,t_i-1}, v_{i,t_i}, u_{i,s_i-1}, u_{i,s_i}\} \subseteq \mathcal{V}$;*
- (F3) *for all $1 \leq j \leq 3$ we have $v_{i,j} = (i, j)$ and there exists $\sigma_i \in \{(132), (213)\}$ such that $u_{i,\sigma_i(1)} = v_{i,1}$, $u_{i,\sigma_i(2)} = v_{i,2}$ and $u_{i,\sigma_i(3)} = v_{i,3}$.*

Then G contains a collection $\mathcal{P} := \{P_1, \ldots, P_{2\ell}\}$ of vertex-disjoint square paths such that, for all $1 \leq s \leq \ell$, the following hold.

- (P1) *$|P_s|, |P_{\ell+s}| \leq 11/\eta$;*
- (P2) *P_s is η-tail-heavy and $P_{\ell+s}$ is η-head-heavy;*
- (P3) *for $k = 1, 2$, there are sets $A^s_k \subseteq V_{s,k}$ such that $|A^s_k| \geq cm/2$, with the following property: for any $x_k \in A^s_k$ where $x_1 x_2 \in E(G)$ we have that $x_1 x_2 P_s$ is a square path;*
- (P4) *for $j = 3, 1$, there are sets $B^s_j \subseteq V_{s,j}$ such that $|B^s_j| \geq cm/2$, with the following property: for any $y_j \in B^s_j$ where $y_3 y_1 \in E(G)$ we have that $P_{\ell+s} y_3 y_1$ is a square path.*

Proof. Suppose, for some $1 \leq r \leq 2\ell$, we have obtained a collection $\mathcal{P}' = \{P_1, \ldots, P_{r-1}\}$ of vertex-disjoint square paths, such that each P_i with $1 \leq i \leq r - 1$ satisfies the required properties. We will find a suitable embedding of P_r into G.

Observe that

$$3m\ell \leq n = 3m\ell + |V_0| \leq 3m\ell + \varepsilon n \leq 4m\ell. \tag{5.22}$$

For $(i, j) \in [\ell] \times [3]$, let $V'_{i,j} := V_{i,j} \setminus \bigcup_{P \in \mathcal{P}'} V(P)$. Then

$$|V_{i,j} \setminus V'_{i,j}| \leq \frac{11}{\eta}(r - 1) \leq \frac{22}{\eta}\ell \leq \frac{22\varepsilon^2}{\eta}\frac{n}{\ell} \overset{(5.22)}{\leq} \frac{88\varepsilon^2}{\eta}m \leq \frac{\varepsilon m}{3}. \tag{5.23}$$

Proposition 4.3(i) implies that $G[V'_{i,j}, V'_{i',j'}]$ is $(2\varepsilon, d/2)$-regular whenever $(i, j)(i', j') \in E(R)$.

Define V'_0 so that $\{V'_0\} \cup \{V'_{i,j} : (i, j) \in [\ell] \times [3]\}$ is a partition of $V(G)$. Thus, (5.23) implies that $|V'_0| \leq \varepsilon n + \varepsilon m\ell \leq 2\varepsilon n$. We can view the vertices (i, j) in R as corresponding to the clusters $V'_{i,j}$. In particular, if $d^{2c}_{R,G}(V_{i,j}) \geq (2/3 + \eta)3\ell$ then (5.23) implies that $d^{c}_{R,G}(V'_{i,j}) \geq (2/3 + \eta)3\ell$.

We will consider three cases, depending on the value of r, i.e. depending on the properties required of P_r.

Case 1. $1 \leq r \leq \ell$.

Apply Lemma 5.7 to G with $V_0', V_{i,j}', 2\varepsilon, c/2, d/2, \eta, F_r$ playing the roles of $V_0, V_i, \varepsilon, c, d, \eta, F$. Thus Lemma 5.7(1) implies that G contains an η-tail-heavy square path P_r with $|P_r| \leq 11/\eta$ and sets $A_k^s \subseteq V_{s,k}' \setminus V(P_r)$ for $k = 1, 2$ with $|A_k^s| \geq cm/2$ such that for any $x_k \in A_k^s$ where $x_1x_2 \in E(G)$, we have that $x_1x_2P_r$ is a square path. Note that P_r shares no vertex with any square path we have previously embedded (since it is disjoint from V_0'). Therefore P_r has the required properties.

Case 2. *$\ell + 1 \leq r \leq 2\ell$ and $\sigma_{r-\ell} = (132)$.*

Let $s := r - \ell$. The first three vertices in F_s' are $(s, 1), (s, 3), (s, 2)$ in that order. Apply Lemma 5.7 to G with $V_0', V_{i,j}', 2\varepsilon, c/2, d/2, \eta, F_s'$ playing the roles of $V_0, V_i, \varepsilon, c, d, \eta, F$. Thus Lemma 5.7(1) implies that G contains an η-tail-heavy square path Q_s with $|Q_s| \leq 11/\eta$ and sets $B_j^s \subseteq V_{s,j} \setminus V(Q_s)$ for $j = 1, 3$ with $|B_j^s| \geq cm/2$ such that for any $y_j \in B_j^s$ where $y_1y_3 \in E(G)$, we have that $y_1y_3Q_s$ is a square path. Note that Q_s shares no vertex with any square path we have previously embedded (since it is disjoint from V_0'). Finally, observe that $P_r := Q_s^*$ is precisely the required square path.

Case 3. *$\ell + 1 \leq r \leq 2\ell$ and $\sigma_{r-\ell} = (213)$.*

Let $s := r - \ell$. The first three vertices of F_s' are $(s, 2), (s, 1), (s, 3)$ in that order. Apply Lemma 5.7 to G with $V_0', V_{i,j}', 2\varepsilon, c/2, d/2, \eta, F_s'$ playing the roles of $V_0, V_i, \varepsilon, c, d, \eta, F$. Thus Lemma 5.7(2) implies that G contains an η-tail-heavy square path Q_s with $|Q_s| \leq 11/\eta$ and sets $B_j^s \subseteq V_{s,j} \setminus V(Q_s)$ for $j = 1, 3$ such that for any $y_j \in B_j^s$ where $y_1y_3 \in E(G)$, we have that $y_1y_3Q_s$ is a square path. Note that Q_s shares no vertex with any square path we have previously embedded. Finally, observe that $P_r := Q_s^*$ is precisely the required square path (and B_1^s, B_3^s the required sets). □

The final step in this section is to combine Theorem 5.2 and Lemmas 5.6 and 5.8 to prove Lemma 5.1.

Proof of Lemma 5.1. Without loss of generality, we may suppose that $0 < \varepsilon \ll \eta \ll 1$ since proving the lemma for $\varepsilon' \leq \varepsilon$ implies the lemma for ε. Choose further constants d, α with $\varepsilon \ll d \ll \alpha \ll \eta$. Apply Theorem 5.2 to obtain $L_0 \in \mathbb{N}$ such that every $(\eta/2)$-good graph on $L \geq L_0$ vertices contains a perfect K_3-packing. Without loss of generality, we may assume that $1/L_0 \ll \varepsilon$, and that the conclusion of Lemma 5.6 holds with $L_0/2, 7d, \eta/4$ playing the roles of n, γ, η. Apply Lemma 4.2 with parameters $\varepsilon' := \varepsilon^5, L_0$ to obtain M, n_0. Without loss of generality, assume that $1/n_0 \ll 1/M \ll 1/L_0$. We therefore have the hierarchy

$$0 < 1/n_0 \ll 1/M \ll 1/L_0 \ll \varepsilon \ll d \ll \alpha \ll \eta \ll 1.$$

Let G be a graph of order $n \geq n_0$ such that G is η-good. Apply the Regularity lemma (Lemma 4.2) to G with parameters ε', d, L_0 to obtain clusters $V_1, \ldots, V_L$ of size m, an exceptional set V_0, a pure graph G' and a reduced graph R. So $|R| = L$ and $L_0 \leq L \leq M$ and $|V_0| \leq \varepsilon' n$; and $G'[V_j, V_{j'}]$ is (ε', d)-regular whenever $jj' \in E(R)$. By Lemma 4.13(ii), $d^{\alpha}_{R,G}$ and R are both $(\eta/2, L)$-good. Moreover, Lemma 4.13(i) implies that, for all $j \in V(R)$,

$$d_R(j) \geq (1 - 6d) d^{\alpha}_{R,G}(j). \tag{5.24}$$

Theorem 5.2 implies that R contains a perfect K_3-packing $\mathcal{T}$. So there exists an integer ℓ with

$$0 \leq L - 3\ell \leq 2 \tag{5.25}$$

so that $\mathcal{T} := \{T_1, \ldots, T_\ell\}$ contains exactly ℓ triangles. Let $R' := R[V(\mathcal{T})]$. Then $\mathcal{T}$ is a 2-regular spanning subgraph of R'. We have that

$$n = mL + |V_0| \leq mL + \varepsilon' n \overset{(5.25)}{\leq} m(3\ell + 2) + \varepsilon' n \quad \text{and so} \quad n \leq 4m\ell. \tag{5.26}$$

Relabel the vertices in R' so that the ith triangle of $\mathcal{T}$ has vertex set $T_i := \{(i, 1), (i, 2), (i, 3)\}$. So $V(R') = [\ell] \times [3]$. Relabel those clusters of G which correspond to vertices of R' by writing $X_{i,j}$ for the cluster corresponding to (i,j). Choose X_0 so that $\{X_0\} \cup \{X_{i,j} : (i,j) \in [\ell] \times [3]\}$ is a partition of $V(G)$. Note that $|X_0| \le |V_0| + 2m \le 2\varepsilon' n$.

Notice that since G' is the pure graph of G, the definition of core degree implies that for all $X = (i,j) \in V(R')$,

$$d^{\alpha}_{R',G}(X) \ge d^{\alpha}_{R',G'}(X) \ge d^{\alpha}_{R',G}(X) - (d+\varepsilon)|R'| \tag{5.27}$$

and $(d^{\alpha}_{R',G}(X))/|R'| = (d^{\alpha}_{R,G}(X))/|R|$. Thus, Proposition 4.14 implies that $d^{\alpha}_{R',G'}$ and R' are both $(\eta/4, L)$-good. Then (5.24) implies that, for all $X \in V(R')$, we have

$$d_{R'}(X) \ge d_R(X) - 2 \overset{(5.24)}{\ge} (1-6d)d^{\alpha}_{R,G}(X) - 2 \ge (1-7d)d^{\alpha}_{R',G}(X) \overset{(5.27)}{\ge} (1-7d)d^{\alpha}_{R',G'}(X).$$

Let

$$\mathcal{X} := \{(i,j) \in V(R') : d^{\alpha}_{R',G'}((i,j)) \ge (2/3 + \eta/4)L\}. \tag{5.28}$$

For each $1 \le i \le \ell$, apply Lemma 5.6 with $R', 3\ell, T_i, \eta/4, 7d, d^{\alpha}_{R',G'}$ playing the roles of $G, n, T, \eta, \gamma, d'_G$, to show that R' contains a folded path $F_i := v^i_1 \dots v^i_{t_i}$ where $8 \le t_i \le 20/\eta$ and $\{v^i_1, v^i_2, v^i_3\} = T_i$; and $\{v^i_{t_i-1}, v^i_{t_i}\} \subseteq \mathcal{X}$. Without loss of generality, we may assume that

$$v^i_j = (i,j) \quad \text{for} \quad (i,j) \in [\ell] \times [3].$$

Moreover, for each $1 \le i \le \ell$, R' contains a folded path $F'_i := u^i_1 \dots u^i_{s_i}$ where $8 \le s_i \le 20/\eta$, $\{u^i_1, u^i_2, u^i_3\} = T_i$; and $\{u^i_{s_i-1}, u^i_{s_i}\} \subseteq \mathcal{X}$. Further, there exists $\sigma_i \in \{(132), (213)\}$ such that

$$u^i_j = (i, \sigma_i(j)) \quad \text{for} \quad (i,j) \in [\ell] \times [3].$$

Therefore the properties (F1)–(F3) as stated in Lemma 5.8 hold with $\eta/4$ playing the role of η.

Therefore Lemma 5.8 applied with $R', G', X_0, X_{i,j}, \varepsilon', \alpha/2, d, \eta/4, \mathcal{X}, F_i, F'_i$ playing the roles of $R, G, V_0, V_{i,j}, \varepsilon, c, d, \eta, \mathcal{V}, F_i, F'_i$, implies that G' contains a collection $\mathcal{P} := \{P_1, \dots, P_{2\ell}\}$ of vertex-disjoint square paths which satisfy (P1)–(P4) with $\eta/4, \alpha/2$ playing the roles of η, c respectively. In particular, (P1) implies that $|P| \le 44/\eta$ for all $P \in \mathcal{P}$.

For each $1 \le i \le \ell$, write $[P_i]^+_2 =: u_i v_i$ and $[P_{\ell+i}]^-_2 =: v'_i u'_i$. So (P2) implies that

$$\{u_i, v_i, u'_i, v'_i\} \subseteq V(G')_{\eta/4} \subseteq V(G)_{\eta/4}.$$

Now Lemma 5.8 implies that

$$\sum_{P \in \mathcal{P}} |P| \le \frac{88\ell}{\eta} \le \frac{\varepsilon' m}{2}. \tag{5.29}$$

Let $a, b \in V(G)_{\eta/4}$ be arbitrary. By Propositions 4.1(i) and 4.11(i),

$$|N^2_G(a,b)_\eta| \ge \left(\frac{1}{3} + \frac{\eta}{2}\right)n - \frac{n}{3} = \frac{\eta n}{2} \overset{(5.29)}{>} \sum_{P \in \mathcal{P}} |P| + 4\ell.$$

So we can find a collection $\{w_i, x_i, w'_i, x'_i : i \in [\ell]\}$ of distinct vertices disjoint from $\mathcal{P}$ such that $u_i v_i w_i x_i$ is an η-tail-heavy square path in G, and $x'_i w'_i v'_i u'_i$ is an η-head-heavy square path in G. Therefore, for $1 \le i \le \ell$, setting $Q_i := P_i w_i x_i$ and $Q_{\ell+i} := w'_i x'_i P_{\ell+i}$, we have that $\mathcal{Q} := \{Q_1, \dots, Q_{2\ell}\}$ is a collection of vertex-disjoint square paths in G such that $|Q| \le 44/\eta + 4 \le 45/\eta$ for all $Q \in \mathcal{Q}$; for all $1 \le i \le \ell$ we have that Q_i is η-tail-heavy and $Q_{\ell+i}$ is η-head-heavy; and $\mathcal{Q}$ satisfies (P3) and (P4) with $\alpha/2$ playing the role of c. For each $1 \le i \le \ell$ and $k = 1, 2$, let $A^i_k \subseteq X_{i,k}$ be the sets guaranteed by (P3), and for each $j = 3, 1$, let $B^i_j \subseteq X_{i,j}$ be the sets guaranteed by (P4). So $|A^i_k|, |B^i_j| \ge \alpha m/4$.

For $(i,j) \in [\ell] \times [3]$, let $X'_{i,j} := X_{i,j} \setminus \bigcup_{Q \in \mathcal{Q}} V(Q)$. So

$$(1-\varepsilon')m \leq (1-\varepsilon'/2)m - 4\ell \overset{(5.29)}{\leq} |X'_{i,j}| \leq m.$$

Lemma 4.3(i) implies that, whenever $(i,j)(i',j') \in E(R')$, $G'[X'_{i,j}, X'_{i',j'}]$ is $(2\varepsilon', d/2)$-regular.

Recall that $E(\mathcal{T}) = \{(i,j)(i,j') : 1 \leq i \leq \ell, 1 \leq j < j' \leq 3\}$. Apply Lemma 4.5 with $R', G', X'_{i,j}, 3\ell, \mathcal{T}, 2, 2\varepsilon', d/2$ playing the roles of $R, G, V_j, L, H, \Delta, \varepsilon, d$ to obtain a collection $\{Y_{i,j} : (i,j) \in [\ell] \times [3]\}$ of disjoint subsets of $V(G)$ so that, for all $(i,j) \in [\ell] \times [3]$, $Y_{i,j} \subseteq X'_{i,j}$ (so $Y_{i,j} \cap \bigcup_{Q \in \mathcal{Q}} V(Q) = \emptyset$); $G'[Y_{i,j}, Y_{i,j'}]$ is $(\varepsilon'^{1/3}, d/4)$-superregular for all $1 \leq i \leq \ell$ and $1 \leq j < j' \leq 3$; and

$$|Y_{i,j}| =: m' \geq (1-\varepsilon'^{1/3})m \quad \text{for all} \quad (i,j) \in [\ell] \times [3]. \tag{5.30}$$

Lemma 5.8(P3) implies that, for $k \in \{1,2\}$,

$$|A^i_k \cap Y_{i,k}| \geq (\alpha/4 - \varepsilon'^{1/3})m \geq \alpha m'/5,$$

and similarly (P4) implies that, for $j \in \{3,1\}$, $|B^i_j \cap Y_{i,j}| \geq \alpha m'/5$.

Write $P^2_{3m'} = z_1 \dots z_{3m'}$ for the square path on $3m'$ vertices. Let $\phi_i : V(P^2_{3m'}) \to T_i$ be defined as follows. For all integers $0 \leq j < m'$, we set $\phi_i(z_{3j+1}) = (i,3)$, $\phi_i(z_{3j+2}) = (i,1)$, and $\phi_i(z_{3j+3}) = (i,2)$. It is easy to check that, for all $1 \leq i \leq \ell$, ϕ_i is a graph homomorphism; and $|\phi_i^{-1}(x)| = m'$ for all $x \in T_i$.

For each $1 \leq i \leq \ell$ we will (independently) do the following. Apply Theorem 4.8 to the subgraph of G' spanned by $Y_{i,1} \cup Y_{i,2} \cup Y_{i,3}$, with $P^2_{3m'}$ playing the role of H (so $\Delta := 4$) and ϕ_i playing the role of ϕ. (So the remaining parameters are given by $d/4, \alpha/5, \varepsilon'^{1/3}$ playing the roles of d, c, ε.) Identify special vertices $z_1, z_2, z_{3m'-1}, z_{3m'}$ to the corresponding special sets $B^i_3 \cap Y_{i,3}, B^i_1 \cap Y_{i,1}, A^i_1 \cap Y_{i,1}, A^i_2 \cap Y_{i,2}$.

Thus obtain a square path

$$S_i := x^i_{1,3} x^i_{1,1} x^i_{1,2} x^i_{2,3} \dots x^i_{m',3} x^i_{m',1} x^i_{m',2}$$

in G' with $V(S_i) = Y_{i,1} \cup Y_{i,2} \cup Y_{i,3}$ such that

$$x^i_{1,3} \in B^i_3 \cap Y_{i,3}; \quad x^i_{1,1} \in B^i_1 \cap Y_{i,1}; \quad x^i_{m',1} \in A^i_1 \cap Y_{i,1} \quad \text{and} \quad x^i_{m',2} \in A^i_2 \cap Y_{i,2}.$$

Lemma 5.8(P3) implies that $x^i_{m',1} x^i_{m',2} Q_i$ is a square path and (P4) implies that $Q_{\ell+i} x^i_{1,3} x^i_{1,1}$ is a square path.

Let $\mathcal{P}' := \{Q_{\ell+i} S_i Q_i : 1 \leq i \leq \ell\}$. Observe that $\mathcal{P}'$ is a collection of vertex-disjoint square paths. We saw earlier that Q_i is η-tail-heavy and $Q_{\ell+i}$ is η-head-heavy. Therefore each path in $\mathcal{P}'$ is η-heavy. Finally,

$$\sum_{P \in \mathcal{P}'} |P| \quad \geq \quad \sum_{1 \leq i \leq \ell} |S_i| = 3m'\ell \overset{(5.30)}{\geq} 3(1-\varepsilon'^{1/3})m\ell \overset{(5.26)}{\geq} (1-\varepsilon)n.$$

This completes the proof of Lemma 5.1. $\square$

6. Connecting heavy square paths into an almost spanning square cycle

Lemma 5.1 implies that we can obtain a collection $(P_i)_i$ of vertex-disjoint η-heavy square paths, which together cover almost all of the vertices of our η-good graph G. The next stage – the goal of this section – is to connect these paths together into a square cycle, which necessarily covers almost all of the vertices of G. Roughly speaking, we will show that one can connect square paths P and Q into a new square path whose initial segment is P and whose final segment is Q, provided that P is η-tail-heavy and Q is η-head-heavy. This new square path will only contain a small number of vertices which do not lie in P or Q. Then, provided that the additional vertices lie outside of $(P_i)_i$, we can repeat this process to obtain an almost spanning square cycle.

