A random version of Sperner's theorem

József Balogh,* Richard Mycroft† and Andrew Treglown‡

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Abstract

Let $\mathcal{P}(n)$ denote the power set of $[n]$, ordered by inclusion, and let $\mathcal{P}(n,p)$ be obtained from $\mathcal{P}(n)$ by selecting elements from $\mathcal{P}(n)$ independently at random with probability p. A classical result of Sperner [12] asserts that every antichain in $\mathcal{P}(n)$ has size at most that of the middle layer, $\binom{n}{\lfloor n/2 \rfloor}$. In this note we prove an analogous result for $\mathcal{P}(n,p)$: If $pn \to \infty$ then, with high probability, the size of the largest antichain in $\mathcal{P}(n,p)$ is at most $(1+o(1))p\binom{n}{\lfloor n/2 \rfloor}$. This solves a conjecture of Osthus [9] who proved the result in the case when $pn/\log n \to \infty$. Our condition on p is best-possible. In fact, we prove a more general result giving an upper bound on the size of the largest antichain for a wider range of values of p.

We write $[n]$ for the set of natural numbers up to n, and $\mathcal{P}(n)$ for the power set of $[n]$. Also, for any $0 \leq k \leq n$ we write $\binom{[n]}{k}$ for the subset of $\mathcal{P}(n)$ consisting of all sets of size k. A subset $\mathcal{A} \subseteq \mathcal{P}(n)$ is an *antichain* if for any $A, B \in \mathcal{A}$ with $A \subseteq B$ we have $A = B$. So $\binom{[n]}{k}$ is an antichain for any $0 \leq k \leq n$; Sperner's theorem [12] states that in fact no antichain in $\mathcal{P}(n)$ has size larger than $\binom{n}{\lfloor n/2 \rfloor}$. Our main theorem is a random version of Sperner's theorem. For this, let $\mathcal{P}(n,p)$ be the set obtained from $\mathcal{P}(n)$ by selecting elements randomly with probability p and independently of all other choices. Write $m := \binom{n}{\lfloor n/2 \rfloor}$. Roughly speaking, our main result asserts that if $p > C/n$ for some constant C, then with high probability, the largest antichain in $\mathcal{P}(n,p)$ is approximately the same size as the 'middle layer' in $\mathcal{P}(n,p)$.

Theorem 1. *For any $\varepsilon > 0$ there exists a constant C such that if $p > C/n$ then with high probability the largest antichain in $\mathcal{P}(n,p)$ has size at most $(1+\varepsilon)pm$.*

*Department of Mathematics, University of Illinois, Urbana, IL 61801, USA and Bolyai Institute, University of Szeged, Szeged, Hungary jobal@math.uiuc.edu. Research is partially supported by Simons Fellowship, NSF CAREER Grant DMS-0745185, Arnold O. Beckman Research Award (UIUC Campus Research Board 13039) and Marie Curie FP7-PEOPLE-2012-IIF 327763.

†University of Birmingham, United Kingdom, r.mycroft@bham.ac.uk.

‡University of Birmingham, United Kingdom, a.c.treglown@bham.ac.uk.

(Here, by 'with high probability' we mean with probability tending to 1 as n tends to infinity.)

The model $\mathcal{P}(n,p)$ was first investigated by Rényi [10] who determined the probability threshold for the property that $\mathcal{P}(n,p)$ is not itself an antichain, thereby answering a question of Erdős. The size of the largest antichain in $\mathcal{P}(n,p)$ for p above this threshold was first studied by Kohayakawa and Kreuter [6]. In [6] they raised the question of which values of p does the conclusion of Theorem 1 hold. Osthus [9] proved Theorem 1 in the case when $pn/\log n \to \infty$ and conjectured that this can be replaced by $pn \to \infty$. (So Theorem 1 resolves this conjecture.) Moreover, Osthus showed that, for a fixed $c > 0$, if $p = c/n$ then with high probability the largest antichain in $\mathcal{P}(n,p)$ has size at least $(1+o(1))(1+e^{-c/2})p\binom{n}{\lfloor n/2 \rfloor}$. So the bound on p in Theorem 1 is best-possible up to the constant C. There have also been a number of results concerning the length of (the longest) chains in $\mathcal{P}(n,p)$ and related models of random posets (see for example, [2, 7, 8]).

