Maximal sum-free subsets in the integers

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Introduction

Motivating question

Set $[n] := \{1, \ldots, n\}$. What arithmetic structures do subsets of $[n]$ contain?

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Roth (1953)

Every subset of $[n]$ of 'linear' size contains an arithmetic progression of length 3.

Introduction

Definition

A set $S \subseteq [n]$ is sum-free if no solutions to $x + y = z$ in S.

Examples

- $\{1, 2, 4\}$ is not sum-free.
- Set of odds is sum-free.
- {n/2+1, n/2+2,. . . , n} is sum-free.

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Deshouillers, Freiman, Sós and Temkin (1999)

If $S \subseteq [n]$ is sum-free then at least one of the following holds:

(i) $|S| \leq 2n/5 + 1$;

(ii) S consists of odds;

(iii) $|S| \leq min(S)$.

Introduction

Examples of sum-free sets

- Set of odds is sum-free.
- $\{n/2+1, n/2+2, \ldots, n\}$ is sum-free.

These two examples show there are at least $2^{n/2}$ sum-free subsets of $[n]$.

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Cameron-Erdős Conjecture (1990)

The number of sum-free subsets of $[n]$ is $O(2^{n/2})$.

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The number of sum-free subsets of $[n]$ is $2^{(1/2+o(1))n}$.

Green; Sapozhenko c. 2003

There are constants c_e and c_o, s.t. the number of sum-free subsets of $[n]$ is

$$(1+o(1))c_e 2^{n/2}, \text{ or } (1+o(1))c_o 2^{n/2}$$

depending on the parity of n.

Introduction

- The previous result doesn't tell us anything about the distribution of the sum-free sets in $[n]$.
- In particular, recall that $2^{n/2}$ sum-free subsets of $[n]$ lie in a single maximal sum-free subset of $[n]$.

Cameron-Erdős Conjecture (1999)

There is an absolute constant $c > 0$, s.t. the number of maximal sum-free subsets of $[n]$ is $O(2^{n/2-cn})$.

Lower bound construction I

There are at least $2^{\lfloor n/4 \rfloor}$ maximal sum-free subsets of $[n]$.

- Suppose n is even. Let S consist of n together with precisely one number from each pair $\{x, n-x\}$ for odd $x < n/2$.
- Notice distinct S lie in distinct maximal sum-free subsets of $[n]$.
- Roughly $2^{n/4}$ choices for S.

Lower bound construction II

There are at least $2^{\lfloor n/4 \rfloor}$ maximal sum-free subsets of $[n]$.

- Suppose that $4|n$ and set $I_1 := \{n/2+1, \dots, 3n/4\}$ and $I_2 := \{3n/4+1, \dots, n\}$.
- First choose the element $n/4$ and a set $S \subseteq I_2$.
- Then for every $x \in I_2 \setminus S$, choose $x - n/4 \in I_1$. No further element in I_2 can be added.
- Notice distinct S lie in distinct maximal sum-free subsets of $[n]$.
- Roughly $2^{n/4}$ choices for S.

Main result

Denote by $f_{\max}(n)$ the number of maximal sum-free subsets in $[n]$.
Recall that $f_{\max}(n) \geq 2^{\lfloor n/4 \rfloor}$.

Cameron-Erdős Conjecture (1999)

$$\exists c > 0, \quad f_{\max}(n) = O(2^{n/2-cn}).$$

Łuczak-Schoen (2001)

$$f_{\max}(n) \leq 2^{n/2-2^{-28}n} \text{ for large } n$$

Wolfovitz (2009)

$$f_{\max}(n) \leq 2^{3n/8+o(n)}.$$

Balogh-Liu-Sharifzadeh-T. (2014)

$$f_{\max}(n) = 2^{n/4+o(n)}.$$

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Balogh-Liu-Sharifzadeh-T. (2015+)

For each $1 \leq i \leq 4$, there is a constant C_i such that, given any $n \equiv i \mod 4$, $[n]$ contains $(C_i + o(1))2^{n/4}$ maximal sum-free sets.

Tools

From additive number theory:

- Container lemma of Green.
- Removal lemma of Green.
- Structure of sum-free sets by Deshouillers, Freiman, Sós and Temkin.

From extremal graph theory: upper bound on the number of maximal independent sets for

- all graphs by Moon and Moser.
- triangle-free graphs by Hujter and Tuza.
- Not too sparse and almost regular graphs.

Sketch of the proof

Balogh-Liu-Sharifzadeh-T. (2014)

$$f_{\max}(n) = 2^{n/4+o(n)}.$$

Container Lemma [Green]

There exists $\mathcal{F} \subseteq 2^{[n]}$, s.t.