Given a graph G with $ab, cd \in E(G)$, we define an *(ab, cd)-path* to be a square path R in G such that $[R]_2^- = ab$ and $[R]_2^+ = cd$. Note that an (ab, cd)-path is not, for example, a (ba, cd)-path. Given a set of vertices W, we say that a square path P *avoids* W if $V(P) \cap W = \emptyset$.

Definition 6.1. (η-flexibility) *Given $\eta > 0$, we say that a square path P in a graph G is* η-head-flexible *if P is η-head-heavy and $G[(P)_4^-] \cong K_4$. We say that P is* η-tail-flexible *if P is η-tail-heavy and $G[(P)_4^+] \cong K_4$. If P is both η-head- and η-tail-flexible, we say that it is* η-flexible. *We drop the prefix η- if it is clear from the context.*

This concept is useful for the following reason. Suppose that $P = x_1 \dots x_\ell$ is a tail-heavy square path and $\ell \geq 4$. If P is tail-flexible, then $P' := x_1 \dots x_{\ell-2} x_\ell x_{\ell-1}$ is also a tail-heavy square path. So we have more flexibility (in the literal sense) in connecting P (or rather a square path containing the vertices of P) to another square path.

Our first aim will be to extend a tail-heavy square path to a tail-flexible square path.

6.1. **Finding flexible square paths.** Our aim in this subsection is to prove the following lemma, which implies that, given a tail-heavy square path P and a head-heavy square path P', either P and P' can be 'connected' or P and P' can be extended to tail- and head-flexible square paths respectively. Recall that in an η-good graph G on n vertices, $V(G)_\eta$ is the set of all vertices $x \in V(G)$ with $d_G(x) \geq (2/3 + \eta)n$.

Lemma 6.2. *Let $n \in \mathbb{N}$ and $\eta > 0$ such that $0 < 1/n \ll \eta < 1$. Suppose that G is an η-good graph on n vertices. Let a, b, c, d be distinct vertices in $V(G)_\eta$, and suppose that $ab, cd \in E(G)$. Let $W \subseteq V(G) \setminus \{a, b, c, d\}$ with $|W| \leq \eta n/8$. Suppose that G contains no (ab, cd)-path P such that $|P| \leq 17$ and P avoids W. Then there exist square paths S, S' such that all of the following hold.*

(i) *$[S]_2^- = ab$, $[S']_2^+ = cd$ and S, S' avoid W;*
(ii) *$|S|, |S'| \leq 10$ and $V(S) \cap V(S') = \emptyset$;*
(iii) *S is η-tail-flexible and S' is η-head-flexible.*

Proof. Throughout the proof, we will write tail-flexible (head-flexible) for η-tail-flexible (η-head-flexible) and will similarly write tail-heavy and head-heavy. We say that a square path S is *ab-good* if $|S| \leq 10$; $[S]_2^- = ab$; S avoids $W \cup \{c, d\}$; and S is tail-flexible. Analogously, we say that a path S' is *cd-good* if $|S'| \leq 10$; $[S']_2^+ = cd$; S' avoids $W \cup \{a, b\}$; and S' is head-flexible. Suppose that G contains no pair S, S' of vertex-disjoint square paths such that S is ab-good and S' is cd-good. We must show that this implies that G contains an (ab, cd)-path which has at most 17 vertices and avoids W.

Suppose that there is a square path S in G that is ab-good. By our assumption any cd-good square path S' in G is such that $V(S) \cap V(S') \neq \emptyset$. By adding the vertices in $V(S) \setminus \{a, b\}$ to W we now have that $|W| \leq \eta n/7$ and there is no cd-good square path S' in G. Otherwise, we have that there is no square path S in G that is ab-good (and $|W| \leq \eta n/8$). Without loss of generality, assume that $|W| \leq \eta n/7$ and there is no cd-good square path S' in G. (The proof in the other case is essentially identical.)

At every step of the proof, we will have two vertex-disjoint square paths P, P' such that $[P]_2^- = ab$ and $[P']_2^+ = cd$, and $|P|, |P'| \leq 8$; and a set $U := V(G) \setminus (W \cup V(P) \cup V(P'))$ which we call the *surround* of P, P'. Initially, we take $P := ab$ and $P' := cd$. In each step, we modify P, P' so that any new additional vertices were taken from U, and P, P' still satisfy the specified properties. Then we update the surround U of the new P, P'. Note that P' is not head-flexible at any stage (otherwise it is cd-good). Further, in every step we have $|U| \geq (1 - \eta/4)n$. Proposition 4.14 implies that the graph with vertex set $V(G)$ containing every edge of G with at least one endpoint in U is

$(\eta/2,n)$-good. Moreover, for all $x \in V(G)$,

$$d_G(x,U) \geq d_G(x) - \eta n/4. \tag{6.1}$$

Assume, for a contradiction, that there is no (ab,cd)-path in G which has at most 17 vertices and avoids W.

Claim 1. *Suppose that P, P' are vertex-disjoint square paths that avoid W with $|P|, |P'| \leq 8$ and $[P]_2^- = ab, [P']_2^+ = cd$. Let U be the surround of P, P'. Then the following hold:*

(A) *for any 4-segment $x_1x_2y_1y_2$ of P' with $x_1, x_2 \in V(G)_\eta$, we have $N_U^2(x_1,x_2) \cap N_U^2(y_1,y_2) = \emptyset$;*

(B) *for any 2-segment x_1x_2 of P' we have that $N_U^2(x_1,x_2)_\eta$ is an independent set in G;*

(C) *for any 2-segments x_1x_2, y_2y_1 of P, P' respectively, where $x_2y_2 \in E(G)$, we have that $N_U^2(x_1,x_2) \cap N_U^2(y_1,y_2) = \emptyset$.*

We now prove Claim 1. If (A) does not hold, there is some $u \in N_U^2(x_1,x_2) \cap N_U^2(y_1,y_2)$ and then G contains a cd-good path Q (with $|Q| \leq |P'|+1 \leq 9$ and $[Q]_5^- = x_1x_2uy_1y_2$), a contradiction. If (B) does not hold, there is an edge $uv \in E(G[N_U^2(x_1,x_2)_\eta])$ and then G contains a cd-good path Q (with $|Q| \leq |P'|+2 \leq 10$ and $[Q]_4^- = uvx_1x_2$), a contradiction. If (C) does not hold, there is some $z \in N_U^2(x_1,x_2) \cap N_U^2(y_1,y_2)$ and then G contains an (ab,cd)-path Q with $|Q| \leq |P|+|P'|+1 \leq 17$ which avoids W, a contradiction. This completes the proof of the claim.

Observe that, by Propositions 4.1(i) and 4.11(i), for all distinct $u, v \in V(G)_\eta$,

$$|N_U^2(u,v)| \overset{(6.1)}{\geq} 2(2/3+3\eta/4)n - n \geq (1/3+\eta)n \quad \text{and} \quad |N_U^2(u,v)_\eta| \geq \eta n. \tag{6.2}$$

Claim 2. *There exist vertex-disjoint square paths T, T' in G such that $|T|, |T'| \leq 5$; T is tail-heavy and T' is head-heavy; $[T]_2^- = ab$, $[T']_2^+ = cd$; T, T' avoid W; and the final vertex of T is adjacent to the initial vertex of T'.*

We now prove Claim 2. Let U be the surround of ab, cd. By (6.2), there exist $d' \in N_U^2(c,d)_\eta$ and $c' \in N_U^2(d',c)_\eta$ (which are necessarily distinct). Then ab and $c'd'cd$ are vertex-disjoint square paths avoiding W. Remove c', d' from U. So U is the surround of $ab, c'd'cd$. Since $c', d' \in V(G)_\eta$, Claim 1(A) applied to $c'd'cd$ implies that $N_U^2(c',d') \cap N_U^2(c,d) = \emptyset$. Let

$$N := N_U^2(a,b) \quad \text{and} \quad N' := N_U^2(c',d') \cup N_U^2(c,d).$$

Then

$$|N'| = |N_U^2(c',d')| + |N_U^2(c,d)| \overset{(6.2)}{\geq} (2/3+2\eta)n.$$

Proposition 4.11(i) implies that

$$|N'_\eta| \geq (1/3+2\eta)n. \tag{6.3}$$

Let $y \in N_\eta$ be arbitrary ($N_\eta \neq \emptyset$ by (6.2)). Then

$$d_G(y, N'_\eta) \geq d_G(y) - n + |N'_\eta| \overset{(6.3)}{\geq} (2/3+\eta)n - (2/3-2\eta)n = 3\eta n.$$

So there is some $z \in N'_\eta \cap N_G(y)$. Set $T := aby$ and take $T' := zcd$ if $z \in N_U^2(c,d)$, or $T' := zc'd'cd$ if $z \in N_U^2(c',d')$. This completes the proof of Claim 2.

Let T and T' be as in Claim 2. Write $[T]_2^+ := wx$ and $[T']_2^- := x'w'$, where $xx' \in E(G)$ (see Figure 2). Let $t := |T| \leq 5$ and $t' := |T'| \leq 5$. Let U be the surround of T, T' and let $Y := N_U^2(w,x)$ and $Y' := N_U^2(x',w')$. Claim 1(C) applied with $T, T', wx, x'w'$ playing the roles of P, P', x_1x_2, y_2y_1 implies that $Y \cap Y' = \emptyset$. Therefore, by Proposition 4.11(i),

$$|(Y \cup Y')_\eta| \geq |Y \cup Y'| - n/3 = |Y| + |Y'| - n/3 \overset{(6.2)}{\geq} (1/3+2\eta)n.$$

Now $G[(Y \cup Y')_\eta]$ contains no isolated vertices by Proposition 4.11(ii). Observe that $Y'_\eta \neq \emptyset$ by (6.2). Moreover, Claim 1(B) implies that Y'_η is an independent set in G. Therefore every vertex of Y'_η has a neighbour in Y_η. Choose $y' \in Y'_\eta$ and $y \in Y_\eta$ with $y'y \in E(G)$.

We have obtained vertex-disjoint square paths

$$Ty = [T]^-_{t-2}wxy \quad \text{and} \quad y'T' = y'x'w'[T']^+_{t'-2} \quad \text{such that} \quad xx', yy' \in E(G), \tag{6.4}$$

and T is tail-heavy and T' is head-heavy. Remove y, y' from U. So U is the surround of $Ty, y'T'$. Let $Z := N^2_U(x, y)$ and $Z' := N^2_U(y', x')$. Claim 1(C) applied with $Ty, y'T', xy, y'x'$ playing the roles of P, P', x_1x_2, y_2y_1 implies that

$$Z \cap Z' = \emptyset. \tag{6.5}$$

Let $A^{xy} := Z \cap N^1_U(y', x')$ be the set of vertices in U adjacent to both x, y and at least one of y', x'. Define $A^{y'x'} := Z' \cap N^1_U(x, y)$ similarly. So certainly

$$A^{y'x'}_\eta \subseteq Z'_\eta. \tag{6.6}$$

Claim 3. $E(G[A^{xy}_\eta, Z'_\eta]) \neq \emptyset$.

Now (6.5) and (4.1) imply that

$$\emptyset = N^4_U(x, y, y', x') = A^{xy} \cap A^{y'x'} = Z' \cap A^{xy} = Z \cap A^{y'x'}. \tag{6.7}$$

Let $A := N^3_U(x, y, y', x')$. Observe that $A = A^{xy} \cup A^{y'x'}$ and $U \cap \{x, y, y', x'\} = \emptyset$. Then

$$(8/3 + 3\eta)n \overset{(6.1)}{\leq} \sum_{v \in \{x,y,y',x'\}} d_G(v, U) = \sum_{u \in U} d_G(u, \{x, y, y', x'\}) \overset{(6.7)}{\leq} 3|A| + 2(n - |A|) = |A| + 2n,$$

and hence $|A| \geq (2/3 + 3\eta)n$. By Proposition 4.11(i),

$$|A_\eta| \geq (1/3 + 3\eta)n. \tag{6.8}$$

(6.2) implies that $|Z'_\eta| \geq \eta n$. Claim 1(B) applied with $Ty, y'T', y'x'$ playing the roles of P, P', x_1x_2 implies that Z'_η is an independent set in G. Suppose that $E(G[A^{xy}_\eta, Z'_\eta]) = \emptyset$. Then no vertex in Z'_η has a neighbour in A^{xy}_η. Therefore, for all $z \in Z'_\eta$, $N_U(z) \cap ((A^{xy} \cup Z')_\eta) = \emptyset$. So

$$\begin{aligned} |A_\eta| &\overset{(6.7)}{=} |A^{xy}_\eta| + |A^{y'x'}_\eta| \overset{(6.6)}{\leq} |A^{xy}_\eta| + |Z'_\eta| \overset{(6.7)}{=} |(A^{xy} \cup Z')_\eta| \leq |U \setminus N_U(Z'_\eta)| \leq n - \max_{z \in Z'_\eta} d_G(z, U) \\ &\overset{(6.1)}{\leq} (1/3 - 3\eta/4)n, \end{aligned}$$

a contradiction to (6.8). This proves Claim 3.

By Claim 3, we may choose $z \in A^{xy}_\eta$ and $z' \in Z'_\eta$ such that $zz' \in E(G)$. We have shown that G contains vertex-disjoint W-avoiding square paths

$$Tyz = [T]^-_{t-2}wxyz \quad \text{and} \quad z'y'T' = z'y'x'w'[T']^+_{t'-2}; \tag{6.9}$$

such that $xx', yy', zz' \in E(G)$ and one of $zy', zx' \in E(G)$; where $\{w, x, y, z, z', y', x', w'\} \subseteq V(G)_\eta$ (see Figure 2). Claim 2 implies that $|Tyz|, |z'y'T'| \leq 7$. Remove z, z' from U. So U is the surround of $Tyz, z'y'T'$.

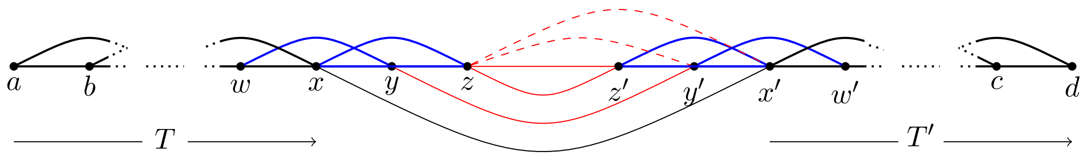

Figure 2: The structure obtained at (6.9). We first obtain the black edges (Claim 2), then the blue edges, then the red edges, where at least one of the dashed red edges is present after (6.9).

We consider two cases, depending on whether $zx' \in E(G)$ or $zy' \in E(G)$.

Case 1. $zx' \in E(G)$.

We will apply Claim 1(A) and (C) with $Tyz, z'y'T'$ playing the roles of P, P'. Claim 1(A) applied with $z'y'x'w'$ playing the role of $x_1x_2y_1y_2$ implies that $N^2_U(z', y') \cap N^2_U(x', w') = \emptyset$. Claim 1(C) applied with $yz, z'y'$ playing the roles of x_1x_2, y_2y_1 implies that $N^2_U(y, z) \cap N^2_U(z', y') = \emptyset$. Claim 1(C) applied with $yz, x'w'$ playing the roles of x_1x_2, y_2y_1 implies that $N^2_U(y, z) \cap N^2_U(x', w') = \emptyset$. Therefore $N^2_U(y, z), N^2_U(z', y'), N^2_U(x', w')$ are pairwise vertex-disjoint subsets of U. But (6.2) implies that each set has size at least $(1/3 + \eta)n$, a contradiction. So we are done in Case 1.

Case 2. $zy' \in E(G)$.

This case is similar to Case 1. Observe that now $Ty, zz'y'T'$ are vertex-disjoint square paths each containing at most eight vertices, and U is the surround of $Ty, zz'y'T'$. We will apply Claim 1(A) and (C) with $Ty, zz'y'T'$ playing the roles of P, P'. Claim 1(A) applied with $zz'y'x'$ playing the role of $x_1x_2y_1y_2$ implies that $N^2_U(z, z') \cap N^2_U(y', x') = \emptyset$. Claim 1(C) applied with xy, zz' playing the roles of x_1x_2, y_2y_1 implies that $N^2_U(x, y) \cap N^2_U(z, z') = \emptyset$. Claim 1(C) applied with $xy, y'x'$ playing the roles of x_1x_2, y_2y_1 implies that $N^2_U(x, y) \cap N^2_U(y', x') = \emptyset$. Therefore $N^2_U(x, y), N^2_U(z, z'), N^2_U(y', x')$ are pairwise vertex-disjoint subsets of U. But (6.2) implies that each set has size at least $(1/3+\eta)n$, a contradiction. So we are done in Case 2.

In both cases we obtain a contradiction to our assumption that there is no (ab, cd)-path in G which has at most 17 vertices and avoids W. This completes the proof of the lemma. $\square$

6.2. **Connecting flexible square paths.** The proof of the next result is similar to that of Lemma 21 in [12] (although there the graph G has minimum degree not much less than $2n/3$ and is 'non-extremal').

Lemma 6.3 (Connecting lemma). *Let $n \in \mathbb{N}$ and $\delta, \eta > 0$ such that $0 < 1/n \ll \delta \ll \eta < 1$. Suppose that G is an η-good graph on n vertices. Let a', b', c', d' be distinct vertices in $V(G)_\eta$ where $a'b', c'd' \in E(G)$. Let $W \subseteq V(G) \setminus \{a', b', c', d'\}$ with $|W| \le \delta n$. Then G contains an $(a'b', c'd')$-path on at most 23 vertices which avoids W.*

Proof. Suppose that G contains no $(a'b', c'd')$-path on at most 17 vertices which avoids W. Apply Lemma 6.2 to obtain vertex-disjoint square paths Q, Q' such that (i)–(iii) hold. (Where $a'b', c'd', Q, Q'$ play the roles of ab, cd, S, S' respectively.) Let $q := |Q| \le 10$ and $q' := |Q'| \le 10$. Write $[Q]^+_2 =: ab$ and $[Q']^-_2 =: cd$ and set $X := \{a, b, c, d\} \subseteq V(G)_\eta$. Let $U := V(G) \setminus (W \cup V(Q) \cup V(Q'))$. Observe that $X \cap U = \emptyset$, and $|U| \ge (1 - 2\delta)n$. Proposition 4.14 implies that $G[U]$ is $(\eta/2, n)$-good. Moreover, for all $x \in V(G)$,

$$d_G(x, U) \ge d_G(x) - \eta n/2. \tag{6.10}$$

Claim. *It suffices to find a path P with $(P)^-_2 = \{a, b\}$, $(P)^+_2 = \{c, d\}$; $V(P) \setminus X \subseteq U$ and $|P| \le 7$.*

To prove the claim, suppose we have such a path P. Note that

$$[Q]^-_{q-2}ab, \quad [Q]^-_{q-2}ba, \quad cd[Q']^+_{q'-2}, \quad dc[Q']^+_{q'-2}$$

are square paths by Lemma 6.2(iii). Then $P' := [Q]^-_{q-2}P[Q']^+_{q'-2}$ is an $(a'b', c'd')$-path which avoids W by Lemma 6.2(i). Finally, Lemma 6.2(ii) implies that $|P'| \leq |Q| + |Q'| + |P| - 4 \leq 23$. This completes the proof of the claim.