Instead of proving Theorem 1 directly we prove the following more general result.

Theorem 2. *For any $\varepsilon > 0$ and $t \in \mathbb{N}$, there exists a constant C such that if $p > C/n^t$ then with high probability the largest antichain in $\mathcal{P}(n,p)$ has size at most $(1+\varepsilon)pmt$.*

Osthus [9] proved this result in the case when $p(n/t)^t/\log n \to \infty$. (In fact, Osthus's result allows for t to be an integer function, see [9] for the precise statement.) Moreover, Osthus showed that, for $1/n^t \ll p \ll 1/n^{t-1}$, with high probability, $\mathcal{P}(n,p)$ has an antichain of size at least $(1+o(1))pmt$ (so Theorem 2 is 'tight' in this window of p).

The method of proof of Theorem 2 also allows us to estimate the number of antichains in $\mathcal{P}(n)$ of certain fixed sizes.

Proposition 3. *Fix any $t \in \mathbb{N}$, and suppose that $m/n^t \ll s \ll m/n^{t-1}$. Then the number of antichains of size s in $\mathcal{P}(n)$ is $\binom{(t+o(1))m}{s}$.*

To prove Theorem 2, let G be the graph with vertex set $\mathcal{P}(n)$ in which sets A and B are adjacent if $A \subsetneq B$ or $B \subsetneq A$. Then an antichain in $\mathcal{P}(n)$ is precisely an independent set in G. We follow the 'hypergraph container' approach (see, for example, [1, 11]): Indeed, we show that all independent sets in G are contained within a fairly small number of low-density sets in G. Crucially, for this method to work, we have to construct our 'containers' in two phases (see Lemma 6). For this we use a result of Kleitman [5] on the minimum number of edges induced by a subset of G with a given fixed size. Define the *centrality order* on the vertices of $\mathcal{P}(n)$ as follows: we begin with the elements of $\binom{[n]}{\lfloor n/2 \rfloor}$, ordered arbitrarily, then the elements of $\binom{[n]}{\lfloor n/2 \rfloor+1}$, then the elements of $\binom{[n]}{\lfloor n/2 \rfloor-1}$, then the elements of $\binom{[n]}{\lfloor n/2 \rfloor+2}$, and so forth until all vertices of $\mathcal{P}(n)$ have been ordered. For any $r \in \mathbb{N}$ let I_r denote the initial segment of this order of length r; Kleitman [5] proved that I_r minimises the number of induced edges over all sets of size r (see also [4], which characterises all the sets U of size r for which $e(G[U])$ is minimised).

Theorem 4 (Kleitman [5])**.** *For any $r \le 2^n$ and any $U \subseteq V(G)$ of size r we have $e(G[U]) \ge e(G[I_r])$.*

We apply this theorem in the form of the following corollary.

Corollary 5. *Let $U \subseteq V(G)$, and suppose that $0 < \varepsilon \leq 1/2$ and $t \in \mathbb{N}$. If $|U| \geq (t+\varepsilon)m$, then $e(G[U]) > \varepsilon n^t |U|/(2t)^{t+1}$.*

Proof. Let $r := |U|$. We have $r \geq (t+\varepsilon)m$, so in particular $r - mt \geq 2\varepsilon r/(1+2t)$. Observe that I_r contains all of the at most mt elements of the t 'middle layers', $\binom{[n]}{\lfloor n/2 \rfloor}$, $\binom{[n]}{\lfloor n/2 \rfloor + 1}$, and so forth. Further, I_r contains at least $r - mt$ elements from outside these layers, each of which has at least $\binom{\lceil n/2 \rceil}{t} \geq (n/2t)^t$ neighbours in the t middle layers. So by Theorem 4 we have

$$e(G[U]) \geq e(G[I_r]) \geq \frac{2\varepsilon r}{1+2t} \cdot \left(\frac{n}{2t}\right)^t \geq \frac{\varepsilon n^t r}{(2t)^{t+1}}.$$

