

Degree sequences forcing Hamilton cycles in directed graphs

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Abstract

We prove the following approximate version of Pósa's theorem for directed graphs: every directed graph on n vertices whose in- and outdegree sequences satisfy $d_i^- \geq i + o(n)$ and $d_i^+ \geq i + o(n)$ for all $i \leq n/2$ has a Hamilton cycle. In fact, we prove that such digraphs are pancyclic (i.e. contain cycles of lengths $2, \dots, n$). We also prove an approximate version of Chvátal's theorem for digraphs. This asymptotically confirms conjectures of Nash-Williams from 1968 and 1975.

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1 Introduction

Since it is unlikely that there is a characterization of all those graphs which contain a Hamilton cycle it is natural to ask for sufficient conditions which ensure Hamiltonicity. One of the most general of these is Chvátal's theorem [3] that characterizes all those degree sequences which ensure the existence of a Hamilton cycle in a graph: Suppose that the degrees of the graph are $d_1 \leq \dots \leq d_n$. If $n \geq 3$ and $d_i \geq i + 1$ or $d_{n-i} \geq n - i$ for all $i < n/2$ then G is Hamiltonian. This condition on the degree sequence is best possible in the sense that for any degree sequence violating this condition there is a corresponding graph with no Hamilton cycle. More precisely, if $d_1 \leq \dots \leq d_n$ is a graphical degree sequence (i.e. there exists a graph with this degree sequence) then there exists a non-Hamiltonian graph G whose degree sequence $d'_1 \leq \dots \leq d'_n$ is such that $d'_i \geq d_i$ for all $1 \leq i \leq n$.

A special case of Chvátal's theorem is Dirac's theorem, which states that every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ has a Hamilton cycle. An analogue of Dirac's theorem for digraphs was proved by Ghouila-Houri [4]. (The digraphs we consider do not have loops and we allow at most one edge in each direction between any pair of vertices.) Nash-Williams [9] raised the question of a digraph analogue of Chvátal's theorem quite soon after the latter was proved.

For a digraph G it is natural to consider both its outdegree sequence $d_1^+ \leq \dots \leq d_n^+$ and its indegree sequence $d_1^- \leq \dots \leq d_n^-$. Note that the terms d_i^+ and d_i^- do not necessarily correspond to the degree of the same vertex of G .

Conjecture 1.1 (Nash-Williams [9]) *Suppose that G is a strongly connected digraph on $n \geq 3$ vertices such that for all $i < n/2$*

- (i) $d_i^+ \geq i + 1$ or $d_{n-i}^- \geq n - i$,
- (ii) $d_i^- \geq i + 1$ or $d_{n-i}^+ \geq n - i$.

Then G contains a Hamilton cycle.

No progress has been made on this conjecture so far. It is even an open problem whether the conditions imply the existence of a cycle through any pair of given vertices (see [1]). At first sight one might also try to replace the degree condition in Chvátal's theorem by

- $d_i^+ \geq i + 1$ or $d_{n-i}^+ \geq n - i$,
- $d_i^- \geq i + 1$ or $d_{n-i}^- \geq n - i$.

However, Bermond and Thomassen [1] observed that the latter conditions do

not guarantee Hamiltonicity. Indeed, consider the digraph obtained from the complete digraph K on $n - 2 \geq 4$ vertices by adding two new vertices v and w which both send an edge to every vertex in K and receive an edge from one fixed vertex $u \in K$.

It is not hard to see that one cannot omit the condition that G is strongly connected in Conjecture 1.1. Section 2 contains an example which shows that degree condition in Conjecture 1.1 would be best possible.

In [6] we prove the following approximate version of Conjecture 1.1 for large digraphs.

Theorem 1.2 *For every $\eta > 0$ there exists an integer $n_0 = n_0(\eta)$ such that the following holds. Suppose G is a digraph on $n \geq n_0$ vertices such that for all $i < n/2$*

- $d_i^+ \geq i + \eta n$ or $d_{n-i-\eta n}^- \geq n - i$,
- $d_i^- \geq i + \eta n$ or $d_{n-i-\eta n}^+ \geq n - i$.

Then G contains a Hamilton cycle.

In fact, we prove that such digraphs are even pancyclic, i.e. for every $\ell = 2, \dots, n$ they contain a cycle of length ℓ . Instead of proving Theorem 1.2 directly, we show the existence of a Hamilton cycle in a digraph satisfying a certain expansion property. Our proof of the latter result uses the Regularity lemma for digraphs and relies on a result from joint work [5] of the first two authors with Keevash on an analogue of Dirac's theorem for oriented graphs. An algorithmic version of Theorem 1.2 was subsequently proved in [2].

The following weakening of Conjecture 1.1 was posed earlier by Nash-Williams [7,8]. It would yield a digraph analogue of Pósa's theorem which states that a graph G on $n \geq 3$ vertices has a Hamilton cycle if its degree sequence d_1, \dots, d_n satisfies $d_i \geq i + 1$ for all $i < (n - 1)/2$ and if additionally $d_{\lceil n/2 \rceil} \geq \lceil n/2 \rceil$ when n is odd [10]. Note this is much stronger than Dirac's theorem but is a special case of Chvátal's theorem.