For all $1 \leq i \leq 4$, let $S_i := \{v \in U : d_G(v, X) = i\}$. Then

$$(8/3 + 2\eta)n \overset{(6.10)}{\leq} \sum_{x \in X} d_G(x, U) = \sum_{u \in U} d_G(u, X) = \sum_{1 \leq i \leq 4} i|S_i| \leq 4|S_4| + 3|S_3| + 2(n - |S_3| - |S_4|),$$

and therefore

$$|S_3| + 2|S_4| \geq (2/3 + \eta)n. \tag{6.11}$$

Suppose that there is some $xy \in E(G[S_4, S_3 \cup S_4])$. Then G contains a square path P with $V(P) = \{a, b, x, y, c, d\}$ which satisfies the claim. (Indeed, if for example $a \notin N_G(y)$, then we can take $P := abxycd$ or $P := abxydc$; or if $c \notin N_G(x)$, then we can take $P := abxydc$ or $P := baxydc$. The other cases are similar.) Therefore we may assume that

$$E(G[S_4, S_3 \cup S_4]) = \emptyset. \tag{6.12}$$

Suppose that $S_4 \neq \emptyset$. Proposition 4.11(iv) applied with $G[U], \eta/2, S_4, S_3 \cup S_4$ playing the roles of G, η, X, Y implies that $|S_4| + (|S_3| + |S_4|) \leq (2/3 - \eta/2)n$, a contradiction to (6.11). Therefore (6.11) implies that

$$S_4 = \emptyset \quad \text{and} \quad |S_3| \geq (2/3 + \eta)n. \tag{6.13}$$

Let

$$T_{ab} := N^2_U(a, b) \cap S_3 \quad \text{and} \quad T_{cd} := N^2_U(c, d) \cap S_3.$$

Suppose that there exists $x \in T_{ab}$ and $y \in T_{cd}$ such that $xy \in E(G)$. Then G contains a square path P with $V(P) = \{a, b, x, y, c, d\}$ which satisfies the claim. (For example, if $x \in N^3_U(a, b, c)$ and $y \in N^3_U(a, c, d)$ then we can take $P := baxycd$, as in Figure 3. Observe that in this case and the other three cases, there is exactly one such P.) So we may assume that

$$T_{ab} \cap T_{cd} = \emptyset \ ; \quad T_{ab} \cup T_{cd} = S_3 \quad \text{and} \quad E(G[T_{ab}, T_{cd}]) = \emptyset. \tag{6.14}$$

(The first two assertions follow from (6.13) and the definitions.) Proposition 4.11(iv) applied with $G[U], \eta/2, T_{ab}, T_{cd}$ playing the roles of G, η, X, Y implies that, if T_{ab}, T_{cd} are both non-empty, then $|T_{ab}| + |T_{cd}| \leq (2/3 - \eta/2)n$, a contradiction to (6.13) and (6.14). Without loss of generality, we may assume that $T_{ab} = \emptyset$. Therefore $|T_{cd}| \geq (2/3 + \eta)n$. Now, by Proposition 4.1(i) and (6.10), we have that $|N^2_U(a, b)| \geq 2(2/3 + \eta/2)n - n \geq (1/3 + \eta)n$. Proposition 4.10(i) implies that there exists $z \in N^2_U(a, b)_\eta$. Then

$$|N_U(z) \cap T_{cd}| \geq d_G(z, U) + |T_{cd}| - n \overset{(6.10)}{\geq} (1/3 + 3\eta/2)n.$$

So there exists $uv \in E(G[N_U(z) \cap T_{cd}])$ by Proposition 4.11(iii). But then we can set $P := abzuvcd$ if $u \in N_G(b)$; or $P := bazuvcd$ if $u \in N_G(a)$. This completes the proof of the lemma. $\square$

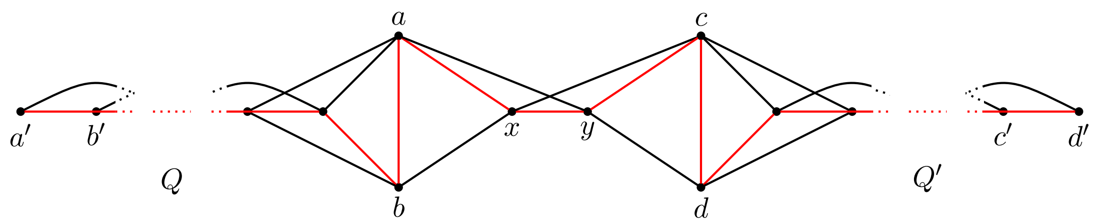

Figure 3: A tail-flexible path Q and head-flexible path Q' with $[Q]_2^+ = ab$ and $[Q']_2^- = cd$ and adjacent vertices $x \in N^3(a,b,c) \subseteq T_{ab}$ and $y \in N^3(a,c,d) \subseteq T_{cd}$. The red line represents the ordering of a square path with vertex set $V(Q) \cup V(Q') \cup \{x,y\}$.

6.3. **An almost spanning square cycle.** The aim of this section is to prove Lemma 6.6, which states that every sufficiently large η-good graph G on n vertices contains a square cycle that covers almost every vertex in G. The idea is to first apply Lemma 5.1 to G to find a collection $\mathcal{P}$ of heavy square paths that cover most of G. Then we repeatedly apply Lemma 6.3 to connect together these square paths into a single almost spanning square cycle in G. If we just apply Lemma 6.3 to G, then when connecting two square paths together we may be forced to use some vertices from other square paths from $\mathcal{P}$. To avoid this problem we in fact connect the square paths from $\mathcal{P}$ together using *only* vertices from a small set $R \subseteq V(G)$ that is disjoint from $\mathcal{P}$. We refer to R as a *reservoir*. R will be constructed in Lemma 6.5 so that $G[R]$ 'inherits' the degree sequence of G (that is $G[R]$ is $(\eta/2, |R|)$-good). This will allow us to apply Lemma 6.3 to $G[R]$ rather that G itself. The idea of connecting paths through a reservoir has been used, for example, in [12, 15, 31, 35].

The hypergeometric random variable X with parameters (n,m,k) is defined as follows. We let N be a set of size n, fix $S \subseteq N$ of size $|S| = m$, pick a uniformly random $T \subseteq N$ of size $|T| = k$, then define $X = |T \cap S|$. Note that $\mathbb{E}X = km/n$. To prove Lemma 6.5 we will use the following standard Chernoff-type bound (see e.g. Theorem 2.10 in [24]).

Proposition 6.4. *Suppose X has hypergeometric distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-\frac{a^2}{3}\mathbb{E}X}$.*

Lemma 6.5. (Reservoir lemma) *Let $n \in \mathbb{N}$ and let $\delta, \eta > 0$ such that $0 < 1/n \ll \delta \ll \eta \ll 1$. Suppose that G is an η-good graph on n vertices. Then there exists $R \subseteq V(G)$ such that $|R| = \delta n$ and*

- *for all $v \in V(G)$ we have $d_G(v,R) \geq (d_G(v)/n - \eta/8)|R|$;*
- *$G[R]$ is $(\eta/2, \delta n)$-good.*

Proof. Choose $R \subseteq V(G)$ uniformly at random from all $\binom{n}{\delta n}$ subsets of $V(G)$ with size δn. We first show that the probability that, for all $v \in V(G)$, we have

$$d_G(v,R) \geq \left(1 - \frac{\eta}{8}\right) \delta d_G(v) \tag{6.15}$$

is more than $1/2$. Indeed, for each $v \in V(G)$, we have

$$\mathbb{E}(d_G(v,R)) = d_G(v)|R|/n = \delta d_G(v).$$

Proposition 6.4 implies that

$$\mathbb{P}\left(d_G(v,R) < \left(1 - \frac{\eta}{8}\right)\delta d_G(v)\right) \leq 2e^{-\eta^2 \delta d_G(v)/192} < 2e^{-\eta^2 \delta n/576} \leq e^{-\sqrt{n}}.$$

(For the penultimate inequality, we used the fact that G is η-good and so $\delta(G) > n/3$.) So taking a union bound over all $v \in V(G)$, we see that the probability that some vertex fails to satisfy (6.15) is at most $ne^{-\sqrt{n}} < 1/2$, as required.

Given $j, \lambda > 0$ and $H \subseteq H' \subseteq G$, let

$$T_{j,\lambda}(H, H') := \{x \in V(H) : d_{H'}(x) \geq (1/3 + \lambda)|H'| + j + 1\}.$$

Note that for all $\kappa \in \mathbb{R}$, whenever $j > \kappa|H'|$ and $\lambda + \kappa > 0$, we have

$$T_{j,\lambda}(H, H') = T_{j-\kappa|H'|,\lambda+\kappa}(H, H'). \tag{6.16}$$

Observe that H is λ-good if and only if, for all integers $1 \leq j \leq |H|/3$ we have $|T_{j,\lambda}(H, H)| \geq |H| - j + 1$. So, since G is η-good, for all integers $1 \leq j \leq n/3$ we have $|T_{j,\eta}(G, G)| \geq n - j + 1$. Observe that

$$\mathbb{E}(|T_{j,\eta}(G[R], G)|) = \delta|T_{j,\eta}(G, G)| \geq \delta(n - j).$$

Proposition 6.4 implies that, for fixed $1 \leq j \leq n/3$,

$$\mathbb{P}\left(|T_{j,\eta}(G[R], G)| < \left(1 - \frac{\eta}{8}\right)\delta(n - j)\right) \leq 2e^{-\eta^2\delta(n-j)/192} \leq 2e^{-\eta^2\delta n/288} \leq e^{-\sqrt{n}}.$$

So the probability that $|T_{j,\eta}(G[R], G)| \leq (1 - \eta/8)\delta(n - j)$ for some integer $1 \leq j \leq n/3$ is at most $ne^{-\sqrt{n}}/3 < 1/2$.

Thus there is some choice of R such that, for all $v \in V(G)$, we have

$$d_G(v, R) \geq \left(1 - \frac{\eta}{8}\right)\delta d_G(v) = \frac{(1 - \eta/8)|R|d_G(v)}{n} \geq \left(\frac{d_G(v)}{n} - \frac{\eta}{8}\right)|R|, \tag{6.17}$$

and for all integers $1 \leq j \leq n/3$ we have

$$|T_{j,\eta}(G[R], G)| \geq \left(1 - \frac{\eta}{8}\right)\delta(n - j). \tag{6.18}$$

To complete the proof, it remains to show that $G[R]$ is $(\eta/2, \delta n)$-good. By an earlier observation, it suffices to show that

$$|T_{i,\eta/2}(G[R], G[R])| \geq \delta n - i + 1 \quad \text{for all integers} \ \ 1 \leq i \leq \delta n/3. \tag{6.19}$$

Let $x \in R$ be arbitrary. Then (6.17) and the fact that G is η-good imply that

$$d_G(x, R) \geq \left(\frac{1}{3} + \frac{7\eta}{8}\right)\delta n \geq \left(\frac{1}{3} + \frac{5\eta}{6}\right)\delta n + 1.$$

A simple rearrangement implies that $|T_{\delta\eta n/3,\eta/2}(G[R], G[R])| = |R| = \delta n$. So, for all $1 \leq i \leq \delta\eta n/3$ we have that $|T_{i,\eta/2}(G[R], G[R])| = \delta n \geq \delta n - i + 1$. Thus, to show (6.18), we may assume that $\delta\eta n/3 < i \leq \delta n/3$ for the remainder of the proof.

By definition, for all $x \in T_{i/\delta,2\eta/3}(G[R], G)$, we have that $d_G(x) \geq (1/3 + 2\eta/3)n + i/\delta + 1$. Therefore (6.17) implies that for such x,

$$d_G(x, R) \geq \left(1 - \frac{\eta}{8}\right)\delta\left(\left(\frac{1}{3} + \frac{2\eta}{3}\right)n + i/\delta + 1\right) \geq \left(\frac{1}{3} + \frac{\eta}{2}\right)\delta n + i + 1.$$

Thus

$$T_{i,\eta/2}(G[R], G[R]) \supseteq T_{i/\delta,2\eta/3}(G[R], G).$$

Therefore for all $\delta\eta n/3 < i \leq \delta n/3$,

$$\begin{aligned} |T_{i,\eta/2}(G[R], G[R])| \quad &\geq \quad |T_{i/\delta,2\eta/3}(G[R], G)| \overset{(6.16)}{=} |T_{i/\delta-\eta n/3,\eta}(G[R], G)| \\ &\overset{(6.18)}{\geq} \quad \left(1 - \frac{\eta}{8}\right)\delta\left(n - \frac{i}{\delta} + \frac{\eta n}{3}\right) = \delta n - i + \frac{\delta\eta n}{24}(5 - \eta) + \frac{\eta i}{8} \geq \delta n - i + 1. \end{aligned}$$

So (6.19) holds, as required. $\square$

We will now combine Lemmas 5.1, 6.3 and 6.5 to prove the main result of this section.

Lemma 6.6. *Let $n \in \mathbb{N}$ and $0 < 1/n \ll \varepsilon \ll \eta \ll 1$. Then every η-good graph G on n vertices contains a square cycle C with $|C| \geq (1-\varepsilon)n$.*

Proof. Apply Lemma 5.1 with $\eta/2, \varepsilon/2$ playing the roles of η, ε to obtain $n_0, M \in \mathbb{N}$ such that every $(\eta/2)$-good graph H on at least n_0 vertices contains a collection of at most M vertex-disjoint $(\eta/2)$-heavy square paths which together cover at least $(1-\varepsilon/2)|H|$ vertices. Note that we may assume that $1/n \ll 1/n_0 \ll 1/M \ll \varepsilon$. Further, choose δ so that $1/M \ll \delta \ll \varepsilon$.

Apply Lemma 6.5 to G to obtain a set R such that $|R| = \delta n$; for all $v \in V(G)$ we have

$$d_G(v, R) \geq \left(\frac{d_G(v)}{n} - \frac{\eta}{8}\right)|R|; \tag{6.20}$$

and $G[R]$ is $(\eta/2, \delta n)$-good.

Note that $|G \setminus R| = (1-\delta)n \geq (1-\eta/4)n$. Proposition 4.14 implies that $G \setminus R$ is $(\eta/2, n)$-good. Lemma 5.1 and the choice of M above implies that $G \setminus R$ contains a collection $\mathcal{P}$ of $m \leq M$ vertex-disjoint $(\eta/2)$-heavy square paths such that

$$\sum_{P \in \mathcal{P}} |P| \geq (1-\varepsilon/2)(1-\delta)n \geq (1-\varepsilon)n. \tag{6.21}$$

Write $\mathcal{P} := \{P_1, \ldots, P_m\}$. Let $P_0 = Q_0 := \emptyset$ and $P_{m+1} := P_1$. For each $0 \leq i \leq m$, we will find a square path Q_i in $G[R]$ such that $P_i Q_i P_{i+1}$ is an $(\eta/2)$-heavy square path in G. Suppose, for some $0 \leq i \leq m-1$, we have obtained vertex-disjoint square paths $Q_0, \ldots, Q_i$ in $G[R]$ such that, for all $0 \leq j \leq i$ we have that $P_j Q_j P_{j+1}$ is an $(\eta/2)$-heavy square path in G, and $|Q_j| \leq 19$. Let $[P_{i+1}]_2^+ = a'b'$ and $[P_{i+2}]_2^- = c'd'$.

Set $G' := G[R \cup \{a', b', c', d'\}]$ and $n' := |G'| = \delta n + 4$. We claim that G' is $(\eta/4, n')$-good. First note that, for all $v \in V(G)_{\eta/2}$, (6.20) implies that

$$d_G(v, V(G')) \geq d_G(v, R) \geq \left(\frac{2}{3} + \frac{3\eta}{8}\right)|R| \geq \left(\frac{2}{3} + \frac{\eta}{4}\right)n' + 1.$$

So, since P_{i+1} and P_{i+2} are $(\eta/2)$-heavy square paths in G, we have $\{a', b', c', d'\} \subseteq V(G')_{\eta/4}$. Fix $1 \leq i \leq n'/3$ and let $X_i \subseteq V(G')$ be such that $|X_i| = i$. We need to show that $\max_{x \in X_i} d_{G'}(x) \geq (1/3 + \eta/4)n' + i + 1$. So we may assume that $\{a', b', c', d'\} \cap X_i = \emptyset$, i.e. $X_i \subseteq R$ (otherwise we are done). Since $G[R]$ is $(\eta/2)$-good, we have that

$$\max_{x \in X_i} d_{G'}(x) \geq \max_{x \in X_i} d_G(x, R) \geq \left(\frac{1}{3} + \frac{\eta}{2}\right)\delta n + i + 1 \geq \left(\frac{1}{3} + \frac{\eta}{4}\right)n' + i + 1,$$

as required. So G' is $(\eta/4, n')$-good.

Let $W := \bigcup_{0 \leq j \leq i} V(Q_j)$. Then $|W| \leq 19M \leq \delta n'$. So we can apply Lemma 6.3 with $G', n', \delta, \eta/4$ playing the roles of G, n, δ, η to find in G' an $(a'b', c'd')$-path P' on at most 23 vertices which avoids W. We take Q_{i+1} to be the square path such that $P' = a'b'Q_{i+1}c'd'$. So $Q_{i+1} \subseteq G[R]$. Then Q_{i+1} is vertex-disjoint from $Q_0, \ldots, Q_i$; $|Q_{i+1}| \leq 19$ and $P_{i+1}Q_{i+1}P_{i+2}$ is an $(\eta/2)$-heavy square path in G.

Follow this procedure until we have obtained $Q_0, \ldots, Q_m$ in $G[R]$ with the required properties. It is easy to see that $C := P_1 Q_1 P_2 \ldots P_m Q_m$ is a square cycle in G. Finally, (6.21) implies that $|C| \geq \sum_{P \in \mathcal{P}} |P| \geq (1-\varepsilon)n$. □

7. An almost spanning triangle cycle

In order to find the square of a Hamilton cycle in G, we will first show that the reduced graph R of G contains an almost spanning subgraph Z_ℓ which itself contains a spanning square cycle, but with some specific additional edges. We call this structure Z_ℓ an 'ℓ-triangle cycle'. The structure Z_ℓ in R will act as a 'framework' for embedding the square of a Hamilton cycle in G. Given $c \in \mathbb{N}$, write C_c^2 for the square cycle on c vertices. So $V(C_c^2) = \{x_1, \dots, x_c\}$, and $x_ix_j \in E(C_c^2)$ whenever $|i-j| \in \{1,2\}$ modulo c. We will often write $C_c^2 =: x_1 \dots x_c$.

Definition 7.1. (ℓ-triangle cycle Z_ℓ) *Write Z_ℓ for the graph with vertex set $[\ell] \times [3]$ such that for all $1 \le i \le \ell$ and distinct $1 \le j, j' \le 3$, we have $(i,j)(i,j') \in E(Z_\ell)$, and $(i,j)(i+1,j') \in E(Z_\ell)$, where addition is modulo ℓ. We call Z_ℓ an* ℓ-triangle cycle.

Let T_ℓ be the spanning subgraph of Z_ℓ such that for all $1 \le i \le i' \le \ell$ and $1 \le j < j' \le 3$, $(i,j)(i',j') \in E(T_\ell)$ whenever $i = i'$. So T_ℓ is a collection of ℓ vertex-disjoint triangles.

So Z_ℓ consists of a cyclically ordered collection of ℓ vertex-disjoint triangles T_ℓ, and between any pair of consecutive triangles, there is a complete bipartite graph minus a perfect matching. We observe the following properties of Z_ℓ:

- $|Z_\ell| = 3\ell$ and Z_ℓ is 6-regular;
- $Z_\ell \supseteq C_{3\ell}^2$, i.e. Z_ℓ contains the square of a Hamilton cycle;
- Z_ℓ is a 3-partite graph (where the vertex (i,j) belongs to the jth colour class);
- Z_ℓ is invariant under permutation of the second index j.

This final property will be crucial when using a copy of Z_ℓ in R to embed the square of a Hamilton cycle in G. We explain this further in Section 8.

The following lemma states that a large η-good graph G contains a copy of Z_ℓ which covers almost every vertex of G. Its proof is a consequence of Theorem 4.8 and Lemma 6.6.

