Matchings in 3-uniform hypergraphs of large minimum vertex degree

Andrew Treglown [1]

School of Mathematics
University of Birmingham
Birmingham, UK

Daniela Kühn & Deryk Osthus [2]

School of Mathematics
University of Birmingham
Birmingham, UK

Abstract

We determine the minimum vertex degree that ensures a perfect matching in a 3-uniform hypergraph. More precisely, suppose that H is a sufficiently large 3-uniform hypergraph whose order n is divisible by 3. If the minimum vertex degree of H is greater than $\binom{n-1}{2}-\binom{2n/3}{2}$, then H contains a perfect matching. This bound is tight and answers a question of Hàn, Person and Schacht. More generally, we determine the minimum vertex degree threshold that ensures that H contains a matching of size $d \leq n/3$.

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[1] Email: `treglowa@maths.bham.ac.uk`
[2] Email: `kuehn,osthus@maths.bham.ac.uk`

1 Introduction

A *perfect matching* in a hypergraph H is a collection of vertex-disjoint edges of H which cover the vertex set $V(H)$ of H. Tutte's theorem gives a characterisation of all those graphs which contain a perfect matching. On the other hand, the decision problem whether an r-uniform hypergraph contains a perfect matching is NP-complete for $r \geq 3$. (See, for example, [5] for complexity results in the area.) It is natural therefore to seek simple sufficient conditions that ensure a perfect matching in an r-uniform hypergraph.

Given an r-uniform hypergraph H and distinct vertices $v_1, \dots, v_\ell \in V(H)$ (where $1 \leq \ell \leq r-1$) we define $d_H(v_1, \dots, v_\ell)$ to be the number of edges containing each of $v_1, \dots, v_\ell$. The *minimum* ℓ*-degree* $\delta_\ell(H)$ of H is the minimum of $d_H(v_1, \dots, v_\ell)$ over all ℓ-element sets of vertices in H. Of these parameters the two most natural to consider are the *minimum vertex degree* $\delta_1(H)$ and the *minimum collective degree* or *minimum codegree* $\delta_{r-1}(H)$. Rödl, Ruciński and Szemerédi [15] determined the minimum codegree that ensures a perfect matching in an r-uniform hypergraph. This improved bounds given in [8,14]. An r-partite version was proved by Aharoni, Georgakopoulos and Sprüssel [1].

Much less is known about minimum vertex degree conditions for perfect matchings in r-uniform hypergraphs H. Hàn, Person and Schacht [4] showed that the threshold in the case when $r = 3$ is $(1 + o(1))\frac{5}{9}\binom{|H|}{2}$. This improved an earlier bound given by Daykin and Häggkvist [3]. In [10] we prove the following result which determines this threshold exactly, thereby answering a question from [4].

Theorem 1.1 *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-uniform hypergraph whose order $n \geq n_0$ is divisible by 3. If*

$$\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2}$$

then H has a perfect matching.

After submitting [10] we learned that Khan [6] has given a proof of Theorem 1.1 using different arguments. The following example shows that the result is best possible: let H^* be the 3-uniform hypergraph whose vertex set is partitioned into two vertex classes V and W of sizes $2n/3 + 1$ and $n/3 - 1$ respectively and whose edge set consists precisely of all those edges with at least one endpoint in W. Then H^* does not have a perfect matching and $\delta_1(H) = \binom{n-1}{2} - \binom{2n/3}{2}$.

The example generalises in the obvious way to r-uniform hypergraphs.

This leads to the following conjecture, which is implicit in several earlier papers (see e.g. [4,9]). Partial results were proved by Hàn, Person and Schacht [4] as well as Markström and Ruciński [11].

Conjecture 1.2 *For each integer $r \geq 3$ there exists an integer $n_0 = n_0(r)$ such that the following holds. Suppose that H is an r-uniform hypergraph whose order $n \geq n_0$ is divisible by r. If $\delta_1(H) > \binom{n-1}{r-1} - \binom{(r-1)n/r}{r-1}$, then H has a perfect matching.*

Very recently Khan [7] proved Conjecture 1.2 in the case when $r = 4$. It is also natural to ask about the minimum (vertex) degree which guarantees a matching of given size d. Bollobás, Daykin and Erdős [2] solved this problem for the case when d is small compared to the order of H. We state the 3-uniform case of their result here. The above hypergraph H^* with W of size $d-1$ shows that the minimum degree bound is best possible.

Theorem 1.3 (Bollobás, Daykin and Erdős [2]) *Let $d \in \mathbb{N}$. If H is a 3-uniform hypergraph on $n > 54(d+1)$ vertices and*

$$\delta_1(H) > \binom{n-1}{2} - \binom{n-d}{2}$$

then H contains a matching of size at least d.

In [10] we extend this result to the entire range of d. Note that Theorem 1.4 generalises Theorem 1.1.

Theorem 1.4 *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-uniform hypergraph on $n \geq n_0$ vertices, that $n/3 \geq d \in \mathbb{N}$ and that*

$$\delta_1(H) > \binom{n-1}{2} - \binom{n-d}{2}.$$

Then H contains a matching of size at least d.