□

Let $s \in \mathbb{N}$, $t > 0$ and let S be a set of size $|S| = s$. Define $\binom{S}{\leq t}$ to be the set of all subsets of S of size at most t and $\binom{s}{\leq t} := \left|\binom{S}{\leq t}\right|$. We can now prove the result we need on independent sets, using the following algorithm.

Lemma 6. *Suppose that $t \in \mathbb{N}$, $0 < \varepsilon \leq 1/(2t)^{t+1}$ and n is sufficiently large. Then there exist functions $f : \binom{V(G)}{\leq n^{-(t+0.9)}2^n} \to \binom{V(G)}{\leq (t+1+\varepsilon)m}$ and $g : \binom{V(G)}{\leq (t+2)m/(\varepsilon^2 n^t)} \to \binom{V(G)}{\leq (t+\varepsilon)m}$ such that, for any independent set I in G, there are disjoint subsets $S_1, S_2 \subseteq I$ such that $S_1 \cup S_2$ and $g(S_1 \cup S_2)$ are disjoint, $S_2 \subseteq f(S_1)$, and $I \subseteq S_1 \cup S_2 \cup g(S_1 \cup S_2)$.*

Proof. Fix an arbitrary total order $v_1, \ldots, v_n$ on the vertices of $V(G)$. Given any independent set I in G, define $G_0 := G$, and take S_1 and S_2 to be initially empty. We add vertices to S_1 and S_2 through the following iterative process, beginning at Step 1 in Phase 1.

Phase 1: At Step i, let u be the maximum degree vertex of G_{i-1} (with ties broken by our fixed total order). If $u \notin I$ then define $G_i := G_{i-1} \setminus \{u\}$, and proceed to Step $i+1$ (still in Phase 1). Alternatively, if $u \in I$ and $\deg_{G_{i-1}}(u) \geq n^{t+0.9}$ then add u to S_1, define $G_i := G_{i-1} \setminus (\{u\} \cup N_G(u))$, and proceed to Step $i+1$ (still in Phase 1). Finally, if $u \in I$ and $\deg_{G_{i-1}}(u) < n^{t+0.9}$, then add u to S_1, define $G_i := G_{i-1} \setminus \{u\}$ and $f(S_1) := V(G_i)$, and proceed to Step $i+1$ of Phase 2.

Phase 2: At Step i, let u be the maximum degree vertex of G_{i-1}. If $u \notin I$ then define $G_i := G_{i-1} \setminus \{u\}$, and proceed to Step $i+1$ (still in Phase 2). Alternatively, if $u \in I$ and $\deg_{G_{i-1}}(u) \geq \varepsilon^2 n^t$ then add u to S_2, define $G_i := G_{i-1} \setminus (\{u\} \cup N_G(u))$, and proceed to Step $i+1$ (still in Phase 2). Finally, if $u \in I$ and $\deg_{G_{i-1}}(u) < \varepsilon^2 n^t$, then add u to S_2, define $G_i := G_{i-1} \setminus \{u\}$ and $g(S_1 \cup S_2) := V(G_i)$, and terminate.

Observe first that for any independent set I in G the process defined ensures that S_1 and S_2 are disjoint subsets of I, that $S_1 \cup S_2$ is disjoint from $g(S_1 \cup S_2)$, that $S_2 \subseteq f(S_1)$ and that $I \subseteq S_1 \cup S_2 \cup g(S_1 \cup S_2)$.