Lemma 7.2. *Let $n \in \mathbb{N}$ and $0 < 1/n \ll \varepsilon \ll \eta \ll 1$. Then, for every η-good graph G on n vertices, there exists an integer ℓ with $(1-\varepsilon)n \le 3\ell \le n$ such that $G \supseteq Z_\ell$.*

A structure very similar to Z_ℓ was used in [9] as a framework for embedding spanning subgraphs of small bandwidth and bounded maximum degree. As such, we believe that Lemma 7.2 could also be applied to embed such subgraphs into graphs satisfying the hypothesis of Theorem 1.3 (see Section 10).

Proof of Lemma 7.2. Let $M \in \mathbb{N}$ and let d be a constant such that $1/n \ll 1/M \ll \varepsilon \ll d \ll \eta$. Apply Lemma 4.2 (the Regularity lemma) to G with parameters ε^4, d, M to obtain a reduced graph R with $|R| =: L$ and pure graph G'. So G has a partition into L clusters $V_1, \dots, V_L$ each of size m, and an exceptional set V_0 of size at most $\varepsilon^4 n$. We may assume that n is sufficiently large so that $1/n \ll 1/L \le 1/M$. Therefore we have the hierarchy

$$0 < 1/n \ll 1/L \ll \varepsilon \ll d \ll \eta \ll 1.$$

Moreover,

$$L \ge \frac{(1-\varepsilon^4)n}{m}. \tag{7.1}$$

Lemma 4.13(ii) implies that R is $(\eta/2, L)$-good. Lemma 6.6 applied with $\eta/2, \varepsilon^4, L$ playing the roles of η, ε, n implies that R contains a square cycle C_c^2 with

$$|C_c^2| = c \ge (1-\varepsilon^4)L. \tag{7.2}$$

So each edge $ij \in E(C_c^2)$ corresponds to an (ε^4, d)-regular pair $G'[V_i, V_j]$ in G'. Lemma 4.5 applied with $C_c^2, 4, \varepsilon^4, d$ playing the roles of $H, \Delta, \varepsilon, d$ implies that each V_i contains a set V_i' with $|V_i'| = (1-\varepsilon^2)m$ such that for every edge ij of C_c^2, the graph $G'[V_i', V_j']$ is $(4\varepsilon^2, d/2)$-superregular. Now vertices in R correspond naturally to the clusters V_i'. Choose $\ell \in 3c\mathbb{N}+1$ such that

$$\left(\frac{1}{3} - \frac{\varepsilon}{3}\right) n < \ell < \left(\frac{1}{3} - \frac{\varepsilon}{2}\right) n. \tag{7.3}$$

(This is possible since $3c \le 3L < \varepsilon n/6$.)

Note that it suffices to find a graph homomorphism $\phi : V(Z_\ell) \to V(C_c^2)$ such that at most $(1-\varepsilon^2)m$ vertices of Z_ℓ are mapped to the same vertex of C_c^2, i.e. that $|\phi^{-1}(w)| \leq (1-\varepsilon^2)m$ for all $w \in V(C_c^2)$. Then Theorem 4.8 (the Blow-up lemma) with $Z_\ell, V_i', (1-\varepsilon^2)m, C_c^2$ playing the roles of H, V_i, n_i, J implies that G contains a copy of Z_ℓ.

We will find ϕ in two stages. We define graph homomorphisms $\phi_1 : V(C_{3c}^2) \to V(C_c^2)$ and $\phi_2 : V(Z_\ell) \to V(C_{3c}^2)$. Then $\phi := \phi_1 \circ \phi_2 : V(Z_\ell) \to V(C_c^2)$ is a graph homomorphism.

Write $C_c^2 := w_1 \dots w_c$ and $C_{3c}^2 := x_1 \dots x_{3c}$. Given integers k, N, write $[k]_N$ for the unique integer in $[N]$ such that $k \equiv [k]_N \mod N$. Let $\phi_1 : V(C_{3c}^2) \to V(C_c^2)$ be defined setting $\phi_1(x_i) = w_{[i]_c}$ for all $1 \leq i \leq 3c$. Then ϕ_1 is a graph homomorphism, and

$$|\phi_1^{-1}(w)| = 3 \quad \text{for all } w \in V(C_c^2). \tag{7.4}$$

For each $1 \leq j \leq 3c$, relabel the vertex x_j of C_{3c}^2 by the ordered pair $(\lceil j/3 \rceil, [j]_3)$. (So the new vertex set is $[c] \times [3]$.) For each $1 \leq j \leq 3c$, let T_j be the triangle in C_{3c}^2 spanned by x_j, x_{j+1}, x_{j+2} (where $x_{3c+1} := x_1$ and $x_{3c+2} := x_2$). So

$$V(T_j) = \{x_j, x_{j+1}, x_{j+2}\} = \left\{ \left(\left\lceil \frac{j}{3} \right\rceil, [j]_3 \right), \left(\left\lceil \frac{j+1}{3} \right\rceil, [j+1]_3 \right), \left(\left\lceil \frac{j+2}{3} \right\rceil, [j+2]_3 \right) \right\}.$$

So for any j, T_j and T_{j+1} have exactly two vertices in common. Observe that $\{[j]_3, [j+1]_3, [j+2]_3\} = [3]$. Let $\phi_2 : V(Z_\ell) \to V(C_{3c}^2)$ be the map that takes a vertex (i,j) to the unique vertex in $T_{[i]_{3c}}$ whose second index is j. This is illustrated in Figure 4.

To see why ϕ_2 is a graph homomorphism, consider an edge $uv \in E(Z_\ell)$. Let S_i be the triangle in Z_ℓ spanned by $(i,1), (i,2), (i,3)$. So ϕ_2 maps each of the vertices of S_i to a distinct vertex in $T_{[i]_{3c}}$. Suppose first that there exists $1 \leq i \leq \ell$ such that u and v both lie in S_i. Then ϕ_2 maps both of u and v to different vertices of the same triangle $T_{[i]_{3c}}$ in C_{3c}^2. So $\phi_2(u)\phi_2(v) \in E(C_{3c}^2)$. Suppose instead that u and v do not lie in the same triangle S_i. Then, since $uv \in E(Z_\ell)$, u and v lie in consecutive triangles. More precisely, there exist $1 \leq i \leq \ell$ and distinct $1 \leq j, j' \leq 3$ such that $u = (i,j)$ and $v = (i+1, j')$ (where $(\ell+1, j') := (1, j')$).

Suppose first that $i \leq \ell - 1$. Then by definition ϕ_2 maps u and v to consecutive triangles T_k and T_{k+1} respectively. It is not hard to see that every pair of the four vertices in $T_k \cup T_{k+1}$ is joined by an edge whenever their second index is different. But the second indices of $\phi_2(u)$ and $\phi_2(v)$ are indeed different since $j \neq j'$. So $\phi_2(u)\phi_2(v) \in E(C_{3c}^2)$.

Suppose instead that $i = \ell$ (observe that we cannot have $i > \ell$). So $u = (\ell, j)$ and $v = (1, j')$ for some distinct $1 \leq j, j' \leq 3$. Since $\ell \in 3c\mathbb{N} + 1$, we have $[\ell]_{3c} = 1 = [1]_{3c}$. So, by the definition of ϕ_2, u is mapped to the unique vertex in T_1 with second index j and v is mapped to the unique vertex in T_1 with second index j'. Since $j \neq j'$, we have $\phi_2(u)\phi_2(v) = (1,j)(1,j') \in E(C_{3c}^2)$.

Therefore ϕ_2, and hence ϕ, is a graph homomorphism. It remains to check that the preimage of each vertex of C_c^2 under ϕ is not too large. First note that

$$\lfloor \ell/c \rfloor \leq |\phi_2^{-1}(x)| \leq \lceil \ell/c \rceil \quad \text{for all } x \in V(C_{3c}^2). \tag{7.5}$$

Thus, for each $1 \leq j \leq c$ we have that

$$\begin{aligned} |\phi^{-1}(w_j)| \quad &\leq \quad \left| \max_{x \in V(C_{3c}^2)} \phi_2^{-1}(x) \right| \left| \max_{w \in V(C_c^2)} \phi_1^{-1}(w) \right| \overset{(7.4),(7.5)}{\leq} 3\lceil \ell/c \rceil < \frac{3\ell}{c} + 3 \\ &\overset{(7.2),(7.3)}{\leq} \quad \frac{(1-3\varepsilon/2)n}{(1-\varepsilon^4)L} + 3 \overset{(7.1)}{\leq} \frac{(1-\varepsilon)m}{(1-\varepsilon^4)^2} \leq (1-\varepsilon^2)m, \end{aligned}$$

as required. $\square$

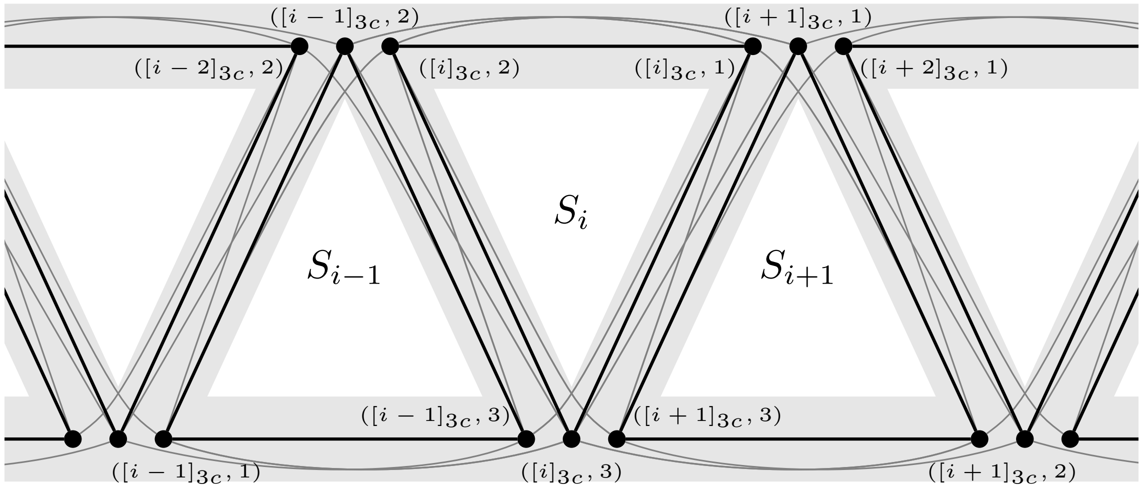

Figure 4: The homomorphism ϕ_2 maps triangles S_i in Z_ℓ (drawn in black) to triangles in C_{3c}^2 (drawn in grey).

8. The square of a Hamilton cycle

The final step in the proof of Theorem 1.3 is to use the almost spanning triangle cycle guaranteed by Lemma 7.2 to obtain the square of a Hamilton cycle.

Let G be a large η-good graph on n vertices. Since the reduced graph R almost inherits the degree sequence of G, we can find an almost spanning ℓ-triangle cycle Z_ℓ in R (whose vertices correspond to clusters and edges to (ε, d)-regular pairs). By removing a small number of vertices, we can ensure that the edges in the triangle packing $T_\ell \subseteq Z_\ell \subseteq R$ are superregular, and each of the 3ℓ clusters has the same size. We say that the collection of clusters now induces a *cycle structure* $\mathcal{C}$ in G. We colour the clusters and vertices in clusters of $\mathcal{C}$ according to the 3-colouring of Z_ℓ (which is unique up to isomorphism), so that both $V_{i,j}$ and $x \in V_{i,j}$ have colour j. It is now a fairly simple consequence of the Blow-up lemma that G contains a square cycle whose vertex set contains precisely the vertices in the clusters of $\mathcal{C}$. In fact, this would still be true as long as the clusters in each triangle in T_ℓ each had the same size (in which case we say that $\mathcal{C}$ is 0-*balanced*).

However, there is a small set V_0 of vertices in G which lie outside any of the clusters of $\mathcal{C}$. We need to incorporate these into the clusters in an appropriate way, and also preserve the structure $\mathcal{C}$ (perhaps with slightly worse parameters). So after any changes to the clusters we require that

(i) regular pairs remain regular;
(ii) superregular pairs remain superregular;
(iii) $\mathcal{C}$ is 0-balanced.

(i) is satisfied as long as no cluster gains or loses too many vertices. For (ii), we need to ensure that, if we insert a vertex v into a cluster V, then v has many neighbours in the two other clusters which lie in the same triangle as V in T_ℓ. (In this case we say that $v \to V$ is *valid*.) It turns out that since $\delta(G) \geq (1/3 + \eta)n$, for each vertex v there are at least $\eta|R|$ clusters V such that $v \to V$ is valid. This appears promising, but recall that a necessary condition for (iii) is that the colour classes in $\mathcal{C}$ are all the same size. However, we may not be able to assign the vertices of $v \in V_0$ so that this is even almost true. For example, every $v \in V_0$ might only have valid clusters in the first colour class.

Given any $v \in V(G)$, we can guarantee more valid clusters V if $d_G(v)$ is larger. In fact, if $v \in V(G)_\eta$, there are $\eta|R|$ triangles $T \in T_\ell$ such that $v \to V$ is valid for every $V \in T$ (see Proposition 8.6). So, if it were true that $V_0 \subseteq V(G)_\eta$, then we could assign each $v \in V_0$ to a triangle in T_ℓ so that no triangle receives too many vertices, and then split the vertices in each triangle among its clusters as equally as possible. Then $\mathcal{C}$ is very close to being 0-balanced (the sizes of clusters in a triangle in T_ℓ differ by at most one).

In order to achieve that $V_0 \subseteq V(G)_\eta$ (see Lemma 8.7), we do the following. Whenever there is $v \in V_0 \setminus V(G)_\eta$, we find many clusters V such that $v \to V$ is valid, and V contains many vertices v' with $d_G(v') \geq d_G(v) + \eta n/4$. Then we swap v and v' without destroying the cycle structure. This process is repeated until no longer possible, in such a way that no cluster X is the location of too many swaps.

Now we have achieved (i) and (ii), and $\mathcal{C}$ is almost 0-balanced. Note that a necessary condition for (iii) is that $3|n$, so assume that this is true. At this stage we appeal to those pairs in $\mathcal{C}$ which correspond to edges in Z_ℓ (not just those in T_ℓ). This is also the stage where having $Z_\ell \subseteq R$ (and not only $C^2_{3\ell} \subseteq R$) is useful. Consider a cluster $V_{i,j}$. Then the fact that $Z_\ell \subseteq R$ ensures that almost every vertex $v \in V_{i,j}$ is such that $v \to V_{i-1,j}$ and $v \to V_{i+1,j}$ are both valid. Applying this repeatedly allows us to make a small number of arbitrary reallocations within a colour class j (see Lemma 8.11).

However, unless the colour classes have equal size (that is, size $n/3$), this procedure can never ensure that $\mathcal{C}$ is 0-balanced. We currently have that the colour classes have close to equal size. Suppose, for example, that colour class 3 is larger than colour class 1, and colour class 2 has exactly the right size. We identify a 'feeder cluster' X_3 in $\mathcal{C}$, whose vertices are all coloured 3, and which has large core degree. Then X_3 contains many vertices of degree at least $(2/3 + \eta)n$. For each of these vertices v, there are many colour 1 clusters V such that $v \to V$ is valid. So we can move a small number of these vertices v to colour 1 clusters so that all the colour classes have the same size (see Lemma 8.9).

8.1. **Cycle structures.** We begin by formally defining a cycle structure.

Definition 8.1. (Cycle structure) *Given an $\ell \times 3$ integer matrix M, integers n, ℓ, a graph G on n vertices, and constants ε, d, we say that G has an $(R, \ell, M, \varepsilon, d)$-*cycle structure *$\mathcal{C}$ if the following hold:*

- (C1) *G has vertex partition $\{V_0\} \cup \{V_{i,j} : (i, j) \in [\ell] \times [3]\}$ where the (i, j)th entry of M is $|V_{i,j}|$ and $|V_0| \leq \varepsilon n$. The sets $V_{i,j}$ are called the* clusters *of $\mathcal{C}$, V_0 is called the* exceptional set *of $\mathcal{C}$, and M is called the* size matrix *of $\mathcal{C}$;*
- (C2) *R has vertex set $[\ell] \times [3]$ and $R \supseteq Z_\ell$ and $G[V_{i,j}, V_{i',j'}]$ is (ε, d)-regular whenever $(i, j)(i', j') \in E(R)$;*
- (C3) *$G[V_{i,j}, V_{i,j'}]$ is (ε, d)-superregular whenever $1 \leq i \leq \ell$ and $1 \leq j < j' \leq 3$.*

We say that $\{V_{i,j} : (i, j) \in [\ell] \times [3]\}$ induces *$\mathcal{C}$. If $V_0 = \emptyset$ we say that $\mathcal{C}$ is* spanning.

Let $\mathcal{C}'$ be the cycle structure obtained from $\mathcal{C}$ by relabelling $V_{i,j}$ by $V_{i,\sigma(j)}$ for all $(i, j) \in [\ell] \times [3]$ and some permutation σ of $[3]$. Since Z_ℓ is invariant under permutation of the second index (as observed immediately after Definition 7.1), $\mathcal{C}'$ is an $(R, \ell, M', \varepsilon, d)$-cycle structure where the (i, j)th entry of M' is $|V_{i,\sigma(j)}|$.

Often we will consider two different cycles structures, say an $(R, \ell, M, \varepsilon, d)$-cycle structure $\mathcal{C}$ and an $(R, \ell, M', \varepsilon', d')$-cycle structure $\mathcal{C}'$. Since the vertex set of R corresponds to both the clusters of $\mathcal{C}$ and $\mathcal{C}'$, it is ambiguous in this case to talk about the core degree $d^\alpha_{R,G}$. Indeed, even though the graph R is the same for both cycle structures $\mathcal{C}$ and $\mathcal{C}'$, the clusters of $\mathcal{C}$ and $\mathcal{C}'$ may be different. We therefore say that $d^\alpha_{R,G}$ is $(\eta, 3\ell)$-*good with respect to* $\mathcal{C}$ to mean that $d^\alpha_{R,G}$ is $(\eta, 3\ell)$-good when considering the vertices of R as corresponding to the clusters of $\mathcal{C}$.

Definition 8.2. (Size matrices) *Given an $n_1 \times n_2$ integer matrix M, we write $M = (m_{i,j})$ if the (i,j)th entry of M is $m_{i,j}$ for all $(i,j) \in [n_1] \times [n_2]$.*

Given integers $k_1 \leq k_2$, we say that M is (k_1,k_2)-bounded if $k_1 \leq m_{i,j} \leq k_2$ for all $(i,j) \in [n_1] \times [n_2]$.

For a non-negative integer k, whenever $|m_{i,j} - m_{i,j'}| \leq k$ for all $1 \leq i \leq n_1$ and $1 \leq j < j' \leq n_2$, we say that M is k-balanced.

If $\sum_{1\leq i\leq n_1} m_{i,j} = \sum_{1\leq i\leq n_1} m_{i,j'}$ for all $1 \leq j, j' \leq n_2$, we say that M has equal columns.

So if $\mathcal{C}$ is an (R,ℓ,M,ε,d)-cycle stucture in which M is (k_1,k_2)-bounded, then

$$(1-\varepsilon)n \leq 3\ell k_2 \quad \text{and} \quad 3\ell k_1 \leq n. \tag{8.1}$$

Observe that, if $\mathcal{C}$ is spanning and M has equal columns, then $3|n$. The columns of M correspond to the colour classes of Z_ℓ.

The purpose of this section is to prove the following lemma, which states that any large η-good graph contains a spanning 0-balanced cycle structure.

Lemma 8.3. *Let $n \in 3\mathbb{N}$, $L_0, L' \in \mathbb{N}$, and let $0 < 1/n \ll 1/L_0 \ll 1/L' \ll \varepsilon \ll d \ll \eta \ll 1$. Suppose that G is an η-good graph on n vertices. Then there exists a spanning subgraph $G' \subseteq G$ and $\ell \in \mathbb{N}$ with $L' \leq \ell \leq L_0$ such that G' has a spanning (R,ℓ,M,ε,d)-cycle structure where M is $((1-\varepsilon)m,(1+\varepsilon)m)$-bounded and 0-balanced.*

The next proposition will be used several times to show that cycle structures are robust in the following sense. If a small number of vertices in a cycle structure are reallocated, so that each of them has many neighbours in appropriate clusters, we still have a cycle structure (with slightly worse parameters). Its proof is a consequence of Proposition 4.4.