It would be interesting to obtain analogous results (i.e. minimum degree conditions which guarantee a matching of size d) for r-uniform hypergraphs and for r-partite hypergraphs (some bounds are given in [3]).

The situation for ℓ-degrees where $1 < \ell < r-1$ is also still open. Pikhurko [12] showed that if $\ell \geq r/2$ and H is an r-uniform hypergraph whose order n is divisible by r then H has a perfect matching provided that $\delta_\ell(H) \geq (1/2 + o(1))\binom{n}{r-\ell}$. This result is best possible up to the $o(1)$-term. In [4], Hàn, Person and Schacht provided conditions on $\delta_\ell(H)$ that ensure a perfect matching in the case when $\ell < r/2$. These bounds were subsequently lowered by Markström and Ruciński [11]. See [13] for further results concerning perfect matchings in hypergraphs.

2 Outline of the proof of Theorem 1.4

Let $d, n \in \mathbb{N}$ such that $d \leq n/3$. Define $H_{n,d}$ to be the 3-uniform hypergraph on n vertices with vertex set $V(H) = V \cup W$ where $|V| = n - d$, $|W| = d$ and whose edge set consists of those triples with precisely one endpoint in V and those triples with precisely one endpoint in W. Thus $H_{n,d}$ has a matching of size d,

$$\delta_1(H_{n,d}) = \binom{n-1}{2} - \binom{n-d-1}{2}$$

and $H_{n,d}$ is very close to the extremal hypergraph which shows that the degree condition in Theorem 1.4 is best possible.

Given a vertex v of a 3-uniform hypergraph H, we write $N_H(v)$ for the *neighbourhood of* v, i.e. the set of all those (unordered) tuples of vertices which form an edge together with v. Given two disjoint sets $A, B \subseteq V(H)$, we define the *link graph* $L_v(AB)$ *of* v *with respect to* A, B to be the bipartite graph whose vertex classes are A and B and in which $a \in A$ is joined to $b \in B$ if and only if $ab \in N_H(v)$.

Our approach towards Theorem 1.4 follows the so-called *stability approach*: we prove an approximate version of the desired result which states that the minimum degree condition implies that either (i) H contains a d-matching or (ii) H is 'close' to the extremal hypergraph. The latter implies that H is 'close' to the hypergraph $H_{n,d}$. This extremal situation (ii) is then dealt with separately.

As mentioned earlier, an approximate version of Theorem 1.1 was proved in [4]. However, we need to proceed somewhat differently as the argument in [4] fails to guarantee the 'closeness' of H to the extremal hypergraph in case (ii). (But we do use the same general approach and a number of ideas from [4].)

We begin by considering a matching M of maximum size and suppose that $|M| < d$. We then carry out a sequence of steps, where in each step we show that we can either find a larger matching (and thus obtain a contradiction), or show that H is successively 'closer' to $H_{n,d}$. Amongst others, the following fact from [4] is used to achieve this.

Fact 2.1 *Let B be a balanced bipartite graph on 6 vertices.*

- *If $e(B) \geq 7$ then B contains a perfect matching.*
- *If $e(B) = 6$ then either B contains a perfect matching or $B \cong B_{033}$.*

- *If* $e(B) = 5$ *then either* B *contains a perfect matching or* $B \cong B_{023}, B_{113}$.

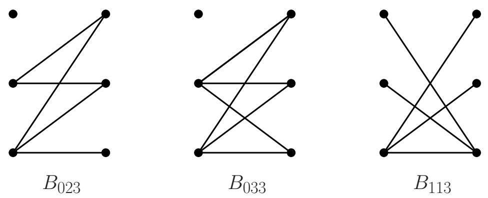

Fig. 1. The graphs B with $e(B) \geq 5$ and no perfect matching

To see how the above fact can be used, suppose for example that x_1, x_2 and x_3 are unmatched vertices, that E and F are edges in M and that the link graphs $L_{x_i}(EF)$ are identical (call this graph B). The minimum degree condition implies that, for almost all unmatched vertices x, we have $e(L_x(EF)) \geq 5$. So let us assume this holds for x_1, x_2, x_3. If B contains a perfect matching, it is easy to see that we can transform M into a (larger) matching which also covers the x_i. If $B = B_{113}$, we can use this to prove that we are 'closer' to $H_{n,d}$. In particular, note that if $H = H_{n,d}$, then in the above example we have $B = B_{113}$. If $B \cong B_{023}, B_{033}$, we need to consider link graphs involving more than 2 edges from M in order to gain further information.

To find a matching which is larger than M, we will often need several vertices whose link graphs with respect to some set of matching edges are identical (as in the above example). We can usually achieve this with a simple application of the pigeonhole principle. But for this to work, we need to be able to assume that the number of vertices not covered by M is fairly large. This may not be true if e.g. we are seeking a perfect matching. To overcome this problem, we apply the 'absorbing method' which was first introduced in [15]. The method (as used in [4]) guarantees the existence of a small matching M^* which can 'absorb' any (very) small set of leftover vertices V' into a matching covering all of $V' \cup V(M^*)$. (The existence of M^* is shown using a probabilistic argument.) So if we are seeking e.g. a perfect matching, it suffices to prove the existence of an almost perfect one outside M^*. In particular, we can always assume that the set of vertices not covered by M is reasonably large.

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