Next, note that for any independent set I, if a vertex u is added to S_1 at step i, u and at least $n^{t+0.9}$ neighbours of u are deleted from G_{i-1} in forming G_i, with a single exception (when u is the final vertex added to S_1). So we must have $|S_1| \leq 1 + |V(G)|/(n^{t+0.9}+1) \leq n^{-(t+0.9)}2^n$. Furthermore, at the end of Phase 1 we know that every vertex v of G_i has $\deg_{G_i}(v) \leq n^{t+0.9}$, and so Corollary 5 implies that $f(S_1)$, the set of all vertices not deleted up to this point, must have size $|f(S_1)| < (t+1+\varepsilon)m$. Then, in Phase 2, if a vertex u is added to S_2 at step i, at least $\varepsilon^2 n^t$ neighbours of u are deleted from G_{i-1} in forming G_i, again with the single exception of the final vertex added to S_2. So we must have $|S_2| \leq 1 + |f(S_1)|/(\varepsilon^2 n^t)$ and thus

$$|S_1 \cup S_2| \leq 1 + (t+1+\varepsilon)m/(\varepsilon^2 n^t) + n^{-(t+0.9)}2^n \leq (t+2)m/(\varepsilon^2 n^t).$$

Moreover, at the end of Phase 2 every vertex v of the final G_i has $\deg_{G_i}(v) \leq \varepsilon^2 n^t$ and so $e(G_i) \leq \varepsilon^2 n^t |G_i| \leq \varepsilon n^t |G_i|/(2t)^{t+1}$. Thus, Corollary 5 implies that $|g(S_1 \cup S_2)| \leq (t+\varepsilon)m$.

So it is sufficient to check that the functions f and g are well-defined. That is, we must check that if the process described above yields the same set S_1 when applied to independent sets I and I', then it should also yield the same set $f(S_1)$, and if additionally the same set S_2 is returned then the sets $g(S_1 \cup S_2)$ should be identical. However, this is a consequence of the fact that we always chose u to be the vertex of I of maximum degree in G_{i-1}. Moreover, if our algorithm produces sets S_1, S_2 for an independent set I and sets S_1', S_2' for an independent set I' such that $S_1 \cup S_2 = S_1' \cup S_2'$ then $S_1 = S_1'$ (and $S_2 = S_2'$). Thus, indeed f and g are well-defined. □

Proof of Theorem 2. Fix $\varepsilon > 0$ and $t \in \mathbb{N}$; we may assume that $\varepsilon < 1/(2t)^{t+1}$. Define $C := 10^{10}\varepsilon^{-5}$ and $\varepsilon_1 := \varepsilon/4$. Let G_p be the graph formed from G by selecting vertices independently at random with probability $p > C/n^t$. Then we must show that, with high probability, G_p has no independent set of size greater than $(1+\varepsilon)pmt$. Apply Lemma 6 with ε_1 playing the role of ε. Suppose for a contradiction that G_p does contain some independent set I with $|I| > (1+\varepsilon)pmt$. Then all vertices of the sets S_1 and S_2 given by Lemma 6 for this I must have been selected for G_p, along with at least $|I| - |S_1 \cup S_2| \geq (1+\varepsilon)pmt - (t+2)m/(\varepsilon_1^2 n^t) \geq (1+\varepsilon/2)pmt$ vertices of $g(S_1 \cup S_2)$ (the second inequality follows from $C = 10^{10}\varepsilon^{-5}$).