Proposition 8.4. *Let $n,\ell,m,r \in \mathbb{N}$ and $0 < 1/n \ll 1/\ell \ll \varepsilon \leq \gamma \ll d < 1$. Suppose that G is a graph on n vertices with an (R,ℓ,M,ε,d)-cycle structure, where $M = (m_{i,j})$ is $(m,(1+\varepsilon)m)$-bounded. Let $\{V_{i,j} : (i,j) \in [\ell] \times [3]\}$ be the set of clusters of $\mathcal{C}$, where $m_{i,j} := |V_{i,j}|$. Suppose that there exists a collection $\mathcal{X} := \{X_{i,j} : (i,j) \in [\ell] \times [3]\}$ of vertex-disjoint subsets of $V(G)$ such that for all $(i,j) \in [\ell] \times [3]$,*

- *$|X_{i,j} \triangle V_{i,j}| \leq \gamma m/2$;*
- *for all $x \in X_{i,j} \setminus V_{i,j}$ we have that $d_G(x,V_{i,j'}) \geq (d-\varepsilon)m$ for all $j' \in [3] \setminus \{j\}$.*

Let $N := (n_{i,j})$ where $n_{i,j} := |X_{i,j}|$. Then, for any $\varepsilon' \geq \varepsilon + 6\sqrt{\gamma}$, we have that $\mathcal{X}$ induces an $(R,\ell,N,\varepsilon',d/2)$-cycle structure $\mathcal{C}'$.

Proof. It is clear that, for all $(i,j) \in [\ell] \times [3]$,

$$(1-\gamma)m \leq |X_{i,j}| \leq (1+2\gamma)m. \tag{8.2}$$

We need to check that $\mathcal{C}'$ satisfies (C1)–(C3). For (C1), it suffices to check that the exceptional set $X_0 := V(G) \setminus \bigcup_{X\in\mathcal{X}} X$ of $\mathcal{C}'$ is such that $|X_0| \leq \varepsilon' n$. Let V_0 be the exceptional set of $\mathcal{C}$. Then $|X_0| \leq |V_0| + \sum_{(i,j)\in[\ell]\times[3]} |V_{i,j} \triangle X_{i,j}| \leq \varepsilon n + 3\ell\gamma m \leq \varepsilon' n$ by (8.1). So (C1) holds.

Note that, since M is $(m,(1+\varepsilon)m)$-bounded, $|X_{i,j} \triangle V_{i,j}| \leq \gamma |X_{i,j}|$. For (C2), let $(i,j)(i',j') \in E(R)$. Then Proposition 4.4 implies that $G[X_{i,j}, X_{i',j'}]$ is $(\varepsilon', d/2)$-regular, as required.

For (C3), let $1 \leq i \leq \ell$ and $1 \leq j < j' \leq 3$. Then, since $(i,j)(i,j') \in E(R)$, Proposition 4.4 implies that it suffices to show that, for all $x \in X_{i,j}$, we have $d_G(x, X_{i,j'}) \geq d|X_{i,j'}|/2$, and for all $y \in X_{i,j'}$, we have $d_G(y, X_{i,j}) \geq d|X_{i,j}|/2$. Let $x \in X_{i,j}$. Suppose first that $x \in V_{i,j}$. Then, since $G[V_{i,j}, V_{i,j'}]$ is (ε,d)-superregular by (C3) for $\mathcal{C}$, we have that $d_G(x,V_{i,j'}) \geq d|V_{i,j'}| \geq dm$. Suppose instead that $x \in X_{i,j} \setminus V_{i,j}$. Then, by hypothesis, $d_G(x,V_{i,j'}) \geq (d-\varepsilon)m$. So for all $x \in X_{i,j}$ we have $d_G(x,V_{i,j'}) \geq (d-\varepsilon)m$. Therefore

$$d_G(x,X_{i,j'}) \geq d_G(x,V_{i,j'}) - |X_{i,j'} \triangle V_{i,j'}| \geq (d-\varepsilon)m - \gamma m \overset{(8.2)}{\geq} \frac{d|X_{i,j'}|}{2},$$

as required. The second assertion follows similarly. This proves (C3). $\square$

Our initial goal is to incorporate each vertex in the exceptional set into a suitable cluster. However, we are only able to do this successfully for vertices with large degree. The following proposition will be used to swap an exceptional vertex with a vertex in a cluster that has larger degree. The cycle structure which remains has the same size matrix M and the exceptional set has the same size. The proposition will be applied repeatedly until every exceptional vertex has degree at least $(2/3+\eta)n$ (see Lemma 8.7).

Proposition 8.5. *Let $n, \ell, m \in \mathbb{N}$ and $0 < 1/n \ll 1/\ell \ll \varepsilon \ll c \ll d \ll \eta < 1 \leq \alpha \leq 1/3\eta + 3/4$. Let G be an η-good graph on n vertices with an $(R, \ell, M, \varepsilon, d)$-cycle structure where M is (m, m)-bounded. Let $\{V_{i,j} : (i,j) \in [\ell] \times [3]\}$ be the set of clusters of $\mathcal{C}$. Suppose further that $d^c_{R,G}$ is $(\eta/2, 3\ell)$-good with respect to $\mathcal{C}$. Let $v \in V(G)$ with $d_G(v) \geq (1/3 + \alpha\eta)n$. Then there exists $I \subseteq V(R)$ with $|I| \geq \eta\ell/10$ such that, for all $(i,j) \in I$, the following hold:*

(i) *for all $j' \neq j$, we have $d_G(v, V_{i,j'}) \geq (d - \varepsilon)m$;*

(ii) *there are at least cm vertices x in $V_{i,j}$ such that $d_G(x) \geq (1/3 + (\alpha + 1/4)\eta)n$.*

Proof. We begin by proving the following claim.

Claim. *Let $I' := \{(i,j) \in V(R) : d_G(v, V_{i,j'}) \geq (d-\varepsilon)m \text{ for all } j' \neq j\}$. Then $|I'| \geq (3\alpha - 1/10)\eta\ell$.*

To prove the claim, define $\overline{d}_G(v) := n - d_G(v)$. For integers $0 \leq p \leq 3$, let

$$K_p := \{1 \leq i \leq \ell : d_G(v, V_{i,j}) \geq (d - \varepsilon)m \text{ for exactly } p \text{ values } j \in [3]\}$$

and $k_p := |K_p|$. Observe that $K_p \cap K_{p'} = \emptyset$ whenever $p \neq p'$. So

$$k_0 + k_1 + k_2 + k_3 = \ell. \tag{8.3}$$

For each $i \in K_2$ there is exactly one $1 \leq j \leq 3$ such that $(i,j) \in I'$, and for each $i \in K_3$ we have $(i,j) \in I'$ for all $1 \leq j \leq 3$. Therefore it suffices to show that $k_2 + 3k_3 \geq (3\alpha - 1/10)\eta\ell$. We have that

$$\begin{aligned}
\overline{d}_G(v) &\geq \sum_{0\leq p\leq 3}\sum_{i\in K_p}\sum_{1\leq j\leq 3}(m - d_G(v, V_{i,j})) \geq \sum_{0\leq p\leq 3}\sum_{i\in K_p}(3-p)(1-d-\varepsilon)m \\
&= (3k_0 + 2k_1 + k_2)(1-d-\varepsilon)m \geq (3k_0 + 2k_1 + k_2)(1-2d)\,m \\
&\overset{(8.3)}{\geq} (3\ell - (k_1+k_2) - (k_2+3k_3))(1-2d)\,m \overset{(8.3)}{\geq} (2\ell - (k_2+3k_3))(1-2d)\,m.
\end{aligned}$$

Suppose that $k_2 + 3k_3 < (3\alpha - 1/10)\eta\ell$. Then

$$\begin{aligned}
\overline{d}_G(v) &\geq (1-2d)\left(2 - 3\alpha\eta + \frac{\eta}{10}\right)m\ell \geq \left(2 - 3\alpha\eta + \frac{\eta}{11}\right)m\ell \overset{(8.1)}{\geq} (1-\varepsilon)\left(\frac{2}{3} - \alpha\eta + \frac{\eta}{33}\right)n \\
&> (2/3 - \alpha\eta)n,
\end{aligned}$$

a contradiction. This completes the proof of the claim.

Recall that $d^c_{R,G}$ is $(\eta/2, 3\ell)$-good. Proposition 4.10(ii) with $R, d^c_{R,G}, I', \eta\ell/10$ playing the roles of G, d'_G, X, k implies that there exists $I \subseteq I'$ with $|I| \geq \eta\ell/10$ such that for every $(i,j) \in I$, we have

$$d^c_{R,G}((i,j)) \geq 3\left(\frac{1}{3} + \frac{\eta}{2}\right)\ell + 3\alpha\eta\ell - \frac{\eta\ell}{5} + 2 \geq 3\left(\frac{1}{3} + \left(\alpha + \frac{1}{4}\right)\eta\right)\ell.$$

The claim together with the fact that $I \subseteq I'$ imply that I satisfies (i). By the definition of core degree, for all $(i,j) \in I$, there are at least $c|V_{i,j}| = cm$ vertices $x \in V_{i,j}$ such that

$$d_G(x) \geq \frac{d^c_{R,G}((i,j))n}{3\ell} \geq \left(\frac{1}{3} + \left(\alpha + \frac{1}{4}\right)\eta\right)n,$$

so I also satisfies (ii). $\square$

The previous proposition will be used to modify our cycle structure slightly so that every exceptional vertex has large degree. The next proposition will be used for incorporating these large degree exceptional vertices into the cycle structure $\mathcal{C}$. It shows that, for each such vertex v, there are many triangles $T \in T_\ell$ such that v can be added to any of the three clusters in T.

Proposition 8.6. *Let $n, \ell, m \in \mathbb{N}$ and $0 < 1/n \ll 1/\ell \ll \varepsilon \ll d \ll \eta < 1$. Suppose that G is a graph on n vertices with an $(R, \ell, M, \varepsilon, d)$-cycle structure $\mathcal{C}$, where M is $(m, (1+\varepsilon)m)$-bounded. Let $\{V_{i,j} : (i,j) \in [\ell] \times [3]\}$ be the set of clusters of $\mathcal{C}$. Let $v \in V(G)$ with $d_G(v) \geq (2/3 + \eta/2)n$. Then there exists $I \subseteq [\ell]$ with $|I| \geq \eta\ell$ such that, for all $i \in I$ and all $j \in [3]$ we have $d_G(v, V_{i,j}) \geq (d - \varepsilon)m$.*

Proof. Let

$$K := \{1 \leq i \leq \ell : \text{ there exists } j \in [3] \text{ such that } d_G(v, V_{i,j}) < (d-\varepsilon)m\}.$$

It suffices to show that $|K| < (1-\eta)\ell$. For all $1 \leq i \leq \ell$, let $U_i := \bigcup_{1 \leq j \leq 3} V_{i,j}$. Then

$$\begin{aligned} d_G(v) &= \sum_{i \in K} d_G(v, U_i) + \sum_{i \notin K} d_G(v, U_i) + d_G(v, V_0) \leq |K|(2 + \varepsilon + d)m + 3(\ell - |K|)(1+\varepsilon)m + \varepsilon n \\ &= 3\ell m - (1 - d + 2\varepsilon)|K|m + 3\varepsilon\ell m + \varepsilon n \overset{(8.1)}{\leq} 3\ell m - (1 - \eta/3)|K|m + 2\varepsilon n. \end{aligned}$$

Suppose, for a contradiction, that $|K| \geq (1-\eta)\ell$. Then

$$d_G(v) \leq \left(3 - \left(1 - \frac{\eta}{3}\right)(1-\eta)\right)\ell m + 2\varepsilon n = 3\left(\frac{2}{3} + \frac{4\eta}{9}\left(1 - \frac{\eta}{4}\right)\right)\ell m + 2\varepsilon n \overset{(8.1)}{<} \left(\frac{2}{3} + \frac{\eta}{2}\right)n,$$

a contradiction. $\square$

The following lemma is used to turn a cycle structure $\mathcal{C}$ which has a constant size matrix and non-empty exceptional set into a spanning 1-balanced cycle structure $\mathcal{C}'$. To prove it, we repeatedly apply Proposition 8.5 to swap vertices in and out of the exceptional set until every exceptional vertex has large degree. We then apply Proposition 8.6 to allocate each of these vertices v to a suitable triangle in T_ℓ, such that v can be placed in any of the three clusters in this triangle. For each triangle, the allocated vertices are then split equally among the clusters so that they have size as equal as possible.

Lemma 8.7. *Let $n, \ell, m \in \mathbb{N}$ and $0 < 1/n \ll 1/\ell \ll \varepsilon \ll c \ll d \ll \eta < 1$. Suppose that G is an η-good graph on n vertices with an $(R, \ell, M, \varepsilon, d)$-cycle structure $\mathcal{C}$, where M is (m, m)-bounded. Suppose further that $d^c_{R,G}$ is $(\eta/2, 3\ell)$-good with respect to $\mathcal{C}$. Then G has a spanning $(R, \ell, N, \varepsilon^{1/3}, d/2)$-cycle structure $\mathcal{C}'$, where N is $(m, (1+\sqrt{\varepsilon})m)$-bounded and 1-balanced. Further, $d^{c/2}_{R,G}$ is $(\eta/2, 3\ell)$-good with respect to $\mathcal{C}'$.*

Proof. Write $V_{i,j}$ for the cluster corresponding to $(i,j) \in V(R)$. Given a vertex $v \in V(G)$ and $(i,j) \in [\ell] \times [3]$, we say that $v \to V_{i,j}$ is *valid* if $d_G(v, V_{i,j'}) \geq (d - \varepsilon)m$ for all $j' \in [3] \setminus \{j\}$. As an initial step, we will prove the following claim.

Claim. *There exist subsets $X_0, X_{i,j}$ of $V(G)$ (for $(i,j) \in [\ell] \times [3]$) so that the following hold:*

(i) *$\{X_0\} \cup \{X_{i,j} : (i,j) \in [\ell] \times [3]\}$ is a partition of $V(G)$;*
(ii) *$|X_0| = |V_0|$ and $|X_{i,j}| = m$ for all $(i,j) \in [\ell] \times [3]$;*
(iii) *$|V_{i,j} \triangle X_{i,j}| \leq 81\varepsilon m/\eta^2$;*
(iv) *for all $v \in X_{i,j} \setminus V_{i,j}$ we have that $v \to V_{i,j}$ is valid;*
(v) *$X_0 \subseteq V(G)_{\eta/2}$.*

To prove the claim, let $K := 4\varepsilon n/3\eta$. Suppose that, for some $0 \leq k \leq K$, we have obtained vertex sets $V_0^k, V_{i,j}^k$ for $(i,j) \in [\ell] \times [3]$ such that the following properties hold:

(α_k) $\{V_0^k\} \cup \{V_{i,j}^k : (i,j) \in [\ell] \times [3]\}$ is a partition of $V(G)$;
(β_k) $|V_0^k| = |V_0|$ and $|V_{i,j}^k| = m$ for all $(i,j) \in [\ell] \times [3]$;
(γ_k) $|V_{i,j} \triangle V_{i,j}^k| \leq 81\varepsilon m/\eta^2$ and $\sum_{(i,j)\in V(R)} |V_{i,j} \triangle V_{i,j}^k| \leq 2k$;
(δ_k) for all $v \in V_{i,j}^k \setminus V_{i,j}$ we have that $v \to V_{i,j}$ is valid;
(ε_k) $S_k := \sum_{v \in V_0^k} d'_G(v)/|V_0| \geq (1/3+\eta)n + k\eta/4\varepsilon$, where $d'_G(v) := \min\{d_G(v), (2/3+\eta)n\}$.

Observe that setting $V_0^0 := V_0$ and $V_{i,j}^0 := V_{i,j}$ for all $(i,j) \in [\ell] \times [3]$ satisfies (α_0)–(ε_0). Indeed, properties (α_0)–(δ_0) are clear; (ε_0) follows from the fact that G is η-good and therefore $\delta(G) \geq (1/3+\eta)n$. So $S_0 \geq (1/3+\eta)n$.

We will show that there is some $k \leq K$ for which we can set $X_0 := V_0^k$ and $X_{i,j} := V_{i,j}^k$ for all $(i,j) \in [\ell] \times [3]$. Observe that ($\alpha_k$)–($\varepsilon_k$) imply that we can do this as long as $V_0^k \subseteq V(G)_{\eta/2}$.

So suppose that $V_0^k \not\subseteq V(G)_{\eta/2}$. In particular, $V_0^k \neq \emptyset$. Let $v_0 \in V_0^k$ be such that $\min_{v \in V_0^k}\{d_G(v)\} = d_G(v_0)$. Then $d_G(v_0) < (2/3+\eta/2)n$, so there is some $1 \leq \alpha < 1/3\eta + 1/2$ such that $d_G(v_0) = (1/3+\alpha\eta)n$. Proposition 8.5 implies that there exists $I \subseteq V(R)$ with $|I| \geq \eta\ell/10$ such that, for all $(i,j) \in I$, the following hold:

- $v_0 \to V_{i,j}$ is valid;
- there are at least cm vertices x in $V_{i,j}$ such that $d_G(x) \geq (1/3 + (\alpha+1/4)\eta)n$.

We claim that $\min_{(i,j)\in I}\{|V_{i,j} \triangle V_{i,j}^k|\} \leq 81\varepsilon m/\eta^2 - 2$. Suppose not. Then

$$\sum_{(i,j)\in I} |V_{i,j} \triangle V_{i,j}^k| \geq |I|\frac{81\varepsilon m}{\eta^2} - 6\ell \overset{(8.1)}{\geq} (1-\varepsilon)\frac{81\varepsilon n}{30\eta} - 6\ell > \frac{8\varepsilon n}{3\eta} = 2K \geq 2k,$$

a contradiction to (γ_k). Therefore we can choose $(i',j') \in I$ with

$$|V_{i',j'} \triangle V_{i',j'}^k| \leq 81\varepsilon m/\eta^2 - 2. \tag{8.4}$$

Let U be the collection of vertices in $V_{i',j'}$ with degree at least $(1/3 + (\alpha+1/4)\eta)n$ in G. Then

$$|U \cap V_{i',j'}^k| \geq |U| - |V_{i',j'}^k \triangle V_{i',j'}| \overset{(8.4)}{\geq} \left(c - \frac{81\varepsilon}{\eta^2}\right)m + 2 \geq \frac{cm}{2} > 0,$$

so we can choose $v_1 \in U \cap V_{i',j'}^k$. For each $(i,j) \in [\ell] \times [3]$, set

$$V_{i,j}^{k+1} := \begin{cases} V_{i,j}^k \cup \{v_0\} \setminus \{v_1\} & \text{if } (i,j) = (i',j') \\ V_{i,j}^k & \text{otherwise;} \end{cases} \tag{8.5}$$

and

$$V_0^{k+1} := V_0 \cup \{v_1\} \setminus \{v_0\}. \tag{8.6}$$

We need to check that (α_{k+1})–(ε_{k+1}) hold. First note that (α_{k+1}) and (β_{k+1}) follow immediately from (α_k) and (β_k) respectively. Property (γ_{k+1}) follows easily from (γ_k), (8.4) and (8.5). To see (δ_{k+1}), (8.5) implies that it suffices to show that $v_0 \to V_{i',j'}$ is valid. But this follows since $(i',j') \in I$.

It remains to prove that (ε_{k+1}) holds. Recall that the choice of v_0 implies that $d_G(v_0) = (1/3+\alpha\eta)n < (2/3+\eta/2)n$. In particular, $d'_G(v_0) = d_G(v_0)$. Suppose first that $d'_G(v_1) = (2/3+\eta)n$. Then $d'_G(v_1) - d'_G(v_0) > \eta n/2$. Suppose instead that $d'_G(v_1) = d_G(v_1)$. Then

$$d'_G(v_1) - d'_G(v_0) \geq \left(\frac{1}{3} + \left(\alpha + \frac{1}{4}\right)\eta\right)n - \left(\frac{1}{3} + \alpha\eta\right)n = \frac{\eta n}{4}.$$

So this latter bound holds in both cases. Therefore

$$S_{k+1} \overset{(8.6)}{=} \sum_{v \in V_0^k} \left(\frac{d'_G(v)}{|V_0|} \right) + \frac{d'_G(v_1) - d'_G(v_0)}{|V_0|} \geq S_k + \frac{\eta n}{4|V_0|} \geq S_k + \frac{\eta}{4\varepsilon} \overset{(\varepsilon_k)}{\geq} \left(\frac{1}{3} + \eta \right) n + (k+1) \frac{\eta}{4\varepsilon},$$

as required.

