ON SOLUTION-FREE SETS OF INTEGERS

ROBERT HANCOCK AND ANDREW TREGLOWN

ABSTRACT. Given a linear equation $\mathcal{L}$, a set $A \subseteq [n]$ is $\mathcal{L}$-free if A does not contain any 'non-trivial' solutions to $\mathcal{L}$. In this paper we consider the following three general questions:

(i) What is the size of the largest $\mathcal{L}$-free subset of $[n]$?
(ii) How many $\mathcal{L}$-free subsets of $[n]$ are there?
(iii) How many maximal $\mathcal{L}$-free subsets of $[n]$ are there?

We completely resolve (i) in the case when $\mathcal{L}$ is the equation $px + qy = z$ for fixed $p, q \in \mathbb{N}$ where $p \geq 2$. Further, up to a multiplicative constant, we answer (ii) for a wide class of such equations $\mathcal{L}$, thereby refining a special case of a result of Green [17]. We also give various bounds on the number of maximal $\mathcal{L}$-free subsets of $[n]$ for three-variable homogeneous linear equations $\mathcal{L}$. For this, we make use of container and removal lemmas of Green [17].

1. INTRODUCTION

Let $[n] := \{1, \dots, n\}$ and consider a fixed linear equation $\mathcal{L}$ of the form

$$a_1 x_1 + \cdots + a_k x_k = b \tag{1}$$

where $a_1, \dots, a_k, b \in \mathbb{Z}$. If $b = 0$ we say that $\mathcal{L}$ is *homogeneous*. If

$$\sum_{i \in [k]} a_i = b = 0$$

then we say that $\mathcal{L}$ is *translation-invariant*. Notice that if $\mathcal{L}$ is translation-invariant then $(x, \dots, x)$ is a 'trivial' solution of (1) for any x. More generally, a solution $(x_1, \dots, x_k)$ to $\mathcal{L}$ is said to be *trivial* if $\mathcal{L}$ is translation-invariant and if there exists a partition $P_1, \dots, P_\ell$ of $[k]$ so that:

(i) $x_i = x_j$ for every i, j in the same partition class P_r;
(ii) For each $r \in [\ell]$, $\sum_{i \in P_r} a_i = 0$.

A set $A \subseteq [n]$ is *$\mathcal{L}$-free* if A does not contain any non-trivial solutions to $\mathcal{L}$. If the equation $\mathcal{L}$ is clear from the context, then we simply say A is *solution-free*.

The notion of an $\mathcal{L}$-free set encapsulates many fundamental topics in combinatorial number theory. Indeed, in the case when $\mathcal{L}$ is $x_1 + x_2 = x_3$ we call an $\mathcal{L}$-free set a *sum-free set*. This is a notion that dates back to 1916 when Schur [33] proved that, if n is sufficiently large, any r-colouring of $[n]$ yields a monochromatic triple x, y, z such that $x + y = z$. *Sidon sets* (when $\mathcal{L}$ is $x_1 + x_2 = x_3 + x_4$) have also been extensively studied. For example, a classical result of Erdős and Turán [15] asserts that the largest Sidon set in $[n]$ has size $(1 + o(1))\sqrt{n}$. In the case when $\mathcal{L}$ is $x_1 + x_2 = 2x_3$ an $\mathcal{L}$-free set is simply a *progression-free* set. Roth's theorem [26] states that the largest progression-free subset of $[n]$ has size $o(n)$.

In this paper we prove a number of results concerning $\mathcal{L}$-free subsets of $[n]$ where $\mathcal{L}$ is a homogeneous linear equation in *three variables*. In particular, our work is motivated by the following general questions:

(i) What is the size of the largest $\mathcal{L}$-free subset of $[n]$?
(ii) How many $\mathcal{L}$-free subsets of $[n]$ are there?

Date: May 9, 2016.
The second author is supported by EPSRC grant EP/M016641/1.

(iii) How many *maximal* $\mathcal{L}$-free subsets of $[n]$ are there?

We make progress on all three of these questions. For each question we use tools from graph theory; for (i) and (ii) our methods are somewhat elementary. For (iii) our method is more involved and utilises container and removal lemmas of Green [17].

1.1. **The size of the largest solution-free set.** As highlighted above, a central question in the study of $\mathcal{L}$-free sets is to establish the size $\mu_{\mathcal{L}}(n)$ of the largest $\mathcal{L}$-free subset of $[n]$. It is not difficult to see that the largest sum-free subset of $[n]$ has size $\lceil n/2 \rceil$, and this bound is attained by the set of odd numbers in $[n]$ and by the interval $[\lfloor n/2 \rfloor + 1, n]$.

When $\mathcal{L}$ is $x_1 + x_2 = 2x_3$, $\mu_{\mathcal{L}}(n) = o(n)$ by Roth's theorem. In fact, Sanders [29] proved that there is a constant C such that every set $A \subseteq [n]$ with $|A| \geq Cn(\log\log n)^5/\log n$ contains a three-term arithmetic progression. On the other hand, Behrend [7] showed that there is a constant $c > 0$ so that $\mu_{\mathcal{L}}(n) \geq n\exp(-c\sqrt{\log n})$. See [14, 18] for the best known lower bound on $\mu_{\mathcal{L}}(n)$ in this case.

More generally, it is known that $\mu_{\mathcal{L}}(n) = o(n)$ if $\mathcal{L}$ is translation-invariant and $\mu_{\mathcal{L}}(n) = \Omega(n)$ otherwise (see [27]). For other (exact) bounds on $\mu_{\mathcal{L}}(n)$ for various linear equations $\mathcal{L}$ see, for example, [27, 28, 6, 13, 20].

In this paper we mainly focus on $\mathcal{L}$-free subsets of $[n]$ for linear equations $\mathcal{L}$ of the form $px+qy = z$ where $p \geq 2$ and $q \geq 1$ are fixed integers. Notice that for such a linear equation $\mathcal{L}$, the interval $[\lfloor n/(p+q) \rfloor + 1, n]$ is an $\mathcal{L}$-free set. Our first result implies that this is the largest such $\mathcal{L}$-free subset of $[n]$. Let $\min(S)$ denote the smallest element in a finite set $S \subseteq \mathbb{N}$.

Theorem 1. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geq q$ and $p \geq 2$, $p, q \in \mathbb{N}$. Let n be sufficiently large. Suppose S is an $\mathcal{L}$-free subset of $[n]$, and let $\min(S) = \lfloor \frac{n}{p+q} \rfloor - t$ where t is a non-negative integer.*

(i) If $0 \leq t < (\frac{p+q-1}{p+q+p/q})\lfloor \frac{n}{p+q} \rfloor$ then $|S| \leq \lceil \frac{(p+q-1)n}{p+q} \rceil - \lfloor \frac{p}{q} t \rfloor$.

(ii) If $t \geq (\frac{p+q-1}{p+q+p/q})\lfloor \frac{n}{p+q} \rfloor$ then $|S| \leq \frac{(q^2+1)n}{q^2+q+1}$.

Corollary 2. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geq q$ and $p \geq 2$, $p, q \in \mathbb{N}$. If n is sufficiently large then $\mu_{\mathcal{L}}(n) = n - \lfloor \frac{n}{p+q} \rfloor$.*

Roughly, Theorem 1 implies that every $\mathcal{L}$-free subset of $[n]$ is 'interval like' or 'small'. In the case of sum-free subsets (i.e. when $p = q = 1$), a result of Deshouillers, Freiman, Sós and Temkin [12] provides very precise structural information on the sum-free subsets of $[n]$. Loosely speaking, they showed that a sum-free subset of $[n]$ is 'interval like', 'small' or consists entirely of odd numbers.

In the case when $p = q$, Corollary 2 was proven by Hegarty [20] (without a lower bound on n).

1.2. **The number of solution-free sets.** Write $f(n, \mathcal{L})$ for the number of $\mathcal{L}$-free subsets of $[n]$. In the case when $\mathcal{L}$ is $x + y = z$, define $f(n) := f(n, \mathcal{L})$.

By considering all possible subsets of $[n]$ consisting of odd numbers, one observes that there are at least $2^{n/2}$ sum-free subsets of $[n]$. Cameron and Erdős [10] conjectured that in fact $f(n) = \Theta(2^{n/2})$. This conjecture was proven independently by Green [16] and Sapozhenko [30]. In fact, they showed that there are constants C_1 and C_2 such that $f(n) = (C_i + o(1))2^{n/2}$ for all $n \equiv i \mod 2$.

Results from [22, 31] imply that there are between $2^{(1.16+o(1))\sqrt{n}}$ and $2^{(6.45+o(1))\sqrt{n}}$ Sidon sets in $[n]$. There are also several results concerning the number of so-called (k, ℓ)-sum-free subsets of $[n]$ (see, e.g., [8, 9, 32]).

More generally, given a linear equation $\mathcal{L}$, there are at least $2^{\mu_{\mathcal{L}}(n)}$ $\mathcal{L}$-free subsets of $[n]$. In light of the situation for sum-free sets one may ask whether, in general, $f(n, \mathcal{L}) = \Theta(2^{\mu_{\mathcal{L}}(n)})$. However, Cameron and Erdős [10] observed that this is false for translation-invariant $\mathcal{L}$.

Green [17] though showed that given a homogeneous linear equation $\mathcal{L}$, $f(n,\mathcal{L}) = 2^{\mu_{\mathcal{L}}(n)+o(n)}$ (where here the $o(n)$ may depend on $\mathcal{L}$). Our next result implies that one can omit the term $o(n)$ in the exponent for certain types of linear equation $\mathcal{L}$.

Theorem 3. *Fix $p, q \in \mathbb{N}$ where (i) $q \geq 2$ and $p > q(3q-2)/(2q-2)$ or (ii) $q = 1$ and $p \geq 3$. Let $\mathcal{L}$ denote the equation $px + qy = z$. Then*

$$f(n,\mathcal{L}) = \Theta(2^{\mu_{\mathcal{L}}(n)}).$$

1.3. **The number of maximal solution-free sets.** Given a linear equation $\mathcal{L}$, we say that $S \subseteq [n]$ is a *maximal $\mathcal{L}$-free subset of* $[n]$ if it is $\mathcal{L}$-free and it is not properly contained in another $\mathcal{L}$-free subset of $[n]$. Write $f_{\max}(n,\mathcal{L})$ for the number of maximal $\mathcal{L}$-free subsets of $[n]$. In the case when $\mathcal{L}$ is $x + y = z$, define $f_{\max}(n) := f_{\max}(n,\mathcal{L})$.

A significant proportion of the sum-free subsets of $[n]$ lie in just two maximal sum-free sets, namely the set of odd numbers in $[n]$ and the interval $[\lfloor n/2 \rfloor + 1, n]$. This led Cameron and Erdős [11] to ask whether $f_{\max}(n) = o(f(n))$ or even $f_{\max}(n) \leq f(n)/2^{\varepsilon n}$ for some constant $\varepsilon > 0$. Łuczak and Schoen [24] answered this question in the affirmative, showing that $f_{\max}(n) \leq 2^{n/2 - 2^{-28}n}$ for sufficiently large n. Later, Wolfovitz [34] proved that $f_{\max}(n) \leq 2^{3n/8+o(n)}$. Very recently, Balogh, Liu, Sharifzadeh and Treglown [2, 3] proved the following: For each $1 \leq i \leq 4$, there is a constant C_i such that, given any $n \equiv i \mod 4$, $f_{\max}(n) = (C_i + o(1))2^{n/4}$.

Except for sum-free sets, the problem of determining the number of maximal solution-free subsets of $[n]$ remains wide open. In this paper we give a number of bounds on $f_{\max}(n,\mathcal{L})$ for homogeneous linear equations $\mathcal{L}$ in three variables. The next result gives a general upper bound for such $\mathcal{L}$. Given a three-variable linear equation $\mathcal{L}$, an *$\mathcal{L}$-triple* is a multiset $\{x, y, z\}$ which forms a solution to $\mathcal{L}$. Let $\mu^*_{\mathcal{L}}(n)$ denote the number of elements $x \in [n]$ that do not lie in *any* $\mathcal{L}$-triple in $[n]$.

