A DEGREE SEQUENCE HAJNAL–SZEMERÉDI THEOREM

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ABSTRACT. We say that a graph G has a perfect H-packing if there exists a set of vertex-disjoint copies of H which cover all the vertices in G. The seminal Hajnal–Szemerédi theorem [12] characterises the minimum degree that ensures a graph G contains a perfect K_r-packing. Balogh, Kostochka and Treglown [4] proposed a degree sequence version of the Hajnal–Szemerédi theorem which, if true, gives a strengthening of the Hajnal–Szemerédi theorem. In this paper we prove this conjecture asymptotically. Another fundamental result in the area is the Alon–Yuster theorem [3] which gives a minimum degree condition that ensures a graph contains a perfect H-packing for an *arbitrary* graph H. We give a wide-reaching generalisation of this result by answering another conjecture of Balogh, Kostochka and Treglown [4] on the degree sequence of a graph that forces a perfect H-packing. We also prove a degree sequence result concerning perfect transitive tournament packings in directed graphs. The proofs blend together the regularity and absorbing methods.

1. INTRODUCTION

1.1. **Perfect clique packings.** A *perfect matching* in a graph G is a collection of vertex-disjoint edges that together cover all the vertices of G. A theorem of Tutte [29] characterises those graphs that contain a perfect matching. Moreover, the decision problem of whether a graph contains a perfect matching is polynomial time solvable [10].

A natural generalisation of the notion of a perfect matching is a so-called perfect packing: Given two graphs H and G, an H-*packing* in G is a collection of vertex-disjoint copies of H in G. An H-packing is called *perfect* if it covers all the vertices of G. Perfect H-packings are also referred to as H-*factors* or *perfect H-tilings*. Hell and Kirkpatrick [13] showed that the decision problem of whether a graph G has a perfect H-packing is NP-complete precisely when H has a component consisting of at least 3 vertices. Therefore, for such graphs H, it is unlikely that there is a complete characterisation of those graphs containing a perfect H-packing.

The following classical result of Hajnal and Szemerédi [12] characterises the minimum degree that ensures a graph contains a perfect K_r-packing.

Theorem 1.1 (Hajnal and Szemerédi [12])**.** *Every graph G whose order n is divisible by r and whose minimum degree satisfies $\delta(G) \geq (1 - 1/r)n$ contains a perfect K_r-packing.*

It is easy to see that the minimum degree condition here cannot be lowered. Earlier, Corrádi and Hajnal [7] proved Theorem 1.1 in the case when $r = 3$. More recently, Kierstead and Kostochka [16] gave a short proof of the Hajnal–Szemerédi theorem.

Over the last twenty years the Hajnal–Szemerédi theorem has been generalised in a number of directions. For example, Kierstead and Kostochka [15] proved an *Ore-type* analogue of the Hajnal–Szemerédi theorem: If G is a graph whose order n is divisible by r, then G contains a perfect K_r-packing provided that $d(x) + d(y) \geq 2(1 - 1/r)n - 1$ for all non-adjacent $x \neq y \in V(G)$. Together with Balogh and Kostochka [4], the author characterised the edge density required to ensure a graph of given minimum degree contains a perfect K_r-packing. In [4] the following conjecture was also proposed.

Conjecture 1.2 (Balogh, Kostochka and Treglown [4]). *Let $n, r \in \mathbb{N}$ such that r divides n. Suppose that G is a graph on n vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that:*

    (α) $d_i \geq (r-2)n/r + i$ for all $i < n/r$;
    (β) $d_{n/r+1} \geq (r-1)n/r$.

Then G contains a perfect K_r-packing.

Note that Conjecture 1.2, if true, is much stronger than the Hajnal–Szemerédi theorem since the degree condition allows for n/r vertices to have degree less than $(r-1)n/r$. Moreover, the degree sequence condition in Conjecture 1.2 is best-possible. Indeed, examples in Section 4 in [4] show that one cannot replace (α) with $d_i \geq (r-2)n/r + i - 1$ for a *single* value of $i < n/r$ and (β) cannot be replaced with $d_{n/r+1} \geq (r-1)n/r - 1$. Chvátal's theorem on Hamilton cycles [6] implies Conjecture 1.2 in the case when $r = 2$. In [4] it was shown that Conjecture 1.2 is true under the additional assumption that no vertex $x \in V(G)$ of degree less than $(r-1)n/r$ lies in a copy of K_{r+1}.