So, for each $0 \leq k < K$, either the procedure has terminated, or we are able to proceed to step $k+1$. Observe that, for all k, we have $S_k \leq (2/3 + \eta)n$. Moreover, $S_k = (2/3 + \eta)n$ if and only if $V_0 \subseteq V(G)_\eta \subseteq V(G)_{\eta/2}$. Suppose that this iteration does not terminate in at most K steps. Then (ε_K) implies that

$$S_K \geq \left(\frac{1}{3} + \eta \right) n + K\eta/4\varepsilon = \left(\frac{2}{3} + \eta \right) n,$$

as required. So the iteration terminates at some $p \leq 4\varepsilon n/3\eta$. Let $X_0 := V_0^p$ and $X_{i,j} := V_{i,j}^p$ for all $(i,j) \in [\ell] \times [3]$. This completes the proof of the claim.

Now we will use the claim to prove the lemma. For each $x \in X_0$, let

$$S_x := \{1 \leq i \leq \ell : x \to V_{i,j} \text{ is valid for all } 1 \leq j \leq 3\}. \tag{8.7}$$

Property (v) of the claim together with Proposition 8.6 imply that $|S_x| \geq \eta\ell$. Therefore, for each $x \in X_0$ we can choose $i_x \in S_x$, such that for each $i \in [\ell]$, there are at most $|X_0|/\eta\ell$ vertices $x \in X_0$ such that $i = i_x$. For the collection of $x \in X_0$ with $i_x = i$, choose j_x as evenly as possible from [3]. More precisely, for each $x \in X_0$, choose $j_x \in [3]$ so that

$$||\{x \in X_0 : i = i_x, j = j_x\}| - |\{x \in X_0 : i = i_x, j' = j_x\}|| \leq 1 \quad \text{for} \quad 1 \leq j, j' \leq 3. \tag{8.8}$$

Define a partition $\{U_{i,j} : (i,j) \in [\ell] \times [3]\}$ of $V(G)$ by setting

$$U_{i,j} := X_{i,j} \cup \{x \in X_0 : (i_x, j_x) = (i,j)\}. \tag{8.9}$$

Then for all $(i,j) \in [\ell] \times [3]$, part (ii) of the claim implies that

$$0 \leq |U_{i,j}| - m \leq \frac{|X_0|}{\eta\ell} \overset{(8.1)}{\leq} \frac{3\varepsilon m}{(1-\varepsilon)\eta} \leq \frac{4\varepsilon m}{\eta} \leq \sqrt{\varepsilon} m. \tag{8.10}$$

Therefore the $[\ell] \times [3]$ matrix $N = (n_{i,j})$ with $n_{i,j} := |U_{i,j}|$ is $(m, (1+\sqrt{\varepsilon})m)$-bounded. Moreover,

$$|U_{i,j} \triangle V_{i,j}| \leq |U_{i,j} \triangle X_{i,j}| + |X_{i,j} \triangle V_{i,j}| \overset{\text{(iii)}}{\leq} \frac{81\varepsilon m}{\eta^2} + \frac{|V_0|}{\eta\ell} \overset{(8.10)}{\leq} \left(\frac{81\varepsilon}{\eta^2} + \frac{4\varepsilon}{\eta} \right) m \leq \frac{82\varepsilon}{\eta^2} m \overset{(8.10)}{\leq} \frac{82\varepsilon}{\eta^2} |U_{i,j}|.$$

Observe that (8.8), (8.9) and part (ii) of the claim imply that N is 1-balanced. Then Proposition 8.4 (with $164\varepsilon/\eta^2$ playing the role of γ) implies that G contains an $(R, \ell, N, \varepsilon^{1/3}, d/2)$-cycle structure $\mathcal{C}'$, which is spanning.

Now the vertices in R correspond to the clusters $U_{i,j}$. Since $|U_{i,j} \triangle V_{i,j}| \leq 82\varepsilon m/\eta^2$ and $\varepsilon \ll c, \eta$, $d_{R,G}^{c/2}$ is $(\eta/2, 3\ell)$-good with respect to $\mathcal{C}'$. □

The following easy fact is a consequence of the triangle inequality.

Fact 8.8. *Let $a_1, \ldots, a_n \in \mathbb{R}$. Then for all $1 \leq i \leq n$,*

$$\left| a_i - \frac{1}{n} \sum_{1 \leq j \leq n} a_j \right| \leq \frac{n-1}{n} \max_{1 \leq j < k \leq n} |a_j - a_k|.$$

In the next lemma, we make some small changes to ensure that the sizes of the colour classes in our cycle structure $\mathcal{C}$ are equal, i.e. the size matrix has equal columns. Note that this is a necessary condition for $\mathcal{C}$ to be 0-balanced. The proof is as follows. We assume that M is 1-balanced. So the sum of entries in each column is almost equal (to within $\pm\ell$). We show that for each of the three colours (columns) $j = 1, 2, 3$, we can find a 'feeder cluster' X_j of this colour which has large core degree. Each feeder cluster has the property that it contains many vertices x such that, for each j, j', there are many clusters $Y_{j'}$ of colour j' for which $x \to Y_{j'}$ is valid. So if the j'th column has sum which is too small, and the jth column has sum which is too large, we remove some vertices of large degree which lie in X_j and add them to a cluster of colour j'.

Lemma 8.9. *Let $n \in 3\mathbb{N}$ and $\ell, m \in \mathbb{N}$ and $0 < 1/n \ll 1/\ell \ll \varepsilon \ll c \ll d \ll \eta < 1$. Suppose that G is a graph on n vertices with a spanning $(R, \ell, M, \varepsilon, d)$-cycle structure $\mathcal{C}$, where M is $(m, (1+\varepsilon)m)$-bounded and 1-balanced. Suppose further that $d^c_{R,G}$ is $(\eta/2, 3\ell)$-good with respect to $\mathcal{C}$. Then G contains a spanning $(R, \ell, N, \varepsilon^{1/3}, d/2)$-cycle structure $\mathcal{C}'$, where N is 2ℓ-balanced, is $((1-\varepsilon)m, (1+2\varepsilon)m)$-bounded, and has equal columns.*

Proof. Write $V_{i,j}$ for the cluster of $\mathcal{C}$ corresponding to $(i, j) \in V(R)$, and $M := (m_{i,j})$, where $m_{i,j} := |V_{i,j}|$. As before, given a vertex $v \in V(G)$ and $(i, j) \in [\ell] \times [3]$, we say that $v \to V_{i,j}$ is *valid* if $d_G(v, V_{i,j'}) \geq (d - \varepsilon)m$ for all $j' \in [3] \setminus \{j\}$.

For $1 \leq j \leq 3$, let $M_j := \sum_{1 \leq i \leq \ell} m_{i,j}$ be the sum of the entries in the jth column of M. Since $\mathcal{C}$ is spanning,

$$M_1 + M_2 + M_3 = n. \tag{8.11}$$

Since M is 1-balanced, for all $1 \leq j < j' \leq 3$ we have

$$|M_j - M_{j'}| \leq \sum_{1 \leq i \leq \ell} |m_{i,j} - m_{i,j'}| \leq \ell.$$

Therefore Fact 8.8 applied with $3, M_j$ playing the roles of n, a_i together with (8.11) imply that

$$\left| M_j - \frac{n}{3} \right| \leq \frac{2\ell}{3} \quad \text{for} \quad 1 \leq j \leq 3. \tag{8.12}$$

Since $d^c_{R,G}$ is $(\eta/2, |R|)$-good, Proposition 4.10(i) applied with $R, d^c_{R,G}, V(R)$ playing the roles of G, d'_G, X implies that there exists $\mathcal{X} \subseteq V(R)$ with $|\mathcal{X}| = 2|R|/3$ and $d^c_{R,G}(X) \geq (2/3 + \eta/2)|R|$ for all $X \in \mathcal{X}$. Proposition 4.10(ii) applied with $R, d^c_{R,G}, V(R) \setminus \mathcal{X}, \eta|R|/4$ playing the roles of G, d'_G, X, k implies that there exists $\mathcal{Y} \subseteq V(R) \setminus \mathcal{X}$ with $|\mathcal{Y}| \geq \eta|R|/4$ such that every $Y \in \mathcal{Y}$ has $d^c_{R,G}(Y) \geq (2/3 + \eta/4)|R| + 2$. Therefore there are at least $(2/3 + \eta/4)|R|$ vertices $U \in V(R)$ with $d^c_{R,G}(U) \geq (2/3 + \eta/4)|R|$. Then, for each $1 \leq j \leq 3$, there is some $i_j \in [\ell]$ and a *feeder cluster* $X_j := V_{i_j,j}$ such that $d^c_{R,G}((i_j, j)) \geq (2/3 + \eta/4)|R|$. Let $I' := \{i_1, i_2, i_3\}$. By definition of core degree, there exists $C_j \subseteq X_j$ such that $|C_j| \geq c|X_j| \geq cm$ and $d_G(x) \geq (2/3 + \eta/4)n$ for all $x \in C_j$. Proposition 8.6 applied with $\eta/2$ playing the role of η implies that for $x \in C_j$, there exists $I_x(j) \subseteq [\ell]$ with $|I_x(j)| \geq \eta\ell/2$ such that, for all $i' \in I_x(j)$ and $j' \in [3]$, we have $d_G(x, V_{i',j'}) \geq (d - \varepsilon)m$.

M has equal columns if $M_1 = M_2 = M_3 = n/3$. By the observation immediately after Definition 8.1, we may suppose without loss of generality that $M_1 \leq M_2 \leq M_3$. So $M_1 \leq M_2, n/3 \leq M_3$. In fact we will assume that

$$M_1 \leq M_2 \leq \frac{n}{3} \leq M_3. \tag{8.13}$$

(The other case is similar.) We wish to move some suitable vertices from the feeder cluster X_3 into clusters of colours 1 and 2 so that the new column sums are equal. Choose $B_{3,2} \subseteq C_3$ with

$$|B_{3,2}| = \frac{n}{3} - M_2 \overset{(8.12)}{\leq} \frac{2\ell}{3} \leq |C_3|. \tag{8.14}$$

For each $x \in B_{3,2}$, we can choose an arbitrary $i_x \in I_x(3) \setminus I'$ so that $x \to V_{i_x,2}$ is valid. We have

$$M_3 - |B_{3,2}| \overset{(8.11),(8.14)}{=} \frac{2n}{3} - M_1 \overset{(8.13)}{\geq} M_1.$$

Choose $B_{3,1} \subseteq C_3 \setminus B_{3,2}$ with

$$|B_{3,1}| = \frac{n}{3} - M_1 \overset{(8.12)}{\leq} \frac{2\ell}{3} \overset{(8.14)}{\leq} |C_3 \setminus B_{3,2}|. \tag{8.15}$$

For each $x \in B_{3,1}$, we can choose an arbitrary $i_x \in I_x(3) \setminus I'$ so that $x \to V_{i_x,1}$ is valid.

For $j = 1, 2$, let $X_{i,j} := V_{i,j} \cup \{x \in B_{3,j} : i_x = i\}$ and let $X_{i,3} := V_{i,3} \setminus (B_{3,1} \cup B_{3,2})$. For all $(i,j) \in [\ell] \times [3]$, let $n_{i,j} := |X_{i,j}|$ and let $N := (n_{i,j})$. Now $\{X_{i,j} : (i,j) \in [\ell] \times [3]\}$ is a partition of $V(G)$. We claim that it induces a spanning cycle structure $\mathcal{C}'$.

Observe that

$$M_1 + |B_{3,1}| = M_2 + |B_{3,2}| = M_3 - |B_{3,1}| - |B_{3,2}| = \frac{n}{3}. \tag{8.16}$$

So, for $j = 1, 2$ we have

$$\sum_{1 \leq i \leq \ell} n_{i,j} = \sum_{1 \leq i \leq \ell} m_{i,j} + |B_{3,j}| = M_j + |B_{3,j}| \overset{(8.16)}{=} \frac{n}{3}$$

and similarly $\sum_{1 \leq i \leq \ell} n_{i,3} = n/3$. So N has equal columns. Note that $X_{i_3,1} = V_{i_3,1}$ and $X_{i_3,2} = V_{i_3,2}$ and $X_{i_3,3} = V_{i_3,3} \setminus (B_{3,1} \cup B_{3,2})$. So

$$|n_{i_3,j} - n_{i_3,j'}| \leq |m_{i_3,j} - m_{i_3,j'}| + |B_{3,1}| + |B_{3,2}| \overset{(8.14),(8.15)}{\leq} 1 + \frac{4\ell}{3} \leq 2\ell.$$

Suppose that $i \neq i_3$. Then $X_{i,3} = V_{i,3}$ and

$$|n_{i,j} - n_{i,j'}| \leq |m_{i,j} - m_{i,j'}| + \max\{|B_{3,1}|, |B_{3,2}|\} \leq 1 + \frac{2\ell}{3} \leq 2\ell.$$

So N is 2ℓ-balanced. Similar calculations show that, for all $(i,j) \in [\ell] \times [3]$,

$$|X_{i,j} \triangle V_{i,j}| \leq |B_{3,1}| + |B_{3,2}| \leq 2\ell.$$

Thus,

$$(1-\varepsilon)m \leq m - 2\ell \leq |X_{i,j}| \leq (1+\varepsilon)m + 2\ell \leq (1+2\varepsilon)m.$$

So N is $((1-\varepsilon)m, (1+2\varepsilon)m)$-bounded. For all $v \in X_{i,j} \setminus V_{i,j}$ we have $i \in I_v(3) \subseteq C_3$, so $d_G(v, V_{i,j'}) \geq (d-\varepsilon)m$ for all $j' \in [3]$. Then Proposition 8.4 implies that the partition into $X_{i,j}$s induces a spanning $(R, \ell, N, \varepsilon^{1/3}, d/2)$-cycle structure $\mathcal{C}'$. □

The next proposition shows that $Z_\ell \subseteq R$ implies that it is easy to slightly change the size of clusters in the same colour class in our cycle structure. That is, given $V_{i,j}$ and $V_{k,j}$, we can increase $|V_{k,j}|$ by b and decrease $|V_{i,j}|$ by b, so long as b is not too large. We achieve this by successively moving vertices from $V_{i,j}$ to $V_{i+1,j}$, then $V_{i+1,j}$ to $V_{i+2,j}$, and so on, until we reach $V_{k,j}$. In terms of size matrices, this means we can redistribute the weight within a column.

Proposition 8.10. *Let $n, \ell, m \in \mathbb{N}$ and $0 < 1/n \ll 1/\ell \ll \varepsilon \ll d \ll \eta < 1$. Suppose that G is a graph on n vertices with a spanning $(R, \ell, M, \varepsilon, d)$-cycle structure $\mathcal{C}$, where M is $((1-\varepsilon)m, (1+2\varepsilon)m)$-bounded. Let $(i,j) \in [\ell] \times [3]$. Then there exist at least $(1-8\varepsilon)m$ vertices $v \in V_{i,j}$ such that $d_G(v, V_{i\pm1,j'}) \geq (d-2\varepsilon)m$ for all $j' \in [3] \setminus \{j\}$ (and addition is modulo ℓ).*

Proof. Recall that, since $Z_\ell \subseteq R$ by (C2), we have that $(i,j)(i\pm 1,j') \in E(R)$ for all $j' \in [3]\setminus\{j\}$. Then the statement $(\star)$ in Subsection 4.2 implies that there exist four sets $X^{\pm}_{j'} \subseteq V_{i,j}$ with $|X^{\pm}_{j'}| \geq (1-\varepsilon)|V_{i,j}| \geq (1-2\varepsilon)m$ such that every $x \in X^{\pm}_{j'}$ has $d_G(x, V_{i\pm 1,j'}) \geq (d-\varepsilon)|V_{i\pm 1,j'}| \geq (d-2\varepsilon)m$. Observe that the intersection of these sets has size at least $(1-8\varepsilon)m$, and every vertex within has the required properties. $\square$

Suppose that, instead of $Z_\ell \subseteq R$, we could only guarantee that $C^2_{3\ell} \subseteq R$. Then the conclusion of the previous proposition may fail to hold. For example, neither $(i,2)(i-1,1)$ nor $(i,2)(i+1,3)$ may be edges of R. Then it could be that every vertex $x \in V_{i,2}$ has $d_G(x,V_{i-1,1}) = d_G(x,V_{i+1,3}) = 0$. So in this case no vertex in $V_{i,2}$ can be moved to $V_{i-1,2}$ or $V_{i+1,2}$.

Now, given a cycle structure that has a 2ℓ-balanced size matrix with equal columns, we repeatedly apply Proposition 8.10 to obtain a 0-balanced cycle structure.

Lemma 8.11. *Let $n \in 3\mathbb{N}$ and $\ell, m \in \mathbb{N}$ and $0 < 1/n \ll 1/\ell \ll \varepsilon \ll d \ll \eta < 1$. Suppose that G is a graph on n vertices with a spanning (R,ℓ,M,ε,d)-cycle structure $\mathcal{C}$, where M is $((1-\varepsilon)m,(1+\varepsilon)m)$-bounded, 2ℓ-balanced, and has equal columns. Then G has a spanning $(R,\ell,N,\varepsilon^{1/3},d/2)$-cycle structure $\mathcal{C}'$ such that N is $((1-2\varepsilon)m,(1+2\varepsilon)m)$-bounded and 0-balanced.*

Proof. Write $\{V_{i,j} : (i,j) \in [\ell]\times[3]\}$ for the collection of clusters in $\mathcal{C}$, and write $M =: (m_{i,j})$, where $m_{i,j} := |V_{i,j}|$. Given a vertex $v \in V(G)$ and $(i,j) \in [\ell]\times[3]$, we say that $v \to V_{i,j}$ is *valid* if $d_G(v,V_{i,j'}) \geq (d-2\varepsilon)m$ for all $j' \in [3]\setminus\{j\}$.

We claim that, for each $1 \leq i \leq \ell$, there exists $n_i \in \mathbb{N}$ so that

$$|n_i - m_{i,j}| \leq 2\ell \quad \text{for all} \quad j = 1,2,3, \quad \text{and} \quad \sum_{1\leq i\leq \ell} n_i = \frac{n}{3}. \tag{8.17}$$

To see this, let $\overline{m_i} := (m_{i,1}+m_{i,2}+m_{i,3})/3$. As an initial try, take $n_i := \lceil \overline{m_i} \rceil$ for all i. Then, since $\mathcal{C}$ is spanning,

$$\frac{n}{3} = \frac{1}{3}\sum_{(i,j)\in[\ell]\times[3]} m_{i,j} \leq \sum_{1\leq i\leq\ell} n_i < \sum_{1\leq i\leq\ell}(\overline{m_i}+1) = \frac{n}{3}+\ell$$

and so $0 \leq \sum_{1\leq i\leq\ell} n_i - n/3 \leq \ell-1$. Since this value is less than the number of n_is, we can reduce exactly $\sum_{1\leq i\leq\ell} n_i - n/3$ of them by one. So, for each i we have $n_i \in \{\lceil\overline{m_i}\rceil, \lceil\overline{m_i}\rceil - 1\}$. Therefore $|n_i - \overline{m_i}| \leq 1$ for all $1 \leq i \leq \ell$. Recall that M is 2ℓ-balanced. Fact 8.8 applied for $1 \leq j \leq 3$ with $3, m_{i,j}$ playing the roles of n, a_j implies that $|m_{i,j} - \overline{m_i}| \leq 4\ell/3$. But then, for each $1 \leq i \leq \ell$ we have

$$|n_i - m_{i,j}| \leq |n_i - \overline{m_i}| + |\overline{m_i} - m_{i,j}| \leq 1 + 4\ell/3 \leq 2\ell,$$

proving the claim.