Theorem 4. *Let $\mathcal{L}$ be a fixed homogenous three-variable linear equation. Then*

$$f_{\max}(n,\mathcal{L}) \leq 3^{(\mu_{\mathcal{L}}(n) - \mu^*_{\mathcal{L}}(n))/3 + o(n)}.$$

Theorem 4 together with the aforementioned result of Green shows that $f_{\max}(n,\mathcal{L})$ is significantly smaller than $f(n,\mathcal{L})$ for all homogeneous three-variable linear equations $\mathcal{L}$ that are not translation-invariant. So in this sense it can be viewed as a generalisation of the result of Łuczak and Schoen. The proof of Theorem 4 is a simple application of container and removal lemmas of Green [17]. The same idea was used to prove results in [5, 2, 3]. Although at first sight the bound in Theorem 4 may seem crude, perhaps surprisingly there are equations $\mathcal{L}$ where the value of $f_{\max}(n,\mathcal{L})$ is close to this bound (see Proposition 22 in Section 5).

On the other hand, the following result shows that there are linear equations where the bound in Theorem 4 is far from tight.

Theorem 5. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geq q \geq 2$ are integers so that $p \leq q^2 - q$ and $\gcd(p, q) = q$. Then*

$$f_{\max}(n,\mathcal{L}) \leq 2^{(\mu_{\mathcal{L}}(n) - \mu^*_{\mathcal{L}}(n))/2 + o(n)}.$$

In the case when $\mathcal{L}$ is the equation $2x + 2y = z$ we provide a matching lower bound. Again though, we suspect there are equations $\mathcal{L}$ where the bound in Theorem 5 is far from tight. The proof of Theorem 5 applies Theorem 1 as well as the container and removal lemmas of Green [17].

We also provide another upper bound on $f_{\max}(n,\mathcal{L})$ for a more general class of linear equations.

Theorem 6. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geq q$, $p \geq 2$ and $p, q \in \mathbb{N}$. Then $f_{\max}(n,\mathcal{L}) \leq 2^{\mu_{\mathcal{L}}(\lfloor \frac{n-p}{q} \rfloor) + o(n)}$.*

In Section 5 we discuss in what cases a bound as in Theorem 6 is stronger than the bound in Theorem 5 (and vice versa). We also provide lower bounds on $f_{\max}(n,\mathcal{L})$ for all equations $\mathcal{L}$ of the form $px + qy = z$ where $p, q \geq 2$ are integers; see Proposition 26.

Our results suggest that, in contrast to the case of $f(n, \mathcal{L})$, it is unlikely there is a 'simple' general asymptotic formula for $f_{\max}(n, \mathcal{L})$ for all homogeneous linear equations $\mathcal{L}$. It would be extremely interesting to make further progress on this problem.

The paper is organised as follows. In the next section we collect together a number of useful tools. In Section 3 we prove Theorem 1. Theorem 3 is proven in Section 4. We prove our results on the number of maximal $\mathcal{L}$-free sets in Section 5.

2. Containers and independent sets in graphs

2.1. **Container and removal lemmas.** Recently the method of *containers* has proven powerful in tackling a range of problems in combinatorics and other areas, in particular due to the work of Balogh, Morris and Samotij [4] and Saxton and Thomason [31]. Roughly speaking this method states that for certain (hyper)graphs G, the independent sets of G lie only in a small number of subsets of $V(G)$ called *containers*, where each container is an 'almost independent set'.

Recall that, given a three-variable linear equation $\mathcal{L}$, an $\mathcal{L}$-triple is a multiset $\{x, y, z\}$ which forms a solution to $\mathcal{L}$. Let H denote the hypergraph with vertex set $[n]$ and edges corresponding to $\mathcal{L}$-triples. Then an independent set in H is precisely an $\mathcal{L}$-free set.

The following container lemma is a special case of a result of Green (Proposition 9.1 of [17]). Lemma 7(i)–(iii) is stated explicitly in [17]. Lemma 7(iv) follows as an immediate consequence of Lemma 7(i) and Lemma 8 below.

Lemma 7. [17] *Fix a three-variable homogeneous linear equation $\mathcal{L}$. There exists a family $\mathcal{F}$ of subsets of $[n]$ with the following properties:*

(i) Every $F \in \mathcal{F}$ has at most $o(n^2)$ $\mathcal{L}$-triples.
(ii) If $S \subseteq [n]$ is $\mathcal{L}$-free, then S is a subset of some $F \in \mathcal{F}$.
(iii) $|\mathcal{F}| = 2^{o(n)}$.
(iv) Every $F \in \mathcal{F}$ has size at most $\mu_{\mathcal{L}}(n) + o(n)$.

Throughout the paper we refer to the elements of $\mathcal{F}$ as *containers*. Notice that Lemma 7(iv) gives a bound on the size of the containers in terms of $\mu_{\mathcal{L}}(n)$ even though, in general, the precise value of $\mu_{\mathcal{L}}(n)$ is not known.

The following removal lemma is a special case of a result of Green (Theorem 1.5 in [17]). This result was also generalised to systems of linear equations by Král', Serra and Vena (Theorem 2 in [23]).

Lemma 8. [17] *Fix a three-variable homogeneous linear equation $\mathcal{L}$. Suppose that $A \subseteq [n]$ is a set containing $o(n^2)$ $\mathcal{L}$-triples. Then there exist B and C such that $A = B \cup C$ where B is $\mathcal{L}$-free and $|C| = o(n)$.*

We will also apply the following bound on the number of $\mathcal{L}$-free sets.

Theorem 9. [17] *Fix a homogeneous linear equation $\mathcal{L}$. Then $f(n, \mathcal{L}) = 2^{\mu_{\mathcal{L}}(n)+o(n)}$.*

We will use the above results to deduce upper bounds on the number of maximal $\mathcal{L}$-free sets (Theorems 4, 5 and 6).

2.2. **Independent sets in graphs.** Let G be a graph and consider any subset $X \subseteq V(G)$. Let $\mathrm{IS}(G)$ denote the number of independent sets in G. Let $G[X]$ denote the induced subgraph of G on the vertex set X and $G \setminus X$ denote the induced subgraph of G on the vertex set $V(G) \setminus X$.

Fact 10. *Let G be a graph and let $A_1, \dots, A_r$ be a partition of $V(G)$. Then $\mathrm{IS}(G) \leq \mathrm{IS}(G[A_1]) \times \cdots \times \mathrm{IS}(G[A_r])$.*

The following simple lemma will be used in the proof of Theorem 3.

Lemma 11. *Let G be a graph on n vertices and M be a matching in G which consists of e edges. Suppose that $v \in V(G)$ lies in M. Then the number of independent sets in G which contain v is at most $3^{e-1} \cdot 2^{n-2e}$.*

Proof. First note that the number of independent sets in G which contain v is at most $\mathrm{IS}(G \setminus X)$ where X consists of v and its neighbour in M. Let $A_1, \dots, A_e$ be a partition of the vertex set $V(G \setminus X)$, where if $1 \leq i \leq e-1$ then A_i contains precisely the two vertices from some edge in M. So $|A_e| = n - 2e$. Clearly $\mathrm{IS}(G[A_i]) = 3$ for $1 \leq i \leq e-1$ and $\mathrm{IS}(G[A_e]) \leq 2^{n-2e}$. The result then follows by Fact 10. □

2.3. **Link graphs and maximal independent sets.** We obtain many of our results by counting the number of maximal independent sets in various auxiliary graphs. Similar techniques were used in [34, 2, 3], and in the graph setting in [5, 1]. To be more precise, let B and S be disjoint subsets of $[n]$ and fix a three-variable linear equation $\mathcal{L}$. The *link graph* $L_S[B]$ *of* S *on* B has vertex set B, and an edge set consisting of the following two types of edges:

(i) Two vertices x and y are adjacent if there exists an element $z \in S$ such that $\{x, y, z\}$ is an $\mathcal{L}$-triple;
(ii) There is a loop at a vertex x if there exists an element $z \in S$ or elements $z, z' \in S$ such that $\{x, x, z\}$ or $\{x, z, z'\}$ is an $\mathcal{L}$-triple.

Notice that since the only possible trivial solutions to a three-variable linear equation $\mathcal{L}$ are of the form $\{x, x, x\}$, all the edges in $L_S[B]$ correspond to non-trivial $\mathcal{L}$-triples.

The following simple lemma was stated in [2, 3] for sum-free sets, but extends to three-variable linear equations.

Lemma 12. *Fix a three-variable linear equation $\mathcal{L}$. Suppose that B, S are disjoint $\mathcal{L}$-free subsets of $[n]$. If $I \subseteq B$ is such that $S \cup I$ is a maximal $\mathcal{L}$-free subset of $[n]$, then I is a maximal independent set in $G := L_S[B]$.*

Let $\mathrm{MIS}(G)$ denote the number of maximal independent sets in G. Suppose we have a container $F \in \mathcal{F}$ as in Lemma 7 and suppose $F = A \cup B$ where B is $\mathcal{L}$-free. Observe that any maximal $\mathcal{L}$-free subset of $[n]$ in F can be found by first choosing an $\mathcal{L}$-free set $S \subseteq A$, and then extending S in B. Note that by Lemma 12, the number of possible extensions of S in B (which we shall refer to as $N(S, B)$) is bounded from above by the number of maximal independent sets in the link graph $L_S[B]$ (i.e. we have $N(S, B) \leq \mathrm{MIS}(L_S[B])$). Hence Lemma 12 is a useful tool for bounding the number of maximal $\mathcal{L}$-free subsets of $[n]$.

In particular, we will apply the following result in combination with Lemma 12. The first part was proven by Moon and Moser [25] and the second part by Hujter and Tuza [21]. We use the first condition in the proof of Theorems 4 and 5.

Theorem 13. *Suppose that G is a graph on n vertices possibly with loops. Then the following bounds hold.*

(i) $\mathrm{MIS}(G) \leq 3^{n/3}$;
(ii) $\mathrm{MIS}(G) \leq 2^{n/2}$ *if G is additionally triangle-free.*

To prove Theorem 5 we will combine Theorem 13(ii) and the following result.

Lemma 14. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geq q \geq 2$ and $p, q \in \mathbb{N}$. Let $A \subseteq [1, u]$ and let $B \subseteq [u+1, n]$ for some $u \in [n]$. Consider the link graph $G := L_A[B]$ of A on B. If $q^2 \geq p+q$ then G is triangle-free.*

Proof. Suppose that $q^2 \geq p+q$ and suppose for a contradiction there is a triangle in G with vertices $b_1 < b_2 < b_3$. By definition of the link graph, there exist $s_1, s_2, s_3 \in A$ such that $\{b_1, b_2, s_1\}, \{b_2, b_3, s_2\}, \{b_1, b_3, s_3\}$ are $\mathcal{L}$-triples.

Since all numbers in A are smaller than all numbers in B we have $1 \leq s_1, s_2, s_3 < b_1 < b_2 < b_3$. Also, since $p \geq q \geq 2$, for each of our $\mathcal{L}$-triples $\{b_i, b_j, s_k\}$ (where $b_i < b_j$) it follows that b_j must play the role of z in $\mathcal{L}$.

Define a multiset $\{r_i \in \{p,q\} : 1 \leq i \leq 6, r_1 \neq r_2, r_3 \neq r_4, r_5 \neq r_6\}$. Consider the three equations $r_1 b_1 + r_2 s_1 = b_2, r_3 b_2 + r_4 s_2 = b_3$ and $r_5 b_1 + r_6 s_3 = b_3$. Combining the second and third gives $b_2 = (r_5 b_1 + r_6 s_3 - r_4 s_2)/r_3$. Then combining this with the first equation gives $(r_1 r_3 - r_5) b_1 + r_2 r_3 s_1 + r_4 s_2 = r_6 s_3$. Now since $s_3 < b_1$ and all terms are at least 1, for such an inequality to hold we must have $r_1 r_3 - r_5 < r_6$. Since $r_5 \neq r_6$ this means we have $r_1 r_3 < p+q$. Hence as $r_1, r_3 \in \{p,q\}$, in order for G to have a triangle at least one of $p^2 < p+q$, $q^2 < p+q$ and $pq < p+q$ must be satisfied. Since $p \geq q \geq 2$, the first and third are not true and so we must have $q^2 < p+q$, a contradiction. $\square$

We also use link graphs as a means to obtain lower bounds on the number of maximal $\mathcal{L}$-free sets. We apply the following result in Propositions 22 and 26.