In this paper we prove the following asymptotic version of Conjecture 1.2.

Theorem 1.3. *Let $\gamma > 0$ and $r \in \mathbb{N}$. Then there exists an integer $n_0 = n_0(\gamma, r)$ such that the following holds. Suppose that G is a graph on $n \geq n_0$ vertices where r divides n and with degree sequence $d_1 \leq \cdots \leq d_n$ such that:*

    • *$d_i \geq (r-2)n/r + i + \gamma n$ for all $i < n/r$.*

Then G contains a perfect K_r-packing.

Keevash and Knox [14] also have announced a proof of Theorem 1.3 in the case when $r = 3$. Conjecture 1.2 considers a 'Pósa-type' degree sequence condition: Pósa's theorem [25] states that a graph G on $n \geq 3$ vertices has a Hamilton cycle if its degree sequence $d_1 \leq \cdots \leq d_n$ satisfies $d_i \geq i+1$ for all $i < (n-1)/2$ and if additionally $d_{\lceil n/2 \rceil} \geq \lceil n/2 \rceil$ when n is odd. The aforementioned result of Chvátal [6] generalises Pósa's theorem by characterising those degree sequences which force Hamiltonicity. It would be interesting to establish an analogous result for perfect K_r-packings. A question in this direction is given in the arXiv version of [4].

1.2. **General graph packings.** The following result of Alon and Yuster [3] initiated the topic of 'generalising' the Hajnal–Szemerédi theorem to perfect H-packings for an *arbitrary* graph H.

Theorem 1.4 (Alon and Yuster [3]). *Suppose that $\gamma > 0$ and H is a graph with $\chi(H) = r$. Then there exists an integer $n_0 = n_0(\gamma, H)$ such that the following holds. If G is a graph whose order $n \geq n_0$ is divisible by $|H|$ and*

$$\delta(G) \geq (1 - 1/r + \gamma)n$$

then G contains a perfect H-packing.

Komlós, Sárközy and Szemerédi [19] proved that the term γn in Theorem 1.4 can be replaced with a constant depending only on H. For some graphs H the minimum degree condition in Theorem 1.4 is best-possible up to the term γn. However, for other graphs H the minimum degree condition here can be substantially reduced. Indeed, Kühn and Osthus [22] characterised, up to an additive constant, the minimum degree which ensures a graph G contains a perfect H-packing for an arbitrary graph H. This characterisation involves the so-called critical chromatic number of H (see [22] for more details). Kühn, Osthus and Treglown [23] characterised, asymptotically, the Ore-type degree condition that guarantees a graph G contains a perfect H-packing for an arbitrary graph H.

In this paper we prove the following result, thereby answering another conjecture of Balogh, Kostochka and Treglown [4].

Theorem 1.5. *Suppose that $\gamma > 0$ and H is a graph with $\chi(H) = r$. Then there exists an integer $n_0 = n_0(\gamma, H)$ such that the following holds. If G is a graph whose order $n \geq n_0$ is divisible by $|H|$, and whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

- *$d_i \geq (r-2)n/r + i + \gamma n$ for all $i < n/r$,*

then G contains a perfect H-packing.

Note that Theorem 1.5 is a strong generalisation of the Alon–Yuster theorem. Further, Theorem 1.3 is a special case of Theorem 1.5 (so we do not prove the former directly). Previously, the author and Knox [17] proved Theorem 1.5 in the case when $r = 2$. (In fact, they proved a much more general result concerning embedding spanning bipartite graphs of small bandwidth.)

At first sight one may ask whether the term γn in Theorem 1.5 can be replaced by a constant dependent on H. However, in Section 3 we show that for many graphs H this is not the case. (In fact, for many graphs H one cannot replace γn with $o(\sqrt{n})$.)

The proof of Theorem 1.5 splits into two main tasks: We use the regularity method to find an 'almost' perfect H-packing in G and find a so-called 'absorbing set' that can be used to cover the remaining vertices with disjoint copies of H (see Section 2.2 for the precise definition of such a set).

1.3. **Perfect packings in directed graphs.**
Recently, there has been a focus on generalising the Hajnal–Szemerédi theorem to the directed graph setting (see for example [5, 8, 9, 28]). Here we give a degree sequence condition that forces a digraph to contain a perfect packing of transitive tournaments of a given size. Before we can state this result we require some notation and definitions.