In the remainder of the proof, we will adjust $\mathcal{C}$ until it has size matrix $N = (n_{i,j})$ where $n_{i,j} := n_i$ for all $(i,j) \in [\ell]\times[3]$. Let $K := 3\ell^2$. Suppose, for some $0 \leq k < K$, we have found for each $(i,j) \in V(R)$ subsets $V^k_{i,j} \subseteq V(G)$ such that the following hold:

- (α_k) $\{V^k_{i,j} : (i,j) \in [\ell]\times[3]\}$ is a partition of $V(G)$;
- (β_k) for all $v \in V^k_{i,j}\setminus V_{i,j}$ we have that $v \to V_{i,j}$ is valid;
- (γ_k) for all $(i,j) \in [\ell]\times[3]$ we have $|V^k_{i,j} \triangle V_{i,j}| \leq 2k$;
- (δ_k) for all $1 \leq j \leq 3$ we have $\sum_{1\leq i\leq\ell}|V^k_{i,j}| = n/3$, and $\sum_{(i,j)\in[\ell]\times[3]} ||V^k_{i,j}| - n_i| \leq 6\ell^2 - 2k$.

Notice that we can set $V^0_{i,j} := V_{i,j}$ for all $(i,j) \in [\ell]\times[3]$. Indeed, since $\mathcal{C}$ is spanning, (α_0) holds. Properties (β_0) and (γ_0) are vacuous. To see (δ_0), note that, for all $1 \leq j \leq 3$, $\sum_{1\leq i\leq\ell}|V^0_{i,j}| =$

$\sum_{1\leq i\leq \ell} m_{i,j} = n/3$ since M has equal columns. Furthermore,

$$\sum_{(i,j)\in[\ell]\times[3]} ||V^0_{i,j}| - n_i| = \sum_{(i,j)\in[\ell]\times[3]} |m_{i,j} - n_i| \overset{(8.17)}{\leq} 6\ell^2.$$

If $|V^k_{i,j}| = n_i$ for all $(i,j) \in [\ell]\times[3]$, then we stop. Otherwise, we will obtain sets $V^{k+1}_{i,j}$ from $V^k_{i,j}$. Since $\sum_{(i,j)\in[\ell]\times[3]} |V^k_{i,j}| = 3\sum_{1\leq i\leq\ell} n_i = n$ by (α_k) and (8.17), and $n_i \in \mathbb{N}$, there exists some $(i^+, j_0) \in [\ell]\times[3]$ such that $|V^k_{i^+,j_0}| \geq n_{i^+}+1$. Now, since $\sum_{1\leq i\leq\ell} |V^k_{i,j_0}| = n/3 = \sum_{1\leq i\leq\ell} n_i$ by (δ_k), there exists $1 \leq i^- \leq \ell$ such that $|V^k_{i^-,j_0}| \leq n_{i^-} - 1$.

Proposition 8.10 applied repeatedly implies that, for all integers $r \geq 0$, there exist $(1-8\varepsilon)m$ vertices $v \in V_{i^++r,j_0}$ such that $v \to V_{i^++r+1,j_0}$ is valid (where, here and for the rest of the proof, addition is modulo ℓ). Let r_0 be the least non-negative integer such that $i^+ + r_0 + 1 \equiv i^- \mod \ell$. So $0 \leq r \leq \ell - 1$. Now $(1-8\varepsilon)m - 2K = (1-8\varepsilon)m - 6\ell^2 > m/2$ so (γ_k) implies that for each $0 \leq r \leq r_0$, we can find $x_r \in V^k_{i^++r,j_0}$ such that $x_r \to V_{i^++r+1,j_0}$ is valid.

For each $(i,j) \in [\ell]\times[3]$, set

$$V^{k+1}_{i,j} := \begin{cases} V^k_{i,j} \setminus \{x_0\} & \text{if } (i,j) = (i^+, j_0) \\ V^k_{i,j} \cup \{x_{i-1}\} \setminus \{x_i\} & \text{if } i^+ + 1 \leq i \leq i^+ + r_0 \text{ and } j = j_0 \\ V^k_{i,j} \cup \{x_{i-1}\} & \text{if } (i,j) = (i^-, j_0) \\ V^k_{i,j} & \text{otherwise.} \end{cases}$$

The definition of r_0 implies that (α_{k+1}) holds. The choice of x_r implies that (β_{k+1}) holds. We have

$$|V^{k+1}_{i,j} \triangle V_{i,j}| \leq |V^{k+1}_{i,j} \triangle V^k_{i,j}| + |V^k_{i,j} \triangle V_{i,j}| \overset{(\gamma_k)}{\leq} 2(k+1),$$

proving (γ_{k+1}). Finally, observe that $||V^{k+1}_{i^\pm,j_0}| - n_{i^\pm}| = ||V^k_{i^\pm,j_0}| - n_{i^\pm}| - 1$ and $|V^{k+1}_{i,j}| = |V^k_{i,j}|$ for all other (i,j). Therefore

$$\sum_{(i,j)\in[\ell]\times[3]} ||V^{k+1}_{i,j}| - n_i| = \sum_{(i,j)\in[\ell]\times[3]} ||V^k_{i,j}| - n_i| - 2 \overset{(\delta_k)}{\leq} 6\ell^2 - 2(k+1),$$

proving (δ_{k+1}).

So, for each $0 \leq k \leq K$, either the procedure has terminated, or we are able to proceed to step $k+1$. Therefore there is some $p \leq K = 3\ell^2$ such that $\sum_{(i,j)\in[\ell]\times[3]} ||V^p_{i,j}| - n_i| = 0$. So $|V^p_{i,j}| = n_i$ for all $(i,j) \in [\ell]\times[3]$. Set $X_{i,j} := V^p_{i,j}$ for all $(i,j) \in [\ell]\times[3]$.

We claim that the partition into $X_{i,j}$s induces a spanning cycle structure $\mathcal{C}'$. Let $N := (n_{i,j})$ where $n_{i,j} := n_i$ for all $(i,j) \in [\ell]\times[3]$. Then N is the size matrix of $\mathcal{C}'$ and is 0-balanced by definition. Note that, by (γ_p), for all $(i,j) \in [\ell]\times[3]$ we have

$$(1-2\varepsilon)m \leq (1-\varepsilon)m - 2K \leq |X_{i,j}| \leq (1+\varepsilon)m + 2K \leq (1+2\varepsilon)m.$$

So N is $((1-2\varepsilon)m, (1+2\varepsilon)m)$-bounded. Finally, Proposition 8.4 implies that $\mathcal{C}'$ is an $(R, \ell, N, \varepsilon^{1/3}, d/2)$-cycle structure $\mathcal{C}'$. $\square$

We are now able to prove the main result of this section, Lemma 8.3.

Proof of Lemma 8.3. Suppose that G is a sufficiently large graph on n vertices as in the statement of the lemma. Apply Lemma 4.2 (the Regularity lemma) with parameters $\varepsilon^{100}, 4L'$ to obtain $L^* \in \mathbb{N}$. Since L^* depends only on ε and L', which appear to the right of L_0 in the hierarchy, we may assume that $1/L_0 \leq 1/L^*$. Apply Lemma 4.2 to G with parameters $\varepsilon^{100}, 16d, 4L'$ to obtain clusters $V_1, \ldots, V_L$ of size m, an exceptional set V_0, a pure graph G' and a reduced graph R'. So $|R'| = L$ where $4L' \leq L \leq L^*$ and $|V_0| \leq \varepsilon^{100}n$; and $G'[V_i, V_j]$ is $(\varepsilon^{100}, 16d)$-regular whenever $ij \in E(R')$.

Lemma 4.2(iv) states that $d_{G'}(x) > d_G(x) - (d+\varepsilon)n$ for all $x \in V(G)$. Then Proposition 4.14 implies that G' is $(\eta/2, n)$-good. Choose α such that $\varepsilon \ll \alpha \ll d$. By Lemma 4.13(ii), $d^\alpha_{R',G}$ and R' are $(\eta/2, L)$-good. Further, Lemma 4.13 applied to G' implies that $d^\alpha_{R',G'}$ is $(\eta/4, L)$-good.

Apply Lemma 7.2 with $L, \eta/2, R', \varepsilon^{100}$ playing the roles of n, η, G, ε to obtain $\ell \in \mathbb{N}$ with $(1-\varepsilon^{100})L \leq 3\ell \leq L$ such that $R' \supseteq Z_\ell$ (where Z_ℓ is the ℓ-triangle cycle). Observe that $L' \leq L/4 \leq \ell \leq L/3 \leq L^* \leq L_0$, as required. Let $R := R'[V(Z_\ell)]$. Let also $V_0'' := V_0 \cup \bigcup_{i \in V(R') \setminus V(R)} V_i$. Then

$$|V_0''| \leq \varepsilon^{100} n + \varepsilon^{100} Lm \leq 2\varepsilon^{100} n. \tag{8.18}$$

Relabel the vertices of Z_ℓ (and hence R) in the canonical way given in Definition 7.1. So $V(R) = [\ell] \times [3]$, and for all $1 \leq i \leq \ell$ and distinct $1 \leq j, j' \leq 3$ we have $(i,j)(i,j') \in E(Z_\ell)$ and $(i,j)(i+1,j') \in E(Z_\ell)$, where addition is modulo ℓ. Then $R \supseteq Z_\ell \supseteq T_\ell$, where T_ℓ consists of the triangles $(i,1)(i,2)(i,3)$ for $1 \leq i \leq \ell$. Given a vertex (i,j) in R write $V_{i,j}$ for the cluster in G corresponding to (i,j).

Apply Lemma 4.5 with $R, 3\ell, \varepsilon^{100}, 16d, G', T_\ell, 2$ playing the roles of $R, L, \varepsilon, d, G, H, \Delta$ to obtain for each $(i,j) \in V(T_\ell) = V(R)$ a subset $V'_{i,j} \subseteq V_{i,j}$ of size

$$m' := (1-\varepsilon^{50})m \tag{8.19}$$

such that for all $1 \leq i \leq \ell$ and $1 \leq j < j' \leq 3$, the graph $G'[V'_{i,j}, V'_{i,j'}]$ is $(4\varepsilon^{50}, 8d)$-superregular. Let $(i,j)(i',j') \in E(R)$ be arbitrary. Then Proposition 4.3(i) with $\varepsilon^{100}, 16d, \varepsilon^{50}$ playing the roles of ε, d, d' implies that $G'[V'_{i,j}, V'_{i',j'}]$ is $(2\varepsilon^{50}, 8d)$-regular and hence $(4\varepsilon^{50}, 8d)$-regular. Let V_0' be the set of all those vertices of G not contained in any $V'_{i,j}$. Then

$$|V_0'| \overset{(8.19)}{\leq} |V_0''| + 3\varepsilon^{50}\ell m \overset{(8.18)}{\leq} (2\varepsilon^{100} + \varepsilon^{50})n \leq 2\varepsilon^{50} n.$$

Let N_0 be the $\ell \times 3$ matrix in which every entry is m'. It is now clear that G' has an $(R, \ell, N_0, 4\varepsilon^{50}, 8d)$-cycle structure $\mathcal{C}_0$ where the $V'_{i,j}$ are the clusters of $\mathcal{C}_0$ and V_0' is the exceptional set. In particular, now we view the vertices in R as corresponding to the clusters $V'_{i,j}$. Recall that $d^\alpha_{R',G'}$ is $(\eta/4, L)$-good when we view the vertices in R' as corresponding to the clusters $V_{i,j}$. Thus, Proposition 4.14 implies that $d^\alpha_{R,G'}$ is $(\eta/8, L)$-good when we view the vertices in R as corresponding to the clusters $V_{i,j}$. So by definition of core degree, $d^{\alpha/2}_{R,G'}$ is $(\eta/8, 3\ell)$-good when we view the vertices in R as corresponding to the clusters $V'_{i,j}$ (i.e. $d^{\alpha/2}_{R,G'}$ is $(\eta/8, 3\ell)$-good with respect to $\mathcal{C}_0$).

We may therefore apply Lemma 8.7 with $n, \eta/4, G', m', 4\varepsilon^{50}, 8d, \alpha/2$ playing the roles of $n, \eta, G, m, \varepsilon, d, c$ to show that G' has a spanning $(R, \ell, N_1, \varepsilon^9, 4d)$-cycle structure $\mathcal{C}_1$, where N_1 is $(m', (1+2\varepsilon^{25})m')$-bounded and 1-balanced. Moreover, $d^{\alpha/4}_{R,G'}$ is $(\eta/8, 3\ell)$-good with respect to $\mathcal{C}_1$.

Apply Lemma 8.9 with $G', \mathcal{C}_1, \alpha/4$ playing the roles of $G, \mathcal{C}, c$ to show that G' has a spanning $(R, \ell, N_2, \varepsilon^3, 2d)$-cycle structure $\mathcal{C}_2$, where N_2 is 2ℓ-balanced, $((1-\varepsilon^9)m', (1+2\varepsilon^9)m')$-bounded, and has equal columns.

Finally, apply Lemma 8.11 with $G', \mathcal{C}_2$ playing the roles of $G, \mathcal{C}$ to show that G' has a spanning $(R, \ell, M, \varepsilon, d)$-cycle structure $\mathcal{C}$ where M is 0-balanced and $((1-2\varepsilon^3)m', (1+2\varepsilon^3)m')$-bounded, and hence $((1-\varepsilon)m, (1+\varepsilon)m)$-bounded by (8.19). $\square$

8.2. **Embedding the square of a Hamilton cycle.** Given $t \in \mathbb{N}$, recall that C^2_{3t} denotes the square cycle on $3t$ vertices. In this section we will always assume implicitly that C^2_{3t} has vertex set $[t] \times [3]$ such that for all $1 \leq i \leq t$ and distinct $1 \leq j, j' \leq 3$, we have $(i,j)(i,j') \in E(C^2_{3t})$ and $(i,2)(i+1,1), (i,3)(i+1,1), (i,3)(i+1,2) \in E(C^2_{3t})$, where addition is modulo t. Observe that $T_t \subseteq C^2_{3t}$.

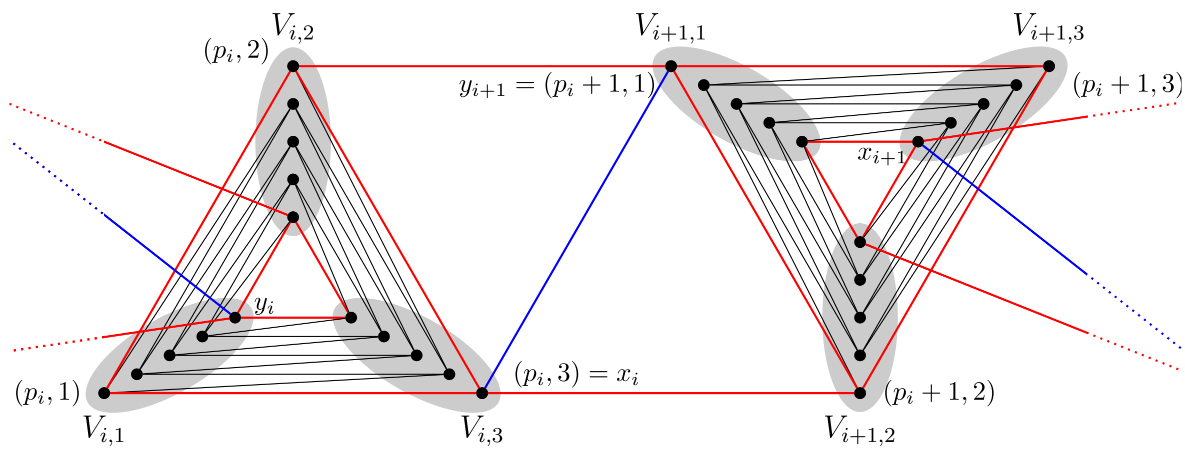

Figure 5: The square path $y_iP_ix_iy_{i+1}P_{i+1}x_{i+1}$ which forms part of the square cycle C_n^2; and the desired embedding into the clusters of G. The edges in $C_n^2[X]$ are coloured blue and the remaining edges in J_ℓ are coloured red.

The following is essentially a special case of an argument in [8], but we prove it here for completeness.

Lemma 8.12. *Let $0 < 1/n \ll 1/\ell \ll \varepsilon \ll d \ll 1$. Suppose that G is a graph on n vertices with a spanning $(R, \ell, M, \varepsilon, d)$-cycle structure $\mathcal{C}$ such that M is $((1-\varepsilon)m, (1+\varepsilon)m)$-bounded and 0-balanced. Then G contains the square of a Hamilton cycle.*

Proof. By (C2), we have that $Z_\ell \subseteq R$. Then $C_{3\ell}^2 \subseteq R$. This is all we require in the proof. Write $\{V_{i,j} : (i,j) \in [\ell] \times [3]\}$ for the collection of clusters of $\mathcal{C}$, where $V_{i,j}$ corresponds to $(i,j) \in V(R)$. So this is a partition of $V(G)$. Since M is 0-balanced, for each $1 \le i \le \ell$ there exists $m_i \in \mathbb{N}$ such that

$$|V_{i,j}| = m_i = (1 \pm \varepsilon)m \quad \text{for all} \quad 1 \le j \le 3. \tag{8.20}$$

Let $p_0 := 0$ and $p_i := \sum_{1 \le r \le i} m_r$ for all $1 \le i \le \ell$. Note that $3p_\ell = n$, and, in particular, n is divisible by 3. To prove the lemma, we will find an embedding $h : V(C_n^2) \to V(G)$, where we write

$$C_n^2 = (1,1)(1,2)(1,3)(2,1)\dots(p_\ell,1)(p_\ell,2)(p_\ell,3).$$

The embedding will map the first $3p_1$ vertices of C_n^2 to distinct vertices in $V_{1,1} \cup V_{1,2} \cup V_{1,3}$, and the $(3p_1+1)$th to $(3p_2)$th vertices of C_n^2 to distinct vertices in $V_{2,1} \cup V_{2,2} \cup V_{2,3}$, and so on. For each $1 \le i \le \ell$, define

$$x_i := (p_i, 3) \quad \text{and} \quad y_i := (p_{i-1}+1, 1). \tag{8.21}$$

Define also

$$\begin{aligned} X_i &:= \{x_i, y_{i+1}\} \quad \text{and} \quad Y_i := \{(p_i,1),(p_i,2),(p_i+1,2),(p_i+1,3)\} = N_{C_n^2}(X_i) \setminus X_i; \\ P_i &:= (p_{i-1}+1,2)(p_{i-1}+1,3)\dots(p_i,1)(p_i,2), \end{aligned} \tag{8.22}$$

where P_i is a square path. Let $X := \bigcup_{1 \le i \le \ell} X_i$ and $Y := \bigcup_{1 \le i \le \ell} Y_i = N_{C_n^2}(X) \setminus X$. Note further that $(P_i)_2^+ \cup (P_{i+1})_2^- = Y_i$. We have that $C_n^2 = y_1P_1x_1y_2P_2x_2y_3P_3\dots P_\ell x_\ell$. (Figure 5 shows the square path $y_iP_ix_iy_{i+1}P_{i+1}x_{i+1}$.)

Our strategy is as follows: first embed the vertices in $X \cup Y$ using the partial embedding lemma (Lemma 4.7), so that there are many choices for the embedding of each $y \in Y$. Then, for each $1 \leq i \leq \ell$, apply the Blow-up lemma (Theorem 4.8) to embed P_i into $V_{i,1} \cup V_{i,2} \cup V_{i,3}$ in such a way that the two embeddings align.

Define $f : V(C_n^2) \to V(C_{3\ell}^2)$ by $f((k,j)) = (g(k), j)$, where $g(k) \in [\ell]$ is such that

$$p_{g(k)-1} < k \leq p_{g(k)}.$$

It is not hard to check that f is a graph homomorphism, i.e. $f(x)f(y) \in E(C_{3\ell}^2)$ whenever $xy \in E(C_n^2)$. By a slight abuse of notation, we will write $V_{f((k,j))}$ for $V_{g(k),j}$. We will find an embedding $h : V(C_n^2) \to V(G)$ such that $h(x) \in V_{f(x)}$ for all $x \in C_n^2$.