Lemma 15. *Fix a three-variable linear equation $\mathcal{L}$. Suppose that B, S are disjoint $\mathcal{L}$-free subsets of $[n]$. Let H be an induced subgraph of the link graph $L_S[B]$. Then $f_{\max}(n, \mathcal{L}) \geq \mathrm{MIS}(H)$.*

Proof. Suppose I and J are different maximal independent sets in H. First note that $S \cup I$ and $S \cup J$ are $\mathcal{L}$-free by definition of the link graph. Both cannot lie in the same maximal $\mathcal{L}$-free subset of $[n]$. To see this, observe by definition of I and J, there exists $i \in I \setminus J$. There must exist $s \in S$, $j \in J$ such that $\{i, j, s\}$ forms an $\mathcal{L}$-triple, else $J \cup \{i\}$ would be an independent set in H, which contradicts the maximality of J. Hence any maximal $\mathcal{L}$-free subset of $[n]$ containing $S \cup J$ does not contain i. Similarly there exists $j \in J \setminus I$ such that any maximal $\mathcal{L}$-free subset of $[n]$ containing $S \cup I$ does not contain j. The result immediately follows. $\square$

3. The size of the largest solution-free set

Throughout this section, $\mathcal{L}$ will denote the equation $px + qy = z$ where $p \geq q$ and $p \geq 2$, $p, q \in \mathbb{N}$. The aim of this section is to determine the size of the largest $\mathcal{L}$-free subset of $[n]$. In fact, we will prove a richer structural result on $\mathcal{L}$-free sets (Theorem 18). For this, we will introduce the following auxiliary graph G_m: Let $m \in [n]$ be fixed. We define the graph G_m to have vertex set $[m, n]$ and edges between c and $pm + qc$ for all $c \in [m, n]$ such that $pm + qc \leq n$. We will also make use of these auxiliary graphs in Section 4.

Fact 16.

(i) *The size of the largest $\mathcal{L}$-free subset S of $[n]$ with $\min(S) = m$ is at most the size of the largest independent set in G_m which contains m.*

(ii) *The number of $\mathcal{L}$-free subsets S of $[n]$ with $\min(S) = m$ is at most the number of independent sets in G_m which contain m.*

Proof. Let S be an $\mathcal{L}$-free subset of $[n]$ with $\min(S) = m$. Since $\{m, c, pm + qc\}$ is an $\mathcal{L}$-triple contained in $[n]$ for all $c \in [m, n]$ such that $pm + qc \leq n$, S cannot contain both c and $pm + qc$. Hence any $\mathcal{L}$-free subset of $[n]$ with minimum element m is also an independent set in G_m which contains m (although the converse does not necessarily hold). This immediately implies (i) and (ii). $\square$

Note that G_m is a union of disjoint paths and isolated vertices. We refer to the connected components of G_m as the *path components*. Given G_m, we define $y_0 := n$, and for $i \geq 1$ define $y_i := \max\{v \in V(G_m) | \, pm + qv \leq y_{i-1}\}$. Thus we have $y_i = \lfloor \frac{y_{i-1} - pm}{q} \rfloor$. For G_m we also define k to be the largest i such that $y_i \in [m, n]$, and refer to k as the *path parameter* of G_m. We define the *size* of a path component to be the number of *vertices* in it, and we define $N(G_m, i)$ to be the number of path components of size i in G_m.

Fact 17. *The graph G_m consists entirely of disjoint path components, where for each $1 \leq i \leq k-1$ there are $y_{i-1} + y_{i+1} - 2y_i$ path components of size i, there are $y_{k-1} - 2y_k + m - 1$ path components of size k and $y_k - m + 1$ path components of size $k+1$.*

Proof. Every vertex $c \in V(G_m)$ satisfying $y_{j+1} < c \leq y_j$ for some $0 \leq j \leq k-1$ is in a path in G_m which contains precisely j vertices which are larger than it, whereas every vertex $c > y_j$ is not in such a path. All the vertices in $[m, y_k]$ are in paths which contain precisely k vertices which are larger than it, all vertices in $[y_k + 1, y_{k-1}]$ are in paths which contain precisely $k-1$ vertices which are larger than it, and so on.

Let A_i be the interval $[y_i + 1, y_{i-1}]$ for $1 \leq i \leq k$ and let A_{k+1} be the interval $[m, y_k]$. There are $|[m, y_k]| = y_k - m + 1$ path components of size $k+1$ in G_m. For $i \leq k$ all vertices in A_i are the smallest vertex in a path on i vertices, however they may not be the smallest vertex in their path component. In fact, by definition of the y_i, all paths which start in A_j for some j must include precisely one vertex from each set $A_{j-1}, A_{j-2}, \ldots, A_1$. This means that for $i \leq k$, the number of path components of size i in G_m is precisely $|A_i| - |A_{i+1}|$. For $i \leq k-1$ this is $y_{i-1} + y_{i+1} - 2y_i$ and for $i = k$ this is $y_{k-1} - 2y_k + m - 1$. $\square$

We now use the graphs G_m and the above facts to bound the size of the largest $\mathcal{L}$-free subset of $[n]$.

Theorem 18. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geq q$ and $p \geq 2$, $p, q \in \mathbb{N}$. Let S be an $\mathcal{L}$-free subset of $[n]$, and let $\min(S) = \lfloor \frac{n}{p+q} \rfloor - t$ where t is a non-negative integer.*

(i) If $0 \leq t < (\frac{p+q-1}{p+q+p/q}) \lfloor \frac{n}{p+q} \rfloor$ then $|S| \leq \lceil \frac{(p+q-1)n}{p+q} \rceil - \lfloor \frac{p}{q} t \rfloor$.

(ii) If $t \geq (\frac{p+q-1}{p+q+p/q}) \lfloor \frac{n}{p+q} \rfloor$ then $|S| \leq \frac{(q^2+1)n}{q^2+q+1}$ provided that

$$n \geq \max\left\{ \frac{3(q^2+q+1)(q^3 + p(q^2+q+1))}{q^2+1}, \frac{5(q^2+q+1)(q^5 + p(q^4+q^3+q^2+q+1))}{q^4 + (p-1)q^3 + q^2 + 1} \right\}.$$

Proof. Let t be a non-negative integer. To prove (i) suppose that $t < (\frac{p+q-1}{p+q+p/q}) \lfloor \frac{n}{p+q} \rfloor$. Suppose S is an $\mathcal{L}$-free set contained in $[\lfloor \frac{n}{p+q} \rfloor - t, n]$ where $m := \lfloor \frac{n}{p+q} \rfloor - t \in S$. By Fact 16(i) we wish to prove that the largest independent set in G_m containing m has size at most $\lceil \frac{(p+q-1)n}{p+q} \rceil - \lfloor \frac{p}{q} t \rfloor$. Since $|V(G_m)| = \lceil \frac{(p+q-1)n}{p+q} \rceil + t + 1$ it suffices to show that any independent set I in G_m satisfies $|V(G_m) \setminus I| \geq \lfloor (p+q)t/q \rfloor + 1$.

For $0 \leq i \leq \lfloor (p+q)t/q \rfloor$, there is an edge between $m + i$ and $(p+q)m + qi$. Note that since $i \leq \lfloor (p+q)t/q \rfloor$ and $q \leq p$ we have that the largest vertex in any of these edges is indeed at most n:

$$(p+q)(\lfloor \tfrac{n}{p+q} \rfloor - t) + qi \leq n - (p+q)t + q\lfloor (p+q)t/q \rfloor \leq n - (p+q)t + q(p+q)t/q = n.$$

Since I can only contain one vertex from each of these edges, we have proven (i), provided that these edges are disjoint. It suffices to show that $\lfloor \frac{n}{p+q} \rfloor + \lfloor pt/q \rfloor < (p+q)m = (p+q)(\lfloor \frac{n}{p+q} \rfloor - t)$ since the left hand side is the largest element of the set $\{m + i : 0 \leq i \leq \lfloor (p+q)t/q \rfloor\}$. But this immediately follows since $t < (\frac{p+q-1}{p+q+p/q}) \lfloor \frac{n}{p+q} \rfloor$.

To prove (ii) let $t \geq (\frac{p+q-1}{p+q+p/q})\lfloor\frac{n}{p+q}\rfloor$ and suppose S is an $\mathcal{L}$-free subset of $[n]$ with $m := \min(S) = \lfloor\frac{n}{p+q}\rfloor - t$. By Fact 16(i) $|S|$ is at most the size of the largest independent set in G_m which contains m. We will first show that G_m has path parameter $k \geq 2$, and then the case $q = 1$ follows easily. Define $\ell := \lfloor k/2 \rfloor$ and

$$C_k := \left(\frac{\sum\limits_{i=0}^{2\ell+1} (-1)(-q)^i + p \sum\limits_{i=0}^{\ell} q^{2i}}{q^{2\ell+1} + p \sum\limits_{i=0}^{2\ell} q^i} \right).$$

We will show that if $q \geq 2$ then the largest independent set in G_m has size at most $C_k n + k$. We then further bound this from above by $(q^2+1)n/(q^2+q+1)$ for n sufficiently large.

Note that by Fact 17, to prove that $k \geq 2$ for G_m it suffices to show that there is a path on 3 vertices in G_m. By definition of k, m lies on a path P on $k+1$ vertices. Write $P = v_0 v_1 \cdots v_k$ where $m = v_0$ and observe that $v_j = (q^j + p\sum\limits_{i=0}^{j-1} q^i)m$ for $0 \leq j \leq k$. To prove $k \geq 2$ it suffices to show that there is indeed a vertex $(q^2+pq+p)m$ in $V(G_m)$, i.e. $(q^2+pq+p)m \leq n$. Note that since $t \geq (\frac{p+q-1}{p+q+p/q})\lfloor\frac{n}{p+q}\rfloor$, we have $m = \lfloor\frac{n}{p+q}\rfloor - t \leq (\frac{p+q+p/q-p-q+1}{p+q+p/q})\lfloor\frac{n}{p+q}\rfloor = (\frac{p+q}{q^2+pq+p})\lfloor\frac{n}{p+q}\rfloor$. Hence $(q^2+pq+p)m \leq n$ as desired.

When $q = 1$ observe that $y_i = y_{i-1} - pm$, so for $i \leq k-1$ by Fact 17 we have $N(G_m, i) = y_{i-1} + y_{i+1} - 2y_i = y_i + pm + y_i - pm - 2y_i = 0$. Hence G_m consists entirely of a union of path components of size either k or $k+1$. Since at most $\lceil i/2 \rceil$ vertices of a path on i vertices can be in an independent set and $k \geq 2$, the largest independent set in G_m has size at most $2n/3 = (q^2+1)n/(q^2+q+1)$ in this case, as desired. So now consider the case when $q \geq 2$. We calculate the maximum size of an independent set in G_m:

$$\begin{aligned}
&\sum_{i=1}^{k+1} \lceil i/2 \rceil \cdot N(G_m, i) \\
&= \left(\sum_{i=1}^{k-1} \lceil i/2 \rceil \cdot (y_{i-1} + y_{i+1} - 2y_i) \right) + \lceil k/2 \rceil (y_{k-1} + m - 1 - 2y_k) + \lceil (k+1)/2 \rceil (y_k - m + 1) \\
&= y_0 + \left(\sum_{i=1}^{k} (\lceil (i-1)/2 \rceil - 2\lceil i/2 \rceil + \lceil (i+1)/2 \rceil) y_i \right) + (m-1)(\lceil k/2 \rceil - \lceil (k+1)/2 \rceil). \qquad (2)
\end{aligned}$$

Here we used Fact 17 in the first equality. For i odd, the coefficient of y_i in (2) is $(i-1)/2 - 2(i+1)/2 + (i+1)/2 = -1$. For i even, the coefficient of y_i in (2) is $i/2 - 2i/2 + (i+2)/2 = 1$.

The following bounds are obtained from the definition of y_i and k:

(a) $\left(n - q^j + 1 - pm\sum\limits_{i=0}^{j-1} q^i\right)/q^j \leq y_j \leq \left(n - pm\sum\limits_{i=0}^{j-1} q^i\right)/q^j$;

(b) $n/\left(q^{k+1} + p\sum\limits_{i=0}^{k} q^i\right) < m \leq n/\left(q^k + p\sum\limits_{i=0}^{k-1} q^i\right)$.