However, the number of possibilities for S_1 is $\binom{2^n}{\leq n^{-(t+0.9)}2^n}$, and for each possibility the probability that $S_1 \subseteq V(G_p)$ is $p^{|S_1|}$. For any fixed S_1 we have $|f(S_1)| \leq (t+2)m$ and $S_2 \subseteq f(S_1)$, so the number of possibilities for S_2 is at most $\binom{(t+2)m}{\leq (t+2)m/(\varepsilon_1^2 n^t)}$, and for each possibility the probability that $S_2 \subseteq V(G_p)$ is $p^{|S_2|}$. Finally, for any fixed S_1 and S_2 we have $g(S_1 \cup S_2) \leq (t+\varepsilon_1)m \leq (1+\varepsilon/4)mt$, so the expected number of vertices of $g(S_1 \cup S_2)$ selected for G_p is at most $(1+\varepsilon/4)pmt$. By a standard Chernoff bound the probability that at least $(1+\varepsilon/2)pmt$ vertices of $g(S_1 \cup S_2)$ are selected for G_p is therefore at most $e^{-\varepsilon^2 pmt/100}$. Taking a union bound, we conclude that the probability that G contains an independent set I of size greater than $(1+\varepsilon)m$ is at most

$$\begin{aligned}\Pi &:= \sum_{0 \leq a \leq n^{-(t+0.9)}2^n} \; \sum_{0 \leq b \leq (t+2)m/(\varepsilon_1^2 n^t)} \binom{2^n}{a} \cdot p^a \cdot \binom{(t+2)m}{b} \cdot p^b \cdot e^{-\varepsilon^2 pmt/100} \\ &\leq (n^{-(t+0.9)}2^n + 1)((t+2)m/(\varepsilon_1^2 n^t) + 1) \binom{2^n}{n^{-(t+0.9)}2^n} \cdot p^{n^{-(t+0.9)}2^n} \binom{(t+2)m}{(t+2)m/(\varepsilon_1^2 n^t)} \\ &\quad \cdot p^{(t+2)m/(\varepsilon_1^2 n^t)} \cdot e^{-\varepsilon^2 pmt/100}.\end{aligned}$$

Note that for large n, with plenty of room to spare we have

$$(n^{-(t+0.9)}2^n + 1)((t+2)m/(\varepsilon_1^2 n^t) + 1) \le e^{\varepsilon^2 pmt/400}$$

and

$$\binom{2^n}{n^{-(t+0.9)}2^n} \cdot p^{n^{-(t+0.9)}2^n} \le e^{\varepsilon^2 pmt/400}.$$

Further, since $C = 10^{10}\varepsilon^{-5}$, for large n we have that

$$\binom{(t+2)m}{(t+2)m/(\varepsilon_1^2 n^t)} \cdot p^{(t+2)m/(\varepsilon_1^2 n^t)} \le e^{\varepsilon^2 pmt/400}.$$

Thus, the upper bound Π on the probability is $o(1)$. □

We conclude with a sketch of the proof of Proposition 3, on the number of antichains of given fixed sizes in $\mathcal{P}(n)$.

Proof sketch of Proposition 3. The lower bound can be obtained by greedily choosing vertices from within the t middle layers of $\mathcal{P}(n)$ to form an antichain of size s, and counting the number of ways to make these choices. For the upper bound, fix any $\varepsilon > 0$ and apply Lemma 6 with this ε and t. Then any independent set in G of size s is uniquely determined by the choice of

1. a set S_1 of size $s_1 \le \ell_1 := 2^n/n^{t+0.9}$, for which there are at most $\binom{2^n}{\le \ell_1}$ choices,
2. a set $S_2 \subseteq f(S_1)$ of size $s_2 \le \ell_2 := (t+2)m/(\varepsilon^2 n^t)$, for which there are at most $\binom{(t+1+\varepsilon)m}{\le \ell_2}$ choices, and
3. a set $S \subseteq g(S_1 \cup S_2)$ of size $s - s_1 - s_2$, for which there are at most $\binom{(t+\varepsilon)m}{s-s_1-s_2}$ choices.

Summing over all these choices by a similar calculation as in the proof of Theorem 2, we find that (for large n) there are at most $\binom{(t+2\varepsilon)m}{s}$ independent sets of size s in G. □

When we completed the project, we were informed that Collares Neto and Morris [3] independently proved Theorem 1. Their method is however different. We used the proof technique of [1], and they followed the method of [11]. In particular, when we constructed containers, we aimed at having few vertices, whilst they aimed at having only few edges.

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