For all $1 \leq i \leq \ell$, since $X_i \cup Y_i$ is a collection of 6 consecutive vertices on C_n^2, we have that $J_\ell := C_n^2[X \cup Y]$ is a collection of ℓ vertex-disjoint square paths of order 6. So $|J_\ell| = 6\ell \leq \varepsilon m$ and $\Delta(J_\ell) = 4$. Choose c such that $\varepsilon \ll c \ll d \ll 1$. Apply Lemma 4.7 with $C_{3\ell}^2, G, \{V_{i,j} : (i,j) \in [\ell] \times [3]\}, J_\ell, c$ playing the roles of $R, G, \{V_i : 1 \leq i \leq L\}, H, c$. Thus obtain an injective mapping $\tau : X \to V(G)$ with $\tau(x) \in V_{f(x)}$ for all $x \in X$, such that for all $y \in Y$ there exist sets $C_y \subseteq V_{f(y)} \setminus \tau(X)$ such that the following hold:

(i) for all $1 \leq i \leq \ell$ (where addition is modulo ℓ), we have that $\tau(x_i)\tau(y_{i+1}) \in E(G)$;
(ii) for all $y \in Y$ we have that $C_y \subseteq N_G(\tau(x))$ for all $x \in N_{C_n^2}(y) \cap X$;
(iii) $|C_y| \geq c|V_{f(y)}|$ for all $y \in Y$.

Note that for each $1 \leq i \leq \ell$, as displayed in Figure 5,

$$V_{i,1} \cap \tau(X) = \{y_i\}, \quad V_{i,2} \cap \tau(X) = \emptyset \quad \text{and} \quad V_{i,3} \cap \tau(X) = \{x_i\}.$$

For all $(i,j) \in [\ell] \times [3]$, let $V'_{i,j} := V_{i,j} \setminus \tau(X)$. So $|V'_{i,j}| = m_i - 1$ for $j = 1, 3$; and $|V'_{i,2}| = m_i$. Proposition 4.3(ii) implies that $G[V'_{i,j}, V'_{i,j'}]$ is $(2\varepsilon, d/2)$-superregular for all $1 \leq i \leq \ell$ and $1 \leq j < j' \leq 3$.

Note that for each $1 \leq i \leq \ell$, P_i is a 3-partite graph with $\Delta(P_i) = 4$ and with vertex classes W_1^i, W_2^i, W_3^i of sizes $m_i - 1, m_i, m_i - 1$ respectively, where $(k,j) \in W_j^i$ for all $(k,j) \in V(P_i)$. Observe that $V(P_i) \cap Y = (P_i)_2^- \cup (P_i)_2^+$. So, by (iii), for each $y \in ((P_i)_2^- \cup (P_i)_2^+) \cap W_j^i$, there is a set $C_y \subseteq V'_{i,j}$ with $|C_y| \geq cm_i$ that satisfies (ii). Let T_i be the triangle in R spanned by $(i,1), (i,2), (i,3)$. Let f_i denote the restriction of f on P_i. So $f_i : V(P_i) \to V(T_i)$ where $f_i((k,j)) = (i,j)$ for all $(k,j) \in V(P_i)$.

For each $1 \leq i \leq \ell$, apply Theorem 4.8 with $3, m_i - 1, m_i, m_i - 1, 2\varepsilon, V'_{i,j}, T_i, d/2, P_i, W_j^i, 4, f_i$ playing the roles of $k, n_1, n_2, n_3, \varepsilon, V_j, J, d, H, W_j, \Delta, \phi$ with special vertices $y \in (P_i)_2^- \cup (P_i)_2^+$ and associated sets C_y playing the role of S_y. Thus obtain an embedding of P_i into $G[V'_{i,1} \cup V'_{i,2} \cup V'_{i,3}]$ such that every $y \in (P_i)_2^- \cup (P_i)_2^+$ is mapped to a vertex in C_y. Note that, for $1 \leq i < i' \leq \ell$, every pair $z_i \in V(P_i)$ and $z_{i'} \in V(P_{i'})$ are mapped to different vertices of G. By considering the union of these embeddings, we obtain a bijective mapping $\sigma : \bigcup_{1 \leq i \leq \ell} V(P_i) \to V(G) \setminus \tau(X)$ such that

$$\sigma(x)\sigma(x') \in E(G) \quad \text{whenever } xx' \in \bigcup_{1 \leq i \leq \ell} E(P_i) \overset{(8.22)}{=} E(C_n^2 \setminus X). \tag{8.23}$$

In particular, we have that

$$\sigma(y) \in C_y \quad \text{for all} \quad y \in Y. \tag{8.24}$$

Let $h : V(C_n^2) \to V(G)$ be defined by

$$h(x) = \begin{cases} \tau(x) & \text{if } x \in X \\ \sigma(x) & \text{if } x \in V(C_n^2) \setminus X. \end{cases} \tag{8.25}$$

It remains to show that h is an embedding of C_n^2 in G. Let $xy \in E(C_n^2)$. We consider three cases. Suppose first that $x, y \in X$. Then, without loss of generality, there is some $1 \le i \le \ell$ such that $x = x_i$ and $y = y_{i+1}$. So $h(x)h(y) = \tau(x_i)\tau(y_{i+1}) \in E(G)$ by (i). Suppose secondly that $x \in X$ and $y \in V(C_n^2) \setminus X$. Then $y \in N_{C_n^2}(x) \setminus X \subseteq Y$, and so

$$h(y) \overset{(8.25)}{=} \sigma(y) \overset{(8.24)}{\in} C_y \overset{\text{(ii)}}{\subseteq} N_G(\tau(x)) \overset{(8.25)}{=} N_G(h(x)),$$

i.e. $h(x)h(y) \in E(G)$. Suppose finally that $x, y \in V(C_n^2) \setminus X$. Then $h(x)h(y) = \sigma(x)\sigma(y) \in E(G)$ by (8.23). $\square$

9. Proof of Theorem 1.3

We will first prove Theorem 1.3 for graphs whose order is divisible by three.

Theorem 9.1. *Let $n \in 3\mathbb{N}$ and let $0 < 1/n \ll \eta \ll 1$. Suppose that G is an η-good graph on n vertices. Then G contains the square of a Hamilton cycle.*

Proof. Choose $L_0, L' \in \mathbb{N}$ and positive constants ε, d so that $0 \ll 1/n \ll 1/L_0 \ll 1/L' \ll \varepsilon \ll d \ll \eta \ll 1$. Apply Lemma 8.3 to show that there exists a spanning subgraph $G' \subseteq G$, and $\ell \in \mathbb{N}$ with $L' \le \ell \le L_0$, such that G' has a spanning $(R, \ell, M, \varepsilon, d)$-cycle structure such that M is 0-balanced and $((1-\varepsilon)m, (1+\varepsilon)m)$-bounded. Now apply Lemma 8.12 to show that G', and hence G, contains the square of a Hamilton cycle. $\square$

The proof of Theorem 1.3 is now a short step away.

Proof of Theorem 1.3. Let $\eta > 0$. Without loss of generality, we may assume that $\eta \ll 1$. Choose $n_0 \in \mathbb{N}$ so that $0 < 1/n_0 \ll \eta$ and the conclusion of Theorem 9.1 holds whenever $n \ge n_0 - 2$ and $\eta/2$ plays the role of η. Let G be a graph on $n \ge n_0$ vertices, whose degree sequence $d_1 \le \ldots \le d_n$ satisfies

$$d_i \ge n/3 + i + \eta n \quad \text{for all} \quad i \le n/3.$$

Note firstly that G is $(2\eta/3)$-good. Then (4.4) with $2\eta/3$ playing the role of η implies that we can find vertex-disjoint edges $x_1y_1, x_2y_2 \in E(G)$ such that $x_1, y_1, x_2, y_2 \in V(G)_{2\eta/3}$.

Let k be the least non-negative integer such that $n \equiv k \mod 3$. So $k \in \{0, 1, 2\}$. Let G' be the graph obtained as follows. If $k = 0$, set $G' := G$. Otherwise, we let z_j be a new vertex for each $1 \le j \le k$, and set

$$V(G') := V(G) \cup \{z_j : 1 \le j \le k\} \setminus \{x_j, y_j : 1 \le j \le k\}$$

and

$$E(G) := E(G \setminus \{x_j, y_j : 1 \le j \le k\}) \cup \{vz_j : 1 \le j \le k \ \text{ and } \ v \in N_G^2(x_j, y_j)\}.$$

So G' is obtained from G by contracting k of the edges x_1y_1, x_2y_2.

Note that, for all $1 \le j \le k$ we have

$$d_{G'}(z_j) = |N_G^2(x_j, y_j)| \ge (1/3 + \eta)n$$

by Proposition 4.1(i). It is easy to see that G' is an $(\eta/2)$-good graph and $|G'| = n - k \equiv 0 \mod 3$. Furthermore, $|G'| \ge n - 2 \ge n_0 - 2$. Then Theorem 9.1 implies that G' contains the square of a Hamilton cycle C'. Since every neighbour of z_j in G' is a neighbour of both x_j and y_j in G, the cycle C obtained from C' by replacing each z_j with the edge x_jy_j (in any order) gives the square of a Hamilton cycle in G. $\square$

10. Concluding remarks

Recall that in Lemma 7.2, we showed that a graph G as in Theorem 1.3 contains an almost spanning copy of a so-called triangle cycle Z_ℓ. We then used this framework to embed the square of a Hamilton cycle. (Roughly speaking, by *framework* we mean a structure in the reduced graph which enables us to embed a subgraph into G.) Frameworks similar to Z_ℓ have been used previously for embedding other spanning structures.

In [30], Kühn, Osthus and Taraz showed that any graph G on n vertices and with $\delta(G) \geq (2/3 + o(1))n$ contains a spanning triangulation, i.e. a maximal planar graph. To embed the triangulation, the framework they used was the square of a Hamilton path. (The error term $o(n)$ here was subsequently removed in [28], yielding an exact result.)

We say a graph H on n vertices has *bandwidth* b if there exists an ordering of the vertices $1, \dots, n$ so that $|i-j| < b$ whenever ij is an edge of H. In [8], Böttcher, Schacht and Taraz considered the more general problem of embedding (possibly spanning) graphs H with small bandwidth. They showed that any graph G on n vertices with $\delta(G) \geq (2/3 + o(1))n$ contains every 3-chromatic graph H on at most n vertices and of bounded maximum degree and bandwidth $o(n)$. Again, the framework used here was the square of a Hamilton path. In later work [9] they generalised this to r-chromatic graphs H and used an analogue of Z_ℓ for their framework.

We believe that a graph as in Theorem 1.3 contains a spanning triangulation.

Conjecture 10.1. *Given any $\eta > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. If G is a graph on $n \geq n_0$ vertices whose degree sequence $d_1 \leq \dots \leq d_n$ satisfies*

$$d_i \geq n/3 + i + \eta n \quad \textit{for all} \quad i \leq n/3,$$

then G contains a spanning triangulation.

If true, Conjecture 10.1 implies the aforementioned result of Kühn, Osthus and Taraz. One approach to prove Conjecture 10.1 could be to use Z_ℓ as a framework for embedding (i.e. apply Lemma 7.2). This approach could also be fruitful in attacking the following more general conjecture.

Conjecture 10.2. *Given any $\eta > 0$ and $\Delta \in \mathbb{N}$, there exists a $\beta > 0$ and an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-chromatic graph on $n \geq n_0$ with $\Delta(H) \leq \Delta$ and bandwidth at most βn. If G is a graph on n vertices whose degree sequence $d_1 \leq \dots \leq d_n$ satisfies*

$$d_i \geq n/3 + i + \eta n \quad \textit{for all} \quad i \leq n/3,$$

then G contains H.

We conclude the paper by discussing degree sequence conditions that force the kth power of a Hamilton cycle in a graph. (The kth power of a Hamilton cycle C is obtained from C by adding an edge between every pair of vertices of distance at most k on C.) A conjecture of Seymour [36] states that every graph G on n vertices with $\delta(G) \geq kn/(k+1)$ contains the kth power of a Hamilton cycle. Thus, Seymour's conjecture is a generalisation of Conjecture 1.1. Komlós, Sárközy and Szemerédi [27] proved Seymour's conjecture for sufficiently large graphs G. In light of Theorem 1.3, we believe the following degree sequence version of Seymour's conjecture is true.

Conjecture 10.3. *Given any $\eta > 0$ and $k \geq 2$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. If G is a graph on $n \geq n_0$ vertices whose degree sequence $d_1 \leq \dots \leq d_n$ satisfies*

$$d_i \geq \frac{(k-1)n}{k+1} + i + \eta n \quad \textit{for all} \quad i \leq \frac{n}{k+1},$$

then G contains the kth power of a Hamilton cycle.

If true, Conjecture 10.3 would be essentially best possible. Indeed, the example in Proposition 17 in [6] shows that one cannot replace the term ηn in the degree sequence condition here with -1.

Note that a necessary condition for a graph G to contain the kth power of a Hamilton cycle is that G contains a perfect K_{k+1}-packing: In [38] it was shown that the hypothesis of Conjecture 10.3 indeed ensures that G contains a perfect K_{k+1}-packing.

We believe that most of the proof of Theorem 1.3 naturally generalises to kth powers of Hamilton cycles. The main difficulty in proving Conjecture 10.3 appears to be in proving a 'connecting lemma' (i.e. an analogue of Lemma 6.3). In particular, the methods we use to prove Lemma 6.3 seem somewhat tailored to the case of the square of a Hamilton cycle.

Acknowledgements

We are grateful to Daniela Kühn and Deryk Osthus for helpful comments on the manuscript.

References

[1] S. Abbasi, Spanning cycles in dense graphs, PhD thesis, Rutgers University, 1999.
[2] M. Aigner and S. Brandt, Embedding arbitrary graphs of maximum degree two, *J. London Math. Soc.* **48** (1993), 39–51.
[3] P. Allen, J. Böttcher, H. Hàn, Y. Kohayakawa and Y. Person, Powers of Hamilton cycles in pseudorandom graphs, submitted.
[4] P. Allen, J. Böttcher and J. Hladký, Filling the gap between Turán's theorem and Pósa's conjecture, *J. London Math. Soc.* **84** (2011), 269–302.
[5] N. Alon and E. Fischer, 2-factors in dense graphs, *Discrete Math.* **152** (1996), 13–23.
[6] J. Balogh, A.V. Kostochka and A. Treglown, On perfect packings in dense graphs, *Electron. J. Combin.* **20** (2013) no. 1 P57.
[7] B. Bollobás and S.E. Eldridge, Packings of graphs and applications to computational complexity, *J. Combin. Theory B* **25** (1978), 105–124.
[8] J. Böttcher, M. Schacht and A. Taraz, Spanning 3-colourable subgraphs of small bandwidth in dense graphs, *J. Combin. Theory B* **98** (2008), 752–777.
[9] J. Böttcher, M. Schacht and A. Taraz, Proof of the bandwidth conjecture of Bollobás and Komlós, *Math. Ann.* **343** (1) (2009), 175–205.
[10] P.A. Catlin, Embedding subgraphs and coloring graphs under extremal conditions, PhD thesis, Ohio State University, 1976.
[11] P. Châu, An Ore-type theorem on hamiltonian square cycles, *Graphs Combin.* **29** (2013), 795–834.
[12] P. Châu, L. DeBiasio and H.A. Kierstead, Pósa's Conjecture for graphs of order at least 2×10^8, *Random Structures and Algorithms* **39** (2011), 507–525.
[13] V. Chvátal, On Hamilton's ideals, *J. Combin. Theory B* **12** (1972), 163–168.
[14] V. Chvátal, V. Rödl, E. Szemerédi and W. R. Trotter Jr, The Ramsey number of a graph with bounded maximum degree. *J. Combin Theory B* **34** (3) (1983), 239–243.
[15] L. DeBiasio, S. Faizullah and I. Khan, Ore-degree threshold for the square of a Hamiltonian cycle, *Discrete Math. Theor. Comput. Sci.* **17** (2015), 13–32.
[16] G.A. Dirac, Some theorems on abstract graphs, *Proc. London. Math. Soc.* **2** (1952), 69–81.
[17] M.H. El-Zahar, On circuits in graphs, *Discrete Math.* **50** (1984), 227–230.
[18] P. Erdős, Problem 9, in: M. Fieldler (Ed.), *Theory of Graphs and its Applications*, Czech. Acad. Sci. Publ., Prague, 1964, p. 159.
[19] G. Fan, R. Häggkvist, The Square of a Hamiltonian Cycle, *SIAM J. Discrete Math.* **7** (1994), 203–212.
[20] G. Fan, H.A. Kierstead, The square of paths and cycles, *J. Combin. Theory B* **63** (1995), 55–64.
[21] G. Fan, H.A. Kierstead, Hamiltonian square-paths, *J. Combin. Theory B* **67** (1996), 167–182.
[22] G. Fan, H.A. Kierstead, Partitioning a graph into two square-cycles, *J. Graph Theory* **23** (1996), 241–256.
[23] R.J. Faudree, R.J. Gould, M.S. Jacobson and R.H. Schelp, On a problem of Paul Seymour, *Recent Advances in Graph Theory* (V. R. Kulli ed.), Vishwa International Publication (1991), 197–215.
[24] S. Janson, T. Łuczak and A. Ruciński, *Random graphs*, Wiley-Interscience, 2000.
[25] J. Komlós, G.N. Sárközy and E. Szemerédi, On the square of a Hamiltonian cycle in dense graphs, *Random Structures and Algorithms* **9** (1996), 193–211.
[26] J. Komlós, G.N. Sárközy and E. Szemerédi, Blow-up lemma, *Combinatorica* **17** (1997), 109–123.
[27] J. Komlós, G.N. Sárközy and E. Szemerédi, Proof of the Seymour conjecture for large graphs, *Annals of Combinatorics* **2** (1998), 43–60.
[28] D. Kühn and D. Osthus, Spanning triangulations in graphs, *J. Graph Theory* **49** (2005), 205–233.

[29] D. Kühn and D. Osthus, On Pósa's conjecture for random graphs, *SIAM J. Discrete Math.* **26** (2012), 1440–1457.
[30] D. Kühn, D. Osthus and A. Taraz, Large planar subgraphs in dense graphs, *J. Combin. Theory B* **95** (2005), 263–281.
[31] I. Levitt, G.N. Sárközy and E. Szemerédi, How to avoid using the Regularity Lemma: Pósa's conjecture revisited, *Discrete Math.* **310** (2010), 630–641.
[32] O. Ore, Note on Hamilton circuits, *Amer. Math. Monthly* **67** (1960), 55.
[33] L. Pósa, A theorem concerning Hamiltonian lines, *Magyar Tud. Akad. Mat. Fiz. Oszt. Kozl.* **7** (1962), 225–226.
[34] V. Rödl, A. Ruciński and E. Szemerédi, A Dirac-type theorem for 3-uniform hypergraphs, *Combin. Probab. Comput.* **15** (2006), 229–251.
[35] V. Rödl, A. Ruciński and E. Szemerédi, Dirac-type conditions for hamiltonian paths and cycles in 3-uniform hypergraphs, *Advances in Mathematics* **227** (2011), 1225–1299.
[36] P. Seymour, Problem section, in *Combinatorics: Proceedings of the British Combinatorial Conference 1973* (T.P. McDonough and V.C. Mavron eds.), 201–202, Cambridge University Press, 1974.
[37] E. Szemerédi, Regular partitions of graphs, *Problémes Combinatoires et Théorie des Graphes Colloques Internationaux CNRS* **260** (1978), 399–401.
[38] A. Treglown, A degree sequence Hajnal-Szemerédi theorem, submitted.

Katherine Staden
Mathematics Institute
University of Warwick
Coventry
CV4 2AL
UK

Andrew Treglown
School of Mathematics
University of Birmingham
Birmingham
B15 2TT
UK

E-mail addresses: k.l.staden@warwick.ac.uk, a.c.treglown@bham.ac.uk