Let $\ell := \lfloor k/2 \rfloor$ (note $k \geq 2$ so $\ell \geq 1$). First suppose k is odd, i.e. $k = 2\ell + 1$. Using (2), the size of the largest independent set in G_m is bounded above by

$$
\begin{aligned}
& y_0 + \Big(\sum_{i=1}^{k} (\lceil (i-1)/2 \rceil - 2\lceil i/2 \rceil + \lceil (i+1)/2 \rceil) y_i \Big) + (m-1)(\lceil k/2 \rceil - \lceil (k+1)/2 \rceil) \\
&= y_0 - y_1 + y_2 - y_3 + \cdots + y_{2\ell} - y_{2\ell+1} \\
&\overset{\text{(a)}}{\leq} n - \Big(\frac{n - pm - q + 1}{q} \Big) + \Big(\frac{n - pm(1+q)}{q^2} \Big) - \Big(\frac{n - pm(1 + q + q^2) - q^3 + 1}{q^3} \Big) \\
&\quad + \cdots - \left(\frac{n - \Big(pm \sum_{i=0}^{2\ell} q^i\Big) - q^{2\ell+1} + 1}{q^{2\ell+1}} \right) \\
&= n\Big(1 - \frac{1}{q} + \frac{1}{q^2} - \cdots - \frac{1}{q^{2\ell+1}}\Big) + m\Big(\frac{p}{q} + \frac{p}{q^3} + \cdots + \frac{p}{q^{2\ell+1}}\Big) + \frac{q-1}{q} + \frac{q^3-1}{q^3} \\
&\quad + \cdots + \frac{q^{2\ell+1} - 1}{q^{2\ell+1}} \\
&\overset{\text{(b)}}{\leq} n\Big(1 - \frac{1}{q} + \frac{1}{q^2} - \cdots - \frac{1}{q^{2\ell+1}}\Big) + \left(\frac{n}{q^{2\ell+1} + p \sum_{i=0}^{2\ell} q^i} \right) \left(\frac{p \sum_{i=0}^{\ell} q^{2i}}{q^{2\ell+1}} \right) + \frac{k+1}{2} \\
&= \left(\frac{\Big[\sum_{i=0}^{2\ell+1} (-1)(-q)^i \Big] (q^{2\ell+1} + p \sum_{i=0}^{2\ell} q^i) + p \sum_{i=0}^{\ell} q^{2i}}{q^{2\ell+1}(q^{2\ell+1} + p \sum_{i=0}^{2\ell} q^i)} \right) n + \frac{k+1}{2} \\
&= \left(\frac{\sum_{i=0}^{2\ell+1} (-q)^{i+2\ell+1} + p \sum_{i=0}^{\ell} q^{2i+2\ell+1}}{q^{2\ell+1}(q^{2\ell+1} + p \sum_{i=0}^{2\ell} q^i)} \right) n + \frac{k+1}{2} = \left(\frac{\sum_{i=0}^{2\ell+1} (-1)(-q)^i + p \sum_{i=0}^{\ell} q^{2i}}{q^{2\ell+1} + p \sum_{i=0}^{2\ell} q^i} \right) n + \frac{k+1}{2} \\
&= C_k n + \frac{k+1}{2} \leq C_k n + k.
\end{aligned}
$$

(Note that some of our calculations above did indeed require $q \geq 2$.) By definition, $m \geq y_{k+1} + 1$ and for k even, we have $C_k = C_{k+1}$. So if k is even ($k = 2\ell$) then we have

$$
\begin{aligned}
& y_0 + \Big(\sum_{i=1}^{k} (\lceil (i-1)/2 \rceil - 2\lceil i/2 \rceil + \lceil (i+1)/2 \rceil) y_i \Big) + (m-1)(\lceil k/2 \rceil - \lceil (k+1)/2 \rceil) \\
&= y_0 - y_1 + y_2 - y_3 + \ldots + y_{2\ell} - m + 1 \leq y_0 - y_1 + y_2 - y_3 + \ldots + y_{2\ell} - y_{2\ell+1} \\
&\leq C_{k+1} n + \frac{k+2}{2} \leq C_k n + k.
\end{aligned}
$$

The penultimate inequality follows by using calculations from the odd case. The last inequality follows since $k \geq 2$ and $C_k = C_{k+1}$. Thus we have shown that $|S| \leq C_k n + k$ and we know that $k \geq 2$. It remains to show that

$$C_k n + k \leq \frac{(q^2+1)n}{q^2+q+1} \tag{3}$$

for $k \geq 2$ and n sufficiently large.

We know that $m \leq n/(q^k + p\sum_{i=0}^{k-1} q^i)$ and so $n \geq q^k + p\sum_{i=0}^{k-1} q^i$, therefore condition (3) is met if

$$\Big(\frac{q^2+1}{q^2+q+1} - C_k\Big)\Big(q^k + p\sum_{i=0}^{k-1} q^i\Big) \geq k. \tag{4}$$

Claim 19. *For $k \geq 6$, (4) holds.*

Since the proof of Claim 19 is just a technical calculation, we defer it to the appendix.

The claim is not a result which generally holds for $2 \leq k \leq 5$ so instead we directly calculate how large n should be to satisfy (3) in these cases. For $k = 3$ and $k = 5$ we obtain $n \geq \frac{3(q^3+p(q^2+q+1))(q^2+q+1)}{q^2+1}$ and $n \geq \frac{5(q^5+p(q^4+q^3+q^2+q+1))(q^2+q+1)}{q^4+(p-1)q^3+q^2+1}$ respectively. For $k = 2$ and $k = 4$ we obtain weaker bounds. Hence taking n to be sufficiently large (larger than these two bounds), we have $C_k n + k \leq \frac{(q^2+1)n}{q^2+q+1}$ for all $k \geq 2$.

$\square$

4. The number of solution-free sets

Recall a theorem of Green [17] states that $f(n, \mathcal{L}) = 2^{\mu_{\mathcal{L}}(n)+o(n)}$ for any fixed homogeneous linear equation $\mathcal{L}$. The aim of this section is to replace the term $o(n)$ here with a constant for many equations $\mathcal{L}$. This will be achieved in Theorem 21, which immediately implies Theorem 3. Denote by $f(n, \mathcal{L}, m)$ the number of $\mathcal{L}$-free subsets of $[n]$ with minimum element m. We first give bounds on $f(n, \mathcal{L}, m)$ for linear equations $\mathcal{L}$ of the form $px + qy = z$.

Lemma 20. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geq q$ and $p \geq 2$, $p, q \in \mathbb{N}$.*

(i) If $m \geq \lfloor \frac{n}{p+q} \rfloor + 1$ then $f(n, \mathcal{L}, m) = 2^{n-m}$.
(ii) If $m = \lfloor \frac{n}{p+q} \rfloor$ then $f(n, \mathcal{L}, m) \leq 2^{\mu_{\mathcal{L}}(n)-1}$.
(iii) If $q \geq 2$, $m = \lfloor \frac{n}{p+q} \rfloor - t$ for some positive integer t and G_m has path parameter 1, then $f(n, \mathcal{L}, m) \leq 2^{\mu_{\mathcal{L}}(n)-3/5+t(3q-2p)/(5q)}$.
(iv) If $q \geq 2$, $m = \lfloor \frac{n}{p+q} \rfloor - t$ for some positive integer t and G_m has path parameter $k \geq 2$, then $f(n, \mathcal{L}, m) \leq (4/3) \cdot 2^{(5q^2-2q+2)n/(5q^2)}$.
(v) If $q = 1$, G_m has path parameter ℓ, and $m = \lfloor \frac{n}{\ell p+1} \rfloor - t$ for some integer t, then $f(n, \mathcal{L}, m) \leq 2^{(7\ell p+3p)n/(10\ell p+10)+t(7-3p)/10}$.

Proof. First note that (i) is trivial since all subsets $S \subseteq [n]$ with $\min(S) \geq \lfloor \frac{n}{p+q} \rfloor + 1$ are $\mathcal{L}$-free. By Fact 16(ii) we know that $f(n, \mathcal{L}, m)$ is at most the number of independent sets in G_m which contain m. For (ii), there is one edge between $m = \lfloor \frac{n}{p+q} \rfloor$ and $(p+q)m \leq n$ in G_m, hence there are at most $2^{n-\lfloor \frac{n}{p+q} \rfloor - 1} = 2^{\mu_{\mathcal{L}}(n)-1}$ independent sets in G_m containing m.

For (iii) suppose $q \geq 2$ and $m = \lfloor \frac{n}{p+q} \rfloor - t$ for some $t \in \mathbb{N}$. Notice that G_m contains a matching on $y_1 - m + 1$ edges, namely there is an edge between c and $pm + qc$ for $c \in [m, y_1]$. Observe that

$3/4 \le 2^{-2/5}$ and also

$$y_1 - m = \left\lfloor \frac{n-pm}{q} \right\rfloor - m \ge \frac{n-(p+q)m-q}{q} \ge \frac{t(p+q)}{q} - 1.$$

Hence by Lemma 11 the total number of independent sets in G_m which contain m is at most

$$\begin{aligned} &2^{n-m-2(y_1-m)-1}3^{y_1-m} \le 2^{\mu_{\mathcal{L}}(n)-1+t}(3/4)^{y_1-m} \\ &\le 2^{\mu_{\mathcal{L}}(n)-1+t}(3/4)^{t(p+q)/q-1} \le 2^{\mu_{\mathcal{L}}(n)-3/5+t(3q-2p)/(5q)}, \end{aligned}$$

as desired.

For (iv) suppose $q \ge 2$, $m = \lfloor \frac{n}{p+q} \rfloor - t$ for some positive integer t and G_m has path parameter $k \ge 2$. First note that

$$\begin{aligned} y_1 - y_2 = &\left\lfloor \frac{n-pm}{q} \right\rfloor - \left\lfloor \frac{\lfloor \frac{n-pm}{q} \rfloor - pm}{q} \right\rfloor \ge \frac{n-pm-q}{q} - \frac{n-pm-qpm}{q^2} \\ =&\frac{(q-1)n+pm-q^2}{q^2} \ge \frac{(q-1)n}{q^2} - 1. \end{aligned}$$

Define $F(i)$ to be the ith Fibonacci number where $F(1) = F(2) = 1$. There are $F(i+2)$ independent sets (including the empty set) in a path of length i. Observe the following Fibonacci identity: $F(i+2)F(i) - F(i+1)^2 = (-1)^{i+1}$. If i is even and $a > b$ then

$$\left(\frac{F(i)F(i+2)}{F(i+1)^2}\right)^a \left(\frac{F(i+1)F(i+3)}{F(i+2)^2}\right)^b = \left(\frac{F(i+1)^2-1}{F(i+1)^2}\right)^a \left(\frac{F(i+2)^2+1}{F(i+2)^2}\right)^b \le 1.$$

Also observe that by omitting $(F(i+1)F(i+3)/F(i+2)^2)^b$ the inequality still holds. By use of Fact 17 and applying the above bounds, we can bound from above the number of independent sets in G_m as required:

$$\begin{aligned} &2^{y_0+y_2-2y_1}3^{y_1+y_3-2y_2}5^{y_2+y_4-2y_3}\dots F(k+1)^{y_{k-2}+y_k-2y_{k-1}}F(k+2)^{y_{k-1}+m-2y_k-1}F(k+3)^{y_k-m+1} \\ &= 2^{y_0+y_2-2y_1}3^{y_1-2y_2}5^{y_2}\left(\frac{3\cdot 8}{5^2}\right)^{y_3}\left(\frac{5\cdot 13}{8^2}\right)^{y_4}\cdots\left(\frac{F(k+1)\cdot F(k+3)}{F(k+2)^2}\right)^{y_k}\left(\frac{F(k+2)}{F(k+3)}\right)^{m-1} \\ &\le 2^{y_0+y_2-2y_1}3^{y_1-2y_2}5^{y_2} \le 2^{y_0+y_2-2y_1+y_2}3^{y_1-y_2} = 2^{y_0}(3/4)^{y_1-y_2} \le 2^n(3/4)^{(q-1)n/q^2-1} \\ &\le (4/3)\cdot 2^{n-2(q-1)n/(5q^2)} = (4/3)\cdot 2^{(5q^2-2q+2)n/(5q^2)}. \end{aligned}$$

For (v), since $y_i = n - ipm$ Fact 17 implies that if G_m has path parameter ℓ, then G_m is a union of paths of length ℓ and $\ell+1$. We use the bound $F(i) \le 2^{(7i-11)/10}$ (a simple proof by induction which holds for $i \ge 2$). Since $m < y_\ell = n - \ell pm$ we can write $m = \lfloor \frac{n}{\ell p+1} \rfloor - t$ for some integer $t \ge 0$. Now using these bounds, we have

$$\begin{aligned} &F(\ell+2)^{y_{\ell-1}-2y_\ell+m}F(\ell+3)^{y_\ell-m} = F(\ell+2)^{(\ell p+p+1)m-n}F(\ell+3)^{n-(\ell p+1)m} \\ &\le 2^{(3+7\ell)((\ell p+p+1)m-n)/10+(10+7\ell)(n-(\ell p+1)m)/10} = 2^{(7n+(3p-7)m)/10} \\ &\le 2^{(7n+(3p-7)(n/(\ell p+1)-t))/10} = 2^{(7\ell p+3p)n/(10\ell p+10)+t(7-3p)/10}. \end{aligned}$$

□

Theorem 21. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p, q \in \mathbb{N}$ and*

(i) $q \geq 2$ *and* $p > q(3q-2)/(2q-2)$ *or;*

(ii) $q = 1$ *and* $p \geq 3$.