(i) $|\mathcal{F}| = 2^{o(n)}$;

(ii) $\forall S \subseteq [n]$ sum-free, $\exists F \in \mathcal{F}$, s.t. $S \subseteq F$;

(iii) $\forall F \in \mathcal{F}$, $|F| \leq (1/2 + o(1))n$ and the number of Schur triples in F is $o(n^2)$.

Sketch of the proof

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By (i) and (ii), it suffices to show that for every container $A \in \mathcal{F}$,

$$f_{\max}(A) \leq 2^{n/4+o(n)}.$$

Deshouillers, Freiman, Sós and Temkin (1999)

If $S \subseteq [n]$ is sum-free then at least one of the following holds:

(i) $|S| \leq 2n/5 + 1$;

(ii) S consists of odds;

(iii) $|S| \leq min(S)$.

Removal lemma [Green]

If A is 'almost' sum-free then $A = B \cup C$ where B is sum-free and $|C| = o(n)$.

Constructing maximal sum-free sets

Removal+Structural lemmas $\Rightarrow$ classify containers $A \in \mathcal{F}$:

- Case 1: small container, $|A| \leq 0.45n$;
- Case 2: 'interval' container, 'most' of A in $[n/2+1, n]$.
- Case 3: 'odd' container, $|A \setminus O| = o(n)$.

Moreover, in all cases $A = B \cup C$ where B is sum-free and $|C| = o(n)$.

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Crucial observation

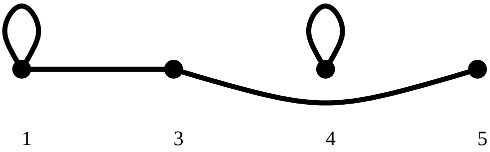
Every maximal sum-free subset in A can be built in two steps:
(1) Choose a sum-free set S in C;
(2) Extend S in B to a maximal one.

maximal sum-free sets $\Rightarrow$ maximal independent sets

Definition

Given $S, B \subseteq [n]$, the link graph of S on B is $L_S[B]$, where $V = B$ and $x \sim y$ iff $\exists z \in S$ s.t. $\{x, y, z\}$ is a Schur triple.

$L_2[1, 3, 4, 5]$



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Lemma

Given $S, B \subseteq [n]$ sum-free and $I \subseteq B$, if $S \cup I$ is a maximal sum-free subset of $[n]$, then I is a maximal independent set in $L_S[B]$.

Case 1: small container, $|A| \leq 0.45n$.

Recall $A = B \cup C$, B sum-free, $|C| = o(n)$.

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Every maximal sum-free subset in A can be built in two steps:
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Crucial observation

Every maximal sum-free subset in A can be built in two steps:
(1) Choose a sum-free set S in C;
(2) Extend S in B to a maximal one.

- Fix a sum-free $S \subseteq C$ (at most $2^{|C|} = 2^{o(n)}$ choices).
- Consider link graph $L_S[B]$.
- Moon-Moser: $\forall$ graphs G, $MIS(G) \leq 3^{|G|/3}$.
- So # extensions in (2) is exactly $MIS(L_S[B])$,

$$MIS(L_S[B]) \leq 3^{|B|/3} \leq 3^{0.45n/3} \ll 2^{0.249n}.$$

- In total, A contains at most $2^{o(n)} \times 2^{0.249n} \ll 2^{n/4}$ maximal sum-free sets.

Cases 2 and 3.

- Now container A could be bigger than $0.45n$.
- This means crude Moon-Moser bound doesn't give accurate bound on $f_{\max}(A)$.
- Instead we obtain more structural information about the link graphs.

Balogh-Liu-Sharifzadeh-T. (2015+)

For each $1 \leq i \leq 4$, there is a constant C_i such that, given any $n \equiv i \mod 4$, $[n]$ contains $(C_i + o(1))2^{n/4}$ maximal sum-free sets.

(i) Count by hand the maximal sum-free sets S that are 'extremal':

- S that contain precisely one even number.
- S where $\min(S) \approx n/4$, $\min_2(S) \approx n/2$.

(ii) Count remaining maximal sum-free sets using the container method.

Open problem

Given an abelian group G let $\mu(G)$ denote the size of the largest sum-free subset of G.

Green–Ruzsa (2005)

There are $2^{\mu(G)+o(|G|)}$ sum-free subsets of G.

Conjecture [Balogh-Liu-Sharifzadeh-T.]

There are at most $2^{\mu(G)/2+o(|G|)}$ maximal sum-free subsets of G.

- Easy to prove $3^{\mu(G)/3+o(|G|)}$ as an upper bound.