Conjecture 1.3 (Nash-Williams [7,8]) *Let G be a digraph on $n \geq 3$ vertices such that $d_i^+, d_i^- \geq i + 1$ for all $i < (n - 1)/2$ and such that additionally $d_{\lceil n/2 \rceil}^+, d_{\lceil n/2 \rceil}^- \geq \lceil n/2 \rceil$ when n is odd. Then G contains a Hamilton cycle.*

Again, the degree condition would be best possible. The assumption of strong connectivity is not necessary in Conjecture 1.3, as it follows from the degree conditions. The following approximate version of Conjecture 1.3 is an immediate consequence of Theorem 1.2.

Corollary 1.4 *For every $\eta > 0$ there exists an integer $n_0 = n_0(\eta)$ such that*

every digraph G on $n \geq n_0$ vertices with $d_i^+, d_i^- \geq i + \eta n$ for all $i < n/2$ contains a Hamilton cycle.

It is a natural question to ask whether the ‘error terms’ in Theorem 1.2 and Corollary 1.4 can be eliminated using an ‘extremal case’ or ‘stability’ analysis. However, this seems quite difficult as there are many different types of digraphs which come close to violating the conditions in Conjectures 1.1 and 1.3 (this is different e.g. to the situation in [5]).

2 Extremal example for Conjecture 1.1

The purpose of this section is to construct a non-Hamiltonian strongly connected digraph G on n vertices, which satisfies the degree conditions in Conjecture 1.1 except (i) for $i = k$. (The example works for all $n \geq 3$ and $k < n/2$ unless n is odd and $k = \lfloor n/2 \rfloor$. In this case both (i) and (ii) fail for $i = k$.)

Suppose $n \geq 5$ and $1 \leq k < n/2$. Let K and K' be complete digraphs on $k - 1$ and $n - k - 2$ vertices respectively. Let G be the digraph on n vertices obtained from the disjoint union of K and K' as follows. Add all possible edges from K' to K (but no edges from K to K') and add new vertices u and v to the digraph such that there are all possible edges from K' to u and v and all possible edges from u and v to K . Finally, add a vertex w that sends and receives edges from all other vertices of G (see Figure 1). Thus G is

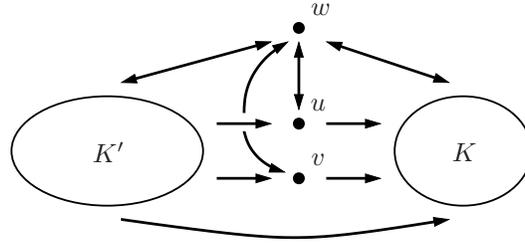


Fig. 1. An extremal example for Conjecture 1.1

strongly connected, not Hamiltonian and has outdegree sequence

$$\underbrace{k - 1, \dots, k - 1}_{k-1 \text{ times}}, k, k, \underbrace{n - 1, \dots, n - 1}_{n-k-1 \text{ times}}$$

and indegree sequence

$$\underbrace{n - k - 2, \dots, n - k - 2}_{n-k-2 \text{ times}}, n - k - 1, n - k - 1, \underbrace{n - 1, \dots, n - 1}_{k \text{ times}}.$$

Suppose that either n is even or, if n is odd, we have that $k < \lfloor n/2 \rfloor$. One can check that G then satisfies the conditions in Conjecture 1.1 except that $d_k^+ = k$ and $d_{n-k}^- = n - k - 1$. (When checking the conditions, it is convenient to note that our assumptions on k and n imply $n - k - 1 \geq \lceil n/2 \rceil$. Hence there are at least $\lceil n/2 \rceil$ vertices of outdegree $n - 1$ and so (ii) holds for all $i < n/2$.) If n is odd and $k = \lfloor n/2 \rfloor$ then conditions (i) and (ii) both fail for $i = k$. We do not know whether a similar construction as above also exists for this case.

References

- [1] J.C. Bermond and C. Thomassen, *Cycles in digraphs—a survey*, J. Graph Theory **5** (1981), 1–43.
- [2] D. Christofides, P. Keevash, D. Kühn and D. Osthus, *Finding Hamilton cycles in robustly expanding digraphs*, submitted.
- [3] V. Chvátal, *On Hamilton’s ideals*, J. Combin. Theory B **12** (1972), 163–168.
- [4] A. Ghouila-Houri, *Une condition suffisante d’existence d’un circuit hamiltonien*, C.R. Acad. Sci. Paris **25** (1960), 495–497.
- [5] P. Keevash, D. Kühn and D. Osthus, *An exact minimum degree condition for Hamilton cycles in oriented graphs*, J. London Math. Soc. **79** (2009), 144–166.
- [6] D. Kühn, D. Osthus and A. Treglown, *Hamiltonian degree sequences in digraphs*, submitted.
- [7] C.St.J.A. Nash-Williams, *Problem 47*, Proceedings of Colloq. Tihany 1966, Academic Press 1968, p. 366.
- [8] C.St.J.A. Nash-Williams, *Hamilton circuits in graphs and digraphs*, *The many facets of graph theory*, Springer Verlag Lecture Notes 110, Springer Verlag 1968, 237–243.
- [9] C.St.J.A. Nash-Williams, *Hamiltonian circuits*, Studies in Math. **12** (1975), 301–360.
- [10] L. Pósa, *A theorem concerning Hamiltonian lines*, Magyar Tud. Akad. Mat. Fiz. Oszt. Kozl. **7** (1962), 225–226.