Then $f(n,\mathcal{L}) \leq (3/2+o(1)+C)2^{\mu_{\mathcal{L}}(n)}$ *where for (i)* $C := \frac{2^{-2p/(5q)}}{1-2^{(3q-2p)/(5q)}}$ *and for (ii)* $C := \frac{2^{(7-3p)/10}}{1-2^{(7-3p)/10}}$.

Proof. For both cases by Lemma 20(i)–(ii) there are at most $3 \cdot 2^{\mu_{\mathcal{L}}(n)-1}$ $\mathcal{L}$-free subsets S of $[n]$ where $\min(S) \geq \lfloor \frac{n}{p+q} \rfloor$. For (i), first consider $\mathcal{L}$-free subsets arising from Lemma 20(iv). Since $k \geq 2$,

$$m < y_2 = \left\lfloor \frac{\lfloor \frac{n-pm}{q} \rfloor - pm}{q} \right\rfloor \leq \frac{n - pm - qpm}{q^2}$$

and so $m \leq n/(q^2 + pq + p)$. Now as $n \to \infty$,

$$\frac{n/(q^2+pq+p) \cdot (4/3) \cdot 2^{(5q^2-2q+2)n/(5q^2)}}{2^{\mu_{\mathcal{L}}(n)}} = \frac{2^{\log_2(4n/(3(q^2+pq+p)))+(5q^2-2q+2)n/(5q^2)}}{2^{\mu_{\mathcal{L}}(n)}} \to 0,$$

as long as we have $2^{(5q^2-2q+2)n/(5q^2)} \ll 2^{\mu_{\mathcal{L}}(n)}$. This is satisfied if $(5q^2-2q+2)/(5q^2) < (p+q-1)/(p+q)$ which when rearranged, gives $p > q(3q-2)/(2q-2)$.

For $\mathcal{L}$-free subsets arising from Lemma 20(iii), set $a := 2^{\mu_{\mathcal{L}}(n)-3/5}$, $r := 2^{(3q-2p)/(5q)}$ and let u be the largest t such that G_m with $m = \lfloor \frac{n}{p+q} \rfloor - t$ has path parameter 1. Then since $p > q(3q-2)/(2q-2) > 3q/2$ we have $|r| < 1$ and so

$$\sum_{t=1}^{u} 2^{\mu_{\mathcal{L}}(n)-3/5+t(3q-2p)/(5q)} \leq \sum_{t=1}^{\infty} ar^t = \sum_{t=0}^{\infty} (ar)r^t = \frac{ar}{1-r} = \frac{2^{\mu_{\mathcal{L}}(n)-2p/(5q)}}{1-2^{(3q-2p)/(5q)}}.$$

Altogether this implies that $f(n,\mathcal{L}) \leq (3/2+o(1)+C)2^{\mu_{\mathcal{L}}(n)}$ where $C := \frac{2^{-2p/(5q)}}{1-2^{(3q-2p)/(5q)}}$.

For (ii), set $a := 2^{(7kp+3p)n/(10kp+10)}$, set $r := 2^{(7-3p)/10}$ and let u be the largest t such that G_m with $m := \lfloor \frac{n}{p+q} \rfloor - t$ has path parameter k for any fixed $k \in \mathbb{N}$. Since $p \geq 3$ we have $|r| < 1$ and so

$$\sum_{t=1}^{u} 2^{(7kp+3p)n/(10kp+10)+t(7-3p)/10} \leq \sum_{t=1}^{\infty} ar^t = \sum_{t=0}^{\infty} (ar)r^t = \frac{ar}{1-r} = \frac{2^{(7kp+3p)n/(10kp+10)+(7-3p)/10}}{1-2^{(7-3p)/10}}.$$

For $k = 1$ the last term is at most $2^{(\mu_{\mathcal{L}}(n)+(7-3p)/10)}/(1-2^{(7-3p)/10})$. For $k \geq 2$ we obtain a term which is $o(2^{\mu_{\mathcal{L}}(n)})$ as n tends to infinity, since $(7kp+3p)n/(10kp+10) < \mu_{\mathcal{L}}(n)$ for $p \geq 3$. Therefore, Lemma 20 implies that $f(n,\mathcal{L}) \leq (3/2+o(1)+C)2^{\mu_{\mathcal{L}}(n)}$ where $C := \frac{2^{(7-3p)/10}}{1-2^{(7-3p)/10}}$. □

5. The number of maximal solution-free sets

5.1. **A general upper bound.** Let $\mathcal{L}$ be a three-variable linear equation. Let $\mathcal{M}_{\mathcal{L}}(n)$ denote the set of elements $x \in [n]$ such that $x \in [n]$ does not lie in *any* $\mathcal{L}$-triple in $[n]$. Define $\mu^*_{\mathcal{L}}(n) := |\mathcal{M}_{\mathcal{L}}(n)|$. For example, if $\mathcal{L}$ is translation-invariant then $\{x, x, x\}$ is an $\mathcal{L}$-triple for all $x \in [n]$ so $\mathcal{M}_{\mathcal{L}}(n) = \emptyset$ and $\mu^*_{\mathcal{L}}(n) = 0$.

Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geq 2$, $p \geq q$ and $p, q \in \mathbb{N}$. Write $t := \gcd(p,q)$. Then notice that $\mathcal{M}_{\mathcal{L}}(n) \supseteq \{s \in [n] : s > \lfloor (n-p)/q \rfloor, t \nmid s\}$. This follows since if $s > \lfloor (n-p)/q \rfloor$ then $ps + q \geq qs + p > n$ and so s cannot play the role of x or y in an $\mathcal{L}$-triple in $[n]$. If $t \nmid s$ then as $t | (px+qy)$ for any $x, y \in [n]$ we have that s cannot play the role of z in an $\mathcal{L}$-triple in $[n]$. Actually, for large enough n we have $\mathcal{M}_{\mathcal{L}}(n) = \{s : s > \lfloor (n-p)/q \rfloor, t \nmid s\}$ for all such $\mathcal{L}$. We omit the proof of this here.

We now prove Theorem 4.

Theorem 4. Let $\mathcal{L}$ be a fixed homogenous three-variable linear equation. Then

$$f_{\max}(n, \mathcal{L}) \leq 3^{(\mu_{\mathcal{L}}(n) - \mu^*_{\mathcal{L}}(n))/3 + o(n)}.$$

Proof. Let $\mathcal{F}$ denote the set of containers obtained by applying Lemma 7. Since every $\mathcal{L}$-free subset of $[n]$ lies in at least one of the $2^{o(n)}$ containers, it suffices to show that every $F \in \mathcal{F}$ houses at most $3^{(\mu_{\mathcal{L}}(n) - \mu^*_{\mathcal{L}}(n))/3 + o(n)}$ maximal $\mathcal{L}$-free subsets.

Let $F \in \mathcal{F}$. By Lemmas 7(i) and 8, $F = A \cup B$ where $|A| = o(n)$, $|B| \leq \mu_{\mathcal{L}}(n)$ and B is $\mathcal{L}$-free. Note that we can add all the elements of $\mathcal{M}_{\mathcal{L}}(n)$ to B (and thus F) whilst ensuring that $|B| \leq \mu_{\mathcal{L}}(n)$ and B is $\mathcal{L}$-free. So we may assume that $\mathcal{M}_{\mathcal{L}}(n) \subseteq B$.

Each maximal $\mathcal{L}$-free subset of $[n]$ in F can be found by picking a subset $S \subseteq A$ which is $\mathcal{L}$-free, and extending it in B. The number of ways of doing this is the number of ways of choosing the subset S multiplied by the number of ways of extending a fixed S in B, which we denote by $N(S, B)$. Since $|A| = o(n)$, there are $2^{o(n)}$ choices for S. It therefore suffices to show that for any $S \subseteq A$, we have $N(S, B) \leq 3^{(\mu_{\mathcal{L}}(n) - \mu^*_{\mathcal{L}}(n))/3}$.

Consider the link graph $G := L_S[B]$. Then by definition, $\mathcal{M}_{\mathcal{L}}(n)$ is an independent set in G. Thus, $\mathrm{MIS}(G) = \mathrm{MIS}(G \setminus \mathcal{M}_{\mathcal{L}}(n))$. Further, Lemma 12 and Theorem 13(i) imply that

$$N(S, B) \leq \mathrm{MIS}(G) = \mathrm{MIS}(G \setminus \mathcal{M}_{\mathcal{L}}(n)) \leq 3^{|B \setminus \mathcal{M}_{\mathcal{L}}(n)|/3} \leq 3^{(\mu_{\mathcal{L}}(n) - \mu^*_{\mathcal{L}}(n))/3},$$

as desired. □

As mentioned in the introduction, Theorem 4 together with Theorem 9 shows that $f_{\max}(n, \mathcal{L})$ is significantly smaller than $f(n, \mathcal{L})$ for all homogeneous three-variable linear equations $\mathcal{L}$ that are not translation-invariant. So in this sense it can be viewed as a generalisation of a result of Łuczak and Schoen [24] on sum-free sets.

Let $\mathcal{L}$ denote the equation $px + y = z$ for some $p \in \mathbb{N}$. Notice that in this case we have $\mu^*_{\mathcal{L}}(n) = 0$ for $n > p$. The next result implies that if p is large then $f_{\max}(n, \mathcal{L})$ is close to the bound in Theorem 4. So for such equations $\mathcal{L}$, Theorem 4 is close to best possible.

Proposition 22. *Given $p, n \in \mathbb{N}$ where $p \geq 2$, let $\mathcal{L}$ denote the equation $px + y = z$. Then*

$$f_{\max}(n, \mathcal{L}) \geq 3^{\mu_{\mathcal{L}}(n)/3 - 2pn/(3(p+1)(3p^2-1)) - p - 5}.$$

Proof. Given $p, n \in \mathbb{N}$, let $\mathcal{L}$ denote the equation $px + y = z$. Set $s := \lfloor \frac{(p-1)n}{3p^2-1} \rfloor$ and $a := \lfloor \frac{n-s}{p} \rfloor$. Consider the link graph $G := L_{\{s,2s\}}[a+1, a+3ps]$. Observe that:

$$\begin{aligned}
&2s \leq \frac{(2p-2)n}{3p^2-1} < \frac{n}{p+1} < \frac{(3p-1)n}{3p^2-1} = \frac{n}{p} - \frac{(p-1)n}{3p^3-p} \leq \frac{n-s}{p} < a+1; \\
&a + 3ps = \left\lfloor \frac{n-s}{p} \right\rfloor + 3ps \leq \frac{n}{p} + \left(3p - \frac{1}{p}\right)s = \frac{n}{p} + \frac{3p^2-1}{p} \left\lfloor \frac{(p-1)n}{3p^2-1} \right\rfloor \leq \frac{n + n(p-1)}{p} = n.
\end{aligned}$$

As a consequence, the sets $\{s, 2s\}$ and $[a+1, a+3ps]$ (a subset of $[\lfloor \frac{n}{p+1} \rfloor + 1, n]$) are disjoint $\mathcal{L}$-free sets in $[n]$, and so Lemma 15 implies that $f_{\max}(n, \mathcal{L}) \geq \mathrm{MIS}(G)$. It remains to show that G contains at least $3^{\mu_{\mathcal{L}}(n)/3 - 2pn/(3(p+1)(3p^2-1)) - 6}$ maximal independent sets.

Observe that for each $i \in [ps]$ there is an edge in G between $a+i$ and $a+ps+i$ (since $\{s, a+i, a+i+ps\}$ is an $\mathcal{L}$-triple), an edge between $a+i+ps$ and $a+i+2ps$ (since $\{s, a+i+ps, a+i+2ps\}$ is an $\mathcal{L}$-triple) and an edge between $a+i$ and $a+i+2ps$ (since $\{2s, a+i, a+i+2ps\}$ is an $\mathcal{L}$-triple). Also since $a > (n-s)/p - 1$, we have $p(a+1) + s > n$ and hence there are no further edges in G.

Hence G is a collection of ps disjoint triangles, where 4 vertices in G have loops ($(p+1)s$, $(p+2)s$, $(2p+1)s$ and $(2p+2)s$). So G has at least 3^{ps-4} maximal independent sets. Now observe:

$$
\begin{aligned}
ps-4-\frac{\mu_{\mathcal{L}}(n)}{3} &= p\left\lfloor\frac{(p-1)n}{3p^2-1}\right\rfloor-4-\frac{n}{3}+\frac{1}{3}\left\lfloor\frac{n}{p+1}\right\rfloor \geq \left(\frac{p^2-p}{3p^2-1}-\frac{1}{3}+\frac{1}{3(p+1)}\right)n-p-5\\
&= \left(\frac{-2p}{3(p+1)(3p^2-1)}\right)n-p-5,
\end{aligned}
$$

as required. $\square$

5.2. **Upper bounds for $px+qy=z$.** Let $\mathcal{L}$ denote the equation $px+qy=z$ where $p \geq q$, $p \geq 2$ and $p,q \in \mathbb{N}$. For such $\mathcal{L}$, the next simple result provides an alternative bound to Theorem 4.

Lemma 23. *Let $\mathcal{L}$ denote the equation $px+qy=z$ where $p \geq q$, $p \geq 2$ and $p,q \in \mathbb{N}$. Then $f_{\max}(n,\mathcal{L}) \leq f(\lfloor (n-p)/q \rfloor, \mathcal{L})$.*

Proof. Set $C := [\lfloor \frac{n-p}{q} \rfloor]$ and $B := [\lfloor \frac{n-p}{q} \rfloor + 1, n]$. In particular, B is $\mathcal{L}$-free. Notice that every maximal $\mathcal{L}$-free subset of $[n]$ can be found by selecting an $\mathcal{L}$-free subset $S \subseteq C$ and then extending it in B to a maximal one. Suppose we have such an $\mathcal{L}$-free subset S. By Lemma 12, the number of such extensions of S is at most $\mathrm{MIS}(L_S[B])$.

For any $\mathcal{L}$-triple $\{x,y,z\}$ in $[n]$ satisfying $px+qy=z$, since $z \leq n$, we must have $x \leq \frac{n-q}{p}$ and $y \leq \frac{n-p}{q}$. Hence $x,y \in C$. This means that there are no $\mathcal{L}$-triples in $[n]$ which contain more than one element from B. Thus the link graph $L_S[B]$ must only contain isolated vertices and loops. So $L_S[B]$ has precisely one maximal independent set. Hence the number of maximal $\mathcal{L}$-free subsets of $[n]$ is bounded by the number of choices of S in C which are $\mathcal{L}$-free, i.e. $f(\lfloor (n-p)/q \rfloor, \mathcal{L})$. $\square$

Lemma 23 together with Theorems 3 and 9 immediately imply the following result (which itself immediately implies Theorem 6).

Corollary 24. *Let $\mathcal{L}$ denote the equation $px+qy=z$ where $p \geq q$, $p \geq 2$ and $p,q \in \mathbb{N}$. Then*

$$f_{\max}(n,\mathcal{L}) \leq 2^{\mu_{\mathcal{L}}(\lfloor \frac{n-p}{q} \rfloor)+o(n)}.$$

Further, if $q \geq 2$ and $p > q(3q-2)/(2q-2)$ or $q=1$ and $p \geq 3$ then

$$f_{\max}(n,\mathcal{L}) = O(2^{\mu_{\mathcal{L}}(\lfloor \frac{n-p}{q} \rfloor)}).$$

The next result gives a further upper bound on $f_{\max}(n,\mathcal{L})$ for certain linear equations $\mathcal{L}$. Notice that for such $\mathcal{L}$, Theorem 5 yields a better bound than Theorem 4.

Theorem 5. Let $\mathcal{L}$ denote the equation $px+qy=z$ where $p \geq q \geq 2$ are integers so that $p \leq q^2-q$ and $\gcd(p,q)=q$. Then

$$f_{\max}(n,\mathcal{L}) \leq 2^{(\mu_{\mathcal{L}}(n)-\mu^*_{\mathcal{L}}(n))/2+o(n)}.$$

Proof. Let $\mathcal{F}$ denote the set of containers obtained by applying Lemma 7. Since every $\mathcal{L}$-free subset of $[n]$ lies in at least one of the $2^{o(n)}$ containers, it suffices to show that every $F \in \mathcal{F}$ houses at most $2^{(\mu_{\mathcal{L}}(n)-\mu^*_{\mathcal{L}}(n))/2+o(n)}$ $\mathcal{L}$-free sets.

Let $F \in \mathcal{F}$. By Lemmas 7(i) and 8, $F = A \cup B$ where $|A| = o(n)$, $|B| \leq \mu_{\mathcal{L}}(n)$ and B is $\mathcal{L}$-free. Note that we can add all the elements of $\mathcal{M}_{\mathcal{L}}(n)$ to B (and thus F) whilst ensuring that $|B| \leq \mu_{\mathcal{L}}(n)$ and B is $\mathcal{L}$-free. So we may assume that $\mathcal{M}_{\mathcal{L}}(n) \subseteq B$. By Theorem 18, $\min(B) = \lfloor \frac{n}{p+q} \rfloor - t$ for some non-negative integer $t < (\frac{p+q-1}{p+q+p/q})\lfloor \frac{n}{p+q} \rfloor$ and $|B| \leq \lceil \frac{(p+q-1)n}{p+q} \rceil - \lfloor \frac{p}{q} t \rfloor$, or $|B| \leq \frac{(q^2+1)n}{q^2+q+1}$.

Case 1: $\min(B) = \lfloor \frac{n}{p+q} \rfloor - t$ for $0 \leq t < (\frac{p+q-1}{p+q+p/q})\lfloor \frac{n}{p+q} \rfloor$. Write $F = X \cup Y$ where $Y \subseteq [\lfloor \frac{n}{p+q} \rfloor + 1, n]$ is $\mathcal{L}$-free, and $X \subseteq [1, \lfloor \frac{n}{p+q} \rfloor]$. Note that $|X| = t' + o(n)$ and $|Y| \leq \lceil \frac{(p+q-1)n}{p+q} \rceil - \lfloor \frac{p}{q} t \rfloor - t' + o(n)$ where $t' \leq t$. Also $\mathcal{M}_{\mathcal{L}}(n) \subseteq Y$. Choose $S \subseteq X$ to be $\mathcal{L}$-free. Consider the link graph $L_S[Y]$

and observe that by Lemma 12, $N(S,Y) \le \mathrm{MIS}(L_S[Y])$. (Recall $N(S,Y)$ denotes the number of extensions of S in Y to a maximal $\mathcal{L}$-free set.)

Since $p \le q^2 - q$, by Lemma 14 $L_S[Y]$ is triangle-free. By definition, $\mathcal{M}_{\mathcal{L}}(n)$ is an independent set in $L_S[Y]$ and so $\mathrm{MIS}(L_S[Y])$=$\mathrm{MIS}(L_S[Y \setminus \mathcal{M}_{\mathcal{L}}(n)])$. Therefore Theorem 13(ii) implies that $\mathrm{MIS}(L_S[Y]) \le 2^{(|Y|-|\mathcal{M}_{\mathcal{L}}(n)|)/2}$. Overall, this implies that the number of $\mathcal{L}$-free sets contained in F is at most

$$2^{|X|} \times 2^{(|Y|-|\mathcal{M}_{\mathcal{L}}(n)|)/2} \le 2^{t'+o(n)+(\mu_{\mathcal{L}}(n)-\mu^*_{\mathcal{L}}(n)-\lfloor \frac{p}{q}t \rfloor - t')/2} \le 2^{(\mu_{\mathcal{L}}(n)-\mu^*_{\mathcal{L}}(n))/2+o(n)},$$

as desired.

Case 2: $|B| \le \frac{(q^2+1)n}{q^2+q+1}$. In this case $|F| \le \frac{(q^2+1)n}{q^2+q+1} + o(n)$. Choose any $\mathcal{L}$-free $S \subseteq A$ (note there are at most $2^{o(n)}$ choices for S). Consider the link graph $L_S[B]$ and observe by Lemma 12 that $N(S,B) \le \mathrm{MIS}(L_S[B])$. Similarly as in Case 1 we have that $\mathrm{MIS}(L_S[B])$=$\mathrm{MIS}(L_S[B'])$ where $B' := B \setminus \mathcal{M}_{\mathcal{L}}(n)$. By Theorem 13(i),

$$\mathrm{MIS}(L_S[B']) \le 3^{|B'|/3} \le 3^{((q^2+1)n/(3(q^2+q+1))-\mu^*_{\mathcal{L}}(n)/3)} \le 2^{(\mu_{\mathcal{L}}(n)-\mu^*_{\mathcal{L}}(n))/2+o(n)}.$$

The last inequality follows since $\mu_{\mathcal{L}}(n) = n - \lfloor n/(p+q) \rfloor$ and $\mathcal{M}_{\mathcal{L}}(n) = \{s : s > \lfloor (n-p)/q \rfloor, q \nmid s\}$ since $\gcd(p,q) = q$.

To see this, first note that

$$\mu^*_{\mathcal{L}}(n) = \frac{(q-1)^2 n}{q^2} - o(n).$$

Hence for the inequality to hold we require that

$$9^{((q^2+1)/(q^2+q+1)-(q^2-2q+1)/(q^2))} < 8^{((p+q-1)/(p+q)-(q^2-2q+1)/(q^2))}.$$

Let $a := \log_9 8$. This rearranges to give

$$p > \frac{(1-a)(q^4-q)+q^3+q^2}{(2a-1)q^3+(a-1)(q^2+q-1)}.$$

Since $p \ge q$ it suffices to show that $(3a-2)q^3 + (a-2)(q^2+q) + (2-2a) > 0$. This indeed holds since $q \ge 2$.

Overall, this implies that the number of $\mathcal{L}$-free sets contained in F is at most $2^{(\mu_{\mathcal{L}}(n)-\mu^*_{\mathcal{L}}(n))/2+o(n)}$, as desired. $\square$

The proof of Theorem 5 actually generalises to some other equations $px+qy = z$ where $\gcd(p,q) \ne q$ (but still $p \le q^2 - q$). However, in these cases Theorem 6 produces a better upper bound on $f_{\max}(n,\mathcal{L})$. The next result summarises when Theorem 4, 5 or 6 yields the best upper bound on $f_{\max}(n,\mathcal{L})$. We defer the proof to the appendix.

Proposition 25. *Let $\mathcal{L}$ denote the equation $px+qy = z$ where $p \ge q$, $p \ge 2$ and $p,q \in \mathbb{N}$. The best upper bound on $f_{\max}(n,\mathcal{L})$ given by Theorems 4, 5 and 6 is:*

(i) $f_{\max}(n,\mathcal{L}) \le 3^{(\mu_{\mathcal{L}}(n)-\mu^*_{\mathcal{L}}(n))/3+o(n)}$ *if* $\gcd(p,q) = q$, $p \ge q^2$, *and either* $q \le 9$ *or* $10 \le q \le 17$ *and* $p < (a-1)(q^2-q)/(q(2-a)-1)$ *where* $a := \log_3(8)$*;*

(ii) $f_{\max}(n,\mathcal{L}) \le 2^{(\mu_{\mathcal{L}}(n)-\mu^*_{\mathcal{L}}(n))/2+o(n)}$ *if* $\gcd(p,q) = q$ *and* $p \le q^2 - q$*;*

(iii) $f_{\max}(n,\mathcal{L}) \le 2^{\mu_{\mathcal{L}}(\lfloor (n-p)/q \rfloor)+o(n)}$ *otherwise.*

5.3. **Lower bounds for** $px + qy = z$**.** The following result provides lower bounds on $f_{\max}(n,\mathcal{L})$ for all equations $\mathcal{L}$ of the form $px+qy = z$ where $p \ge q \ge 2$.

Proposition 26. *Let $\mathcal{L}$ denote the equation $px+qy = z$ where $p \ge q \ge 2$ are integers. Suppose that $n > 2p$. In each case $f_{\max}(n,\mathcal{L}) \ge 2^{\ell}$ where ℓ is defined as follows:*

(i) $\ell := (n(q-1) - pq + q - 2q^2)/q^2$ *if* $p \ge q^2$*,*

(ii) $\ell := (n(p-q) - p^2 + q^2 - 2pq)/(pq)$ *if* $q < p < q^2$,
(iii) $\ell := (n - 6q)/2q$ *if* $p = q$.

Proof. For each case, we shall let $B := [\lfloor \frac{n}{p+q} \rfloor + 1, n]$, and consider the link graph $G := L_{\{1\}}[B]$. Since B and $\{1\}$ are $\mathcal{L}$-free, by Lemma 15 it suffices to show that there is an induced subgraph of G which contains at least 2^ℓ maximal independent sets. For each case we will find an induced perfect matching on 2ℓ vertices in G. (Note there are 2^ℓ maximal independent sets in such a matching.)

More specifically, for each case we shall find an interval $I := [a, b]$ for some $a, b \in V(G)$ and let $J := \{qi + p | i \in I\}$. Note that all edges in G (other than at most one loop) are of the form $\{i, qi + p\}$ and $\{i, pi + q\}$. By our choice of I and J, $G[I \cup J]$ will form a perfect matching on $2|I|$ vertices if the following conditions hold:

(1) $qa + p > b$ (which ensures that $I \cap J = \emptyset$),
(2) $qb + p \leq n$ (which ensures that $J \subseteq [n]$),
(3) $pa + q > n$ (which ensures that the only edges in G are of the form $\{i, qi + p\}$),
(4) $p + q < a$ (which ensures that there is no loop at a vertex in $G[I \cup J]$).

Notice that actually we do not require condition (3) to hold in the case when $p = q$. Indeed, this is because in this case an edge $\{i, pi + q\}$ in G is the same as the edge $\{i, qi + p\}$. Further, there is at most one loop in G (if $p + q \in B$). So even if (4) does not hold we will obtain an induced matching in G on $2|I| - 2$ vertices.

Thus, to obtain an induced matching in G on $2|I| - 2$ vertices it suffices to choose a and b so that (1)–(3) hold except when $p = q$ when we only require that (1) and (2) hold.

By choosing $b := \lfloor (n - p)/q \rfloor$, (2) holds since $qb + p = q\lfloor (n - p)/q \rfloor + p \leq q(n - p)/q + p = n$.

If $p \geq q^2$ then set $a := \lfloor (n - q)/q^2 \rfloor + 1$. Then $a \in B$ and further $pa + q \geq q^2 a + q > q^2((n - q)/q^2) + q = n$ and $qa + p \geq qa + q^2 > q((n - q)/q^2) + q^2 = n/q - 1 + q^2 > \lfloor (n - p)/q \rfloor = b$. So (1) and (3) hold.

If $q < p < q^2$ then set $a := \lfloor (n - q)/p \rfloor + 1$. So $a \in B$. Further, $pa + q > p((n - q)/p) + q = n$ and $qa + p > q((n - q)/p) + p = qn/p - q^2/p + p > qn/q^2 - q + p > n/q > \lfloor (n - p)/q \rfloor = b$. So (1) and (3) hold.

If $p = q$ set $a := \lfloor n/(p + q) \rfloor + 1 = \lfloor n/(2q) \rfloor + 1 \in B$. Observe that $qa + q > qn/2q + q > n/2 > \lfloor (n - q)/q \rfloor = b$ since $q \geq 2$. So (1) holds.

Now calculating the size of the interval $I = [a, b]$ in each case proves the result:

- If $a = \lfloor (n - q)/q^2 \rfloor + 1$, then $|I| - 1 = \lfloor (n - p)/q \rfloor - (\lfloor (n - q)/q^2 \rfloor + 1) \geq (n - p)/q - 1 - (n - q)/q^2 - 1 = (n(q - 1) - pq + q - 2q^2)/q^2$.
- If $a = \lfloor (n - q)/p \rfloor + 1$, then $|I| - 1 = \lfloor (n - p)/q \rfloor - (\lfloor (n - q)/p \rfloor + 1) \geq (n - p)/q - 1 - (n - q)/p - 1 = (n(p - q) - p^2 + q^2 - 2pq)/(pq)$.
- If $a = \lfloor n/(p+q) \rfloor + 1$ then $|I| - 1 = \lfloor (n-p)/q \rfloor - (\lfloor n/(p+q) \rfloor + 1) \geq (n-p)/q - 1 - n/(p+q) - 1 = (pn - (p + 2q)(p + q))/(q(p + q)) = (qn - 6q^2)/(2q^2) = (n - 6q)/2q$.

□

Although the lower bounds in Proposition 26 do not meet the upper bounds in Theorem 5 and Corollary 24 in general, Theorem 5 and Proposition 26(iii) do immediately imply the following asymptotically exact result.

Theorem 27. *Let $\mathcal{L}$ denote the equation $2x + 2y = z$. Then $f_{\max}(n, \mathcal{L}) = 2^{n/4+o(n)}$.*

Since submitting this paper, we have also given a general upper bound on $f_{\max}(n, \mathcal{L})$ for equations $\mathcal{L}$ of the form $px + qy = rz$ where $p \geq q \geq r$ are fixed positive integers (see [19]). In particular, our result shows that in the case when $p = q \geq 2$, $r = 1$ the lower bound in Proposition 26(iii) is correct up to an error term in the exponent.

6. Concluding remarks

The results in the paper show that the parameter $f_{\max}(n, \mathcal{L})$ can exhibit very different behaviour depending on the linear equation $\mathcal{L}$. Indeed, Theorem 4 gives a 'crude' general upper bound on $f_{\max}(n, \mathcal{L})$ for all homogeneous three-variable linear equations $\mathcal{L}$. (It is crude in the sense that, in the proof, we do not use any structural information about the link graphs.) However, this bound is close to the correct value of $f_{\max}(n, \mathcal{L})$ for certain equations $\mathcal{L}$ (Proposition 22). On the other hand, for many equations this bound is far from tight (Theorem 5). Further, for some equations ($x + y = z$ and $2x + 2y = z$) the value of $f_{\max}(n, \mathcal{L})$ is tied to the property that any triangle-free graph on n vertices contains at most $2^{n/2}$ maximal independent sets. Theorem 6 and upper bounds we have obtained since submitting this paper (see [19]) suggest though that the value of $f_{\max}(n, \mathcal{L})$ for other equations $\mathcal{L}$ may depend on completely different factors. Further progress on understanding the possible behaviour of $f_{\max}(n, \mathcal{L})$ would be extremely interesting.

We conclude by briefly describing some results concerning equations with more than three variables. First observe the following simple proposition.

Proposition 28. *Let $\mathcal{L}_1$ denote the equation $p_1x_1 + \cdots + p_kx_k = b$ where $p_1, \dots, p_k, b \in \mathbb{Z}$ and let $\mathcal{L}_2$ denote the equation $(p_1 + p_2)x_1 + p_3x_2 + \cdots + p_kx_{k-1} = b$. Then $\mu_{\mathcal{L}_1}(n) \leq \mu_{\mathcal{L}_2}(n)$.*

The proposition is just a simple consequence of the observation that any solution to the equation $\mathcal{L}_2$ gives rise to a solution to the equation $\mathcal{L}_1$. So all $\mathcal{L}_1$-free subsets of $[n]$ are also $\mathcal{L}_2$-free. Note that for the equations $\mathcal{L}$ which satisfy the hypothesis of the following corollary, the interval $[\lfloor n/(p+q) \rfloor + 1, n]$ is $\mathcal{L}$-free. Hence by applying the above proposition along with Corollary 2, we attain the following result.

Corollary 29. *Let $\mathcal{L}$ denote the equation $a_1x_1 + \cdots + a_kx_k + b_1y_1 + \cdots + b_\ell y_\ell = c_1z_1 + \cdots + c_mz_m$ where the $a_i, b_i, c_i \in \mathbb{N}$ and $p' := \sum_i a_i$, $q' := \sum_i b_i$ and $r' := \sum_i c_i$. Let $t' := \gcd(p', q', r')$ and write $p := p'/t'$, $q := q'/t'$ and $r := r'/t'$. Suppose that $r = 1$. Then for sufficiently large n, we have $\mu_{\mathcal{L}}(n) = n - \lfloor n/(p+q) \rfloor$.*

One can define a link hypergraph $L_S[B]$ analogous to the notion of a link graph defined in Section 2.3 (i.e. now hyperedges correspond to solutions to $\mathcal{L}$ involving at least one element of S). We remark that the removal and container lemmas of Green [17] that we applied do hold for homogeneous linear equations on more than three variables. By arguing as in Lemma 23 (but by considering a link hypergraph), one can obtain the following simple result.

Proposition 30. *Let $\mathcal{L}$ denote the equation $p_1x_1 + \cdots + p_sx_s = rz$ where $p_1 \geq p_2 \geq \cdots \geq p_s > r \geq 1$ are positive integers. Then $f_{\max}(n, \mathcal{L}) \leq f(\lfloor rn/p_s \rfloor, \mathcal{L})$.*

In [19] we obtain further results concerning the number of maximal solution-free sets for linear equations with more than three variables. However the proof method does not use structural results such as Theorem 13, and only work for *some* linear equations. Obtaining similar structural results for the number of maximal independent sets in (non-uniform) hypergraphs would help to attain (general) upper bounds for the number of maximal solution-free sets.

Acknowledgments

The authors are grateful to Allan Lo for comments that lead to a simplification in the calculations in the proof of Theorem 18. We are also grateful to Deryk Osthus for comments on the manuscript.

References

[1] J. Balogh, H. Liu, S. Petříčková and M. Sharifzadeh, The typical structure of maximal triangle-free graphs, The typical structure of maximal triangle-free graphs, *Forum Math. Sigma*, 3, (2015), e20 (19 pages).

[2] J. Balogh, H. Liu, M. Sharifzadeh and A. Treglown, The number of maximal sum-free subsets of integers, *Proc. Amer. Math. Soc.*, 143, (2015), 4713–4721.
[3] J. Balogh, H. Liu, M. Sharifzadeh and A. Treglown, Sharp bound on the number of maximal sum-free subsets of integers, submitted.
[4] J. Balogh, R. Morris and W. Samotij, Independent sets in hypergraphs, *J. Amer. Math. Soc.*, 28, (2015), 669–709.
[5] J. Balogh and S. Petříčková, The number of the maximal triangle-free graphs, *Bull. London Math. Soc.*, 46, (2014), 1003–1006.
[6] A. Baltz, P. Hegarty, J. Knape, U. Larsson and T. Schoen, The structure of maximum subsets of $\{1,\dots,n\}$ with no solutions to $a+b=kc$, *Electron. J. Combin.*, 12, (2005), R19.
[7] F. Behrend. On sets of integers which contain no three terms in arithmetic progression, *Proc. Nat. Acad. Sci.*, 32, (1946), 331–332.
[8] Y. Bilu, Sum-free sets and related sets, *Combinatorica*, 18, (1998), 449–459.
[9] N.J. Calkin and J.M. Thomason, Counting generalized sum-free sets, *J. Number Theory*, 68, (1996), 151–159.
[10] P. Cameron and P. Erdős, On the number of sets of integers with various properties, in Number Theory (R.A. Mollin, ed.), 61–79, Walter de Gruyter, Berlin, 1990.
[11] P. Cameron and P. Erdős, Notes on sum-free and related sets, *Combin. Probab. Comput.*, 8, (1999), 95–107.
[12] J. Deshouillers, G. Freiman, V. Sós and M. Temkin, On the structure of sum-free sets II, *Astérisque*, 258, (1999), 149–161.
[13] K. Dilcher and L. Lucht, On finite pattern-free sets of integers, *Acta Arith.*, 121, (2006), 313–325.
[14] M. Elkin, An improved construction of progression-free sets, *Israel J. Math.*, 184, (2011), 93–128.
[15] P. Erdős and P. Turán, On a problem of Sidon in additive number theory, and on some related problems, *J. London Math. Soc.*, 1, (1941), 212–215.
[16] B. Green, The Cameron-Erdős conjecture, *Bull. London Math. Soc.*, 36, (2004), 769–778.
[17] B. Green, A Szemerédi-type regularity lemma in abelian groups, with applications, *Geom. Funct. Anal.*, 15, (2005), 340–376.
[18] B. Green and J. Wolf, A note on Elkins improvement of Behrend's construction, *Additive number theory: Festschrift in honor of the sixtieth birthday of Melvyn B. Nathanson*, pages 141–144. Springer-Verlag, 1st edition, 2010.
[19] R. Hancock and A. Treglown, A note on solution-free sets of integers, submitted.
[20] P. Hegarty, Extremal subsets of $\{1,\dots,n\}$ avoiding solutions to linear equations in three variables, *Electron. J. Combin.*, 14, (2007), R74.
[21] M. Hujter and Z. Tuza, The number of maximal independent sets in triangle-free graphs, *SIAM J. Discrete Math.*, 6, (1993), 284–288.
[22] Y. Kohayakawa, S. Lee, V. Rödl, and W. Samotij, The number of Sidon sets and the maximum size of Sidon sets contained in a sparse random set of integers, *Random Structures & Algorithms*, 46, (2015), 1–25.
[23] D. Král', O. Serra and L. Vena, A removal lemma for systems of linear equations over finite fields, *Israel J. Math.*, 187, (2012), 193–207
[24] T. Łuczak and T. Schoen, On the number of maximal sum-free sets, *Proc. Amer. Math. Soc.*, 129, (2001), 2205–2207.
[25] J.W. Moon and L. Moser, On cliques in graphs, *Israel J. Math.*, 3, (1965), 23–28.
[26] K.F. Roth, On certain sets of integers, *J. London Math. Soc.*, 28, (1953), 104–109.
[27] I.Z. Ruzsa, Solving a linear equation in a set of integers I, *Acta Arith.*, 65, (1993), 259–282.
[28] I.Z. Ruzsa, Solving a linear equation in a set of integers II, *Acta Arith.*, 72, (1995), 385–397.
[29] T. Sanders, On Roth's theorem on progressions, *Ann. of Math.*, 174, (2011), 619–636.
[30] A.A. Sapozhenko, The Cameron-Erdős conjecture, (Russian) *Dokl. Akad. Nauk.*, 393, (2003), 749–752.
[31] D. Saxton and A. Thomason, Hypergraph containers, *Invent. Math.*, 201, (2015), 925–992.
[32] T. Schoen, The number of $(2,3)$-sum-free subsets of $\{1,\dots,n\}$, *Acta Arith.*, 98, (2001), 155–163.
[33] I. Schur, Uber die Kongruenz $x^m + y^m \equiv z^m \pmod{p}$, *ber. Deutsch. Mat. Verein.*, 25, (1916), 114–117.
[34] G. Wolfovitz, Bounds on the number of maximal sum-free sets, *European J. Combin.*, 30, (2009), 1718–1723.

Robert Hancock, Andrew Treglown
School of Mathematics
University of Birmingham
Edgbaston
Birmingham
B15 2TT
UK

E-mail addresses: {rah410,a.c.treglown}@bham.ac.uk

Appendix A

In this appendix we give the proof of Claim 19 and Proposition 25.

A.1. **Proof of Claim 19.** We use induction on k. Recall that $p \geq q \geq 2$. For the base case $k = 6$ we directly calculate (4). First note that

$$\begin{aligned}&\frac{q^2+1}{q^2+q+1} - \frac{q^7-q^6+q^5-q^4+q^3-q^2+q-1+p(q^6+q^4+q^2+1)}{q^7+p(q^6+q^5+q^4+q^3+q^2+q+1)}\\ =&\frac{(q^6+(p-1)q^5+q^4+(p-1)q^3+q^2+1)}{(q^2+q+1)(q^7+p(q^6+q^5+q^4+q^3+q^2+q+1))},\end{aligned}$$

and so we have

$$\begin{aligned}&\Big(\frac{q^2+1}{q^2+q+1} - C_6\Big)\Big(q^6+p(q^5+q^4+q^3+q^2+q+1)\Big)\\ =&\frac{(q^6+(p-1)q^5+q^4+(p-1)q^3+q^2+1)(q^6+p(q^5+q^4+q^3+q^2+q+1)}{(q^2+q+1)(q^7+p(q^6+q^5+q^4+q^3+q^2+q+1))}.\end{aligned}$$

Since $p \geq q \geq 2$ every power of q in the numerator has a coefficient of at least 1 in both expressions, hence the numerator as a single polynomial in q has positive coefficients. Hence we can make our fraction smaller by dropping lower powers of q. We then make further use of $p \geq q \geq 2$ to get the desired result:

$$\begin{aligned}&\frac{(q^6+(p-1)q^5+q^4+(p-1)q^3+q^2+1)(q^6+p(q^5+q^4+q^3+q^2+q+1)}{(q^2+q+1)(q^7+p(q^6+q^5+q^4+q^3+q^2+q+1))}\\ \geq&\frac{q^{12}+(2p-1)q^{11}+(p^2+1)q^{10}+(p^2+2p-1)q^9}{(q^2+q+1)(q^7+p(q^6+q^5+q^4+q^3+q^2+q+1))}\\ \geq&\frac{q^{12}+(2p-1)q^{11}+(p^2+1)q^{10}+(p^2+2p-1)q^9}{(p+1)q^{10}}\\ =&\frac{q^2+(2p-1)q+(p^2+1)}{p+1}+\frac{p^2+2p-1}{(p+1)q} \geq \frac{p^2+4p+3}{p+1}+\frac{p^2+p}{(p+1)q} = p+3+p/q \geq 6 = k.\end{aligned}$$

For the inductive step, assume that (4) holds for k. It suffices to show that $C_k \geq C_{k+1}$ as then the result holds for $k+1$:

$$\begin{aligned}&\Big(\frac{q^2+1}{q^2+q+1} - C_{k+1}\Big)\Big(q^{k+1}+p\sum_{i=0}^{k} q^i\Big) \geq \Big(\frac{q^2+1}{q^2+q+1} - C_k\Big)\Big(q^{k+1}+p\sum_{i=0}^{k} q^i\Big)\\ \geq& \, q\Big(\frac{q^2+1}{q^2+q+1} - C_k\Big)\Big(q^k+p\sum_{i=0}^{k-1} q^i\Big) \geq qk \geq k+1.\end{aligned}$$

For k even, we have $C_k = C_{k+1}$ by definition. For k odd, consider the following calculations:

(i) $D_1 := q^{k+2}\Big(\sum_{i=0}^{k}(-1)(-q)^i\Big) - q^k\Big(\sum_{i=0}^{k+2}(-1)(-q)^i\Big) = -q^{k+1}+q^k,$

(ii) $D_2 := pq^{k+2}\Big(\sum_{i=0}^{(k-1)/2} q^{2i}\Big) - pq^k\Big(\sum_{i=0}^{(k+1)/2} q^{2i}\Big) = -pq^k,$

(iii) $D_3 := p\Big(\sum_{i=0}^{k+1} q^i\Big)\Big(\sum_{i=0}^{k}(-1)(-q)^i\Big) - p\Big(\sum_{i=0}^{k-1} q^i\Big)\Big(\sum_{i=0}^{k+2}(-1)(-q)^i\Big) = pq^{k+1} - pq^k,$

(iv) $D_4 := p^2\Big(\sum_{i=0}^{k+1} q^i\Big)\Big(\sum_{i=0}^{(k-1)/2} q^{2i}\Big) - p^2\Big(\sum_{i=0}^{k-1} q^i\Big)\Big(\sum_{i=0}^{(k+1)/2} q^{2i}\Big) = p^2q^k.$

Using these we have

$$
\begin{aligned}
C_k - C_{k+1} =& \frac{\Big(\sum_{i=0}^{k}(-1)(-q)^i\Big) + p\Big(\sum_{i=0}^{(k-1)/2} q^{2i}\Big)}{q^k + p\Big(\sum_{i=0}^{k-1} q^i\Big)} - \frac{\Big(\sum_{i=0}^{k+2}(-1)(-q)^i\Big) + p\Big(\sum_{i=0}^{(k+1)/2} q^{2i}\Big)}{q^{k+2} + p\Big(\sum_{i=0}^{k+1} q^i\Big)}\\
=& \frac{D_1 + D_2 + D_3 + D_4}{\Big(q^k + p\Big(\sum_{i=0}^{k-1} q^i\Big)\Big)\Big(q^{k+2} + p\Big(\sum_{i=0}^{k+1} q^i\Big)\Big)}\\
=& \frac{(p-1)q^{k+1} + (p^2 - 2p + 1)q^k}{\Big(q^k + p\Big(\sum_{i=0}^{k-1} q^i\Big)\Big)\Big(q^{k+2} + p\Big(\sum_{i=0}^{k+1} q^i\Big)\Big)} \geq 0,
\end{aligned}
$$

where the last inequality follows since $p, q \geq 2$. $\square$

A.2. **Proof of Proposition 25.** Suppose that $\gcd(p, q) = q$. To prove (ii) it suffices to show that

$$\mu_{\mathcal{L}}(n) - \mu^*_{\mathcal{L}}(n) \leq 2\mu_{\mathcal{L}}(\lfloor (n-p)/q \rfloor) + o(n).$$

Since $\mu_{\mathcal{L}}(n) = (p + q - 1)n/(p + q) + o(n)$, $\mu_{\mathcal{L}}(\lfloor (n-p)/q \rfloor) = (p + q - 1)n/q(p + q) + o(n)$ and $\mu^*_{\mathcal{L}}(n) = (q-1)^2 n/q^2 + o(n)$, it is easy to check that this inequality holds.

To prove (iii) in the case where $t := \gcd(p, q) \neq q$, it certainly suffices to show that $2\mu_{\mathcal{L}}(\lfloor (n-p)/q \rfloor) \leq \mu_{\mathcal{L}}(n) - \mu^*_{\mathcal{L}}(n) + o(n)$. In this case we have $\mu^*_{\mathcal{L}}(n) = (q-1)(t-1)/(qt) + o(n)$, and hence it suffices to show that $t \leq (pq + q^2 - p - q)/(p + 2q - 2)$. First note that $t \leq q/2$ and so $q \neq 1$. Now observe that $t(p + 2q - 2) \leq q(p + 2q - 2)/2 = pq/2 + q^2 - q \leq pq + q^2 - p - q$ and so our inequality on t holds as required.

To prove (iii) in the case where $\gcd(p, q) = q$ and $p \geq q^2$, it suffices to show that

$$2^{\frac{(p+q-1)n}{(p+q)q}} \leq 3^{\frac{(p+q-1)n}{3(p+q)} - \frac{(q-1)^2 n}{3q^2}}.$$

Let $a := \log_3(8)$. The inequality can be rearranged to give

$$p((2-a)q - 1) \geq (a-1)(q^2 - q).$$

If $q \geq 10$ then $((2-a)q - 1)$ is positive and so we require $p \geq (a-1)(q^2 - q)/((2-a)q - 1)$. Note that for $q \geq 18$ this always holds since $p \geq q^2 \geq (a-1)(q^2 - q)/((2-a)q - 1)$.

To prove (i), suppose that $\gcd(p, q) = q$. It suffices to show that

$$3^{\frac{(p+q-1)n}{3(p+q)} - \frac{(q-1)^2 n}{3q^2}} \leq 2^{\frac{(p+q-1)n}{(p+q)q}},$$

or rearranging

$$p((2-a)q - 1) \leq (a-1)(q^2 - q).$$

If $q \leq 9$ then $((2-a)q-1)$ is negative and so the inequality holds as the right hand side is non-negative. If $10 \leq q \leq 17$ then the inequality holds if $p \leq (a-1)(q^2-q)/((2-a)q-1)$. $\square$