Given a digraph G on n vertices and $x \in V(G)$, let $d_G^+(x)$ and $d_G^-(x)$ denote the out- and indegree of x in G. We call $d_G^*(x) := \max\{d_G^+(x), d_G^-(x)\}$ the *dominant degree* of x in G. The *dominant degree sequence of G* is the sequence $d_1^* \leq \cdots \leq d_n^*$ of the dominant degrees of the vertices of G ordered from smallest to largest. Let T_r denote the transitive tournament on r vertices.

Theorem 1.6. *Let $r \in \mathbb{N}$ and $\gamma > 0$. Then there exists an integer $n_0 = n_0(\gamma, r)$ such that the following holds. Let G be a digraph on $n \geq n_0$ vertices where r divides n. If G has dominant degree sequence $d_1^* \leq d_2^* \leq \cdots \leq d_n^*$ where*

$$d_i^* \geq (r-2)n/r + i + \gamma n \quad \text{for all} \ \ i < n/r$$

then G contains a perfect T_r-packing.

Notice that Theorem 1.6 implies Theorem 1.3. (In particular, since Theorem 1.3 is best-possible up to the term γn, so is Theorem 1.6.) Further, Theorem 1.6 complements a number of existing results on perfect T_r-packings: Let n be divisible by r. Czygrinow, DeBiasio, Kierstead and Molla [8] showed that a digraph on n vertices contains a perfect T_r-packing provided that G has minimum outdegree at least $(r-1)n/r$ or minimum total degree at least $2(r-1)n/r - 1$. Amongst other results, in [28] it was shown that if G has minimum dominant degree at least $(1 - 1/r + o(1))n$ then G contains a perfect T_r-packing. (So Theorem 1.6 generalises this result.) It would be interesting to establish whether the term γn in Theorem 1.6 can be replaced by an additive constant.

1.4. **Organisation of the paper.**
The paper is organised as follows. In the next section we introduce some notation and definitions. In Section 3 we give an extremal example which shows that, in general, one cannot replace the term γn in Theorem 1.5 with $o(\sqrt{n})$. In Section 4 we state the main auxiliary results of the paper and derive Theorems 1.5 and 1.6 from them. Szemerédi's Regularity lemma and related tools are introduced in Section 5. We then prove an 'almost' perfect packing result for Theorem 1.6 in Section 6. In Section 7 we give an overview of our absorbing method and then apply it in Section 8 to prove absorbing results for Theorems 1.5 and 1.6.

2. Notation and preliminaries

2.1. **Definitions and notation.** Let G be a (di)graph. We write $V(G)$ for the vertex set of G, $E(G)$ for the edge set of G and define $|G| := |V(G)|$ and $e(G) := |E(G)|$. Given a subset $X \subseteq V(G)$, we write $G[X]$ for the sub(di)graph of G induced by X. We write $G \setminus X$ for the sub(di)graph of G induced by $V(G) \setminus X$. Given a set $X \subseteq V(G)$ and a (di)graph H on $|X|$ vertices we say that X *spans a copy of H in G* if $G[X]$ contains a copy of H. In particular, this does not necessarily mean that X induces a copy of H in G.

Suppose that G is a graph. The degree of a vertex $x \in V(G)$ is denoted by $d_G(x)$ and its neighbourhood by $N_G(x)$. Given a vertex $x \in V(G)$ and a set $Y \subseteq V(G)$ we write $d_G(x, Y)$ to denote the number of edges xy where $y \in Y$. Given disjoint vertex classes $A, B \subseteq V(G)$, $e_G(A, B)$ denotes the number of edges in G with one endpoint in A and the other in B.

Given two vertices x and y of a digraph G, we write xy for the edge directed from x to y. We denote by $N_G^+(x)$ and $N_G^-(x)$ the out- and the inneighbourhood of x and by $d_G^+(x)$ and $d_G^-(x)$ its out- and indegree. We will write $N^+(x)$ for example, if this is unambiguous. Given a vertex $x \in V(G)$ and a set $Y \subseteq V(G)$ we write $d_G^+(x, Y)$ to denote the number of edges in G with startpoint x and endpoint in Y. We define $d_G^-(x, Y)$ analogously. Recall that $d_G^*(x) := \max\{d_G^+(x), d_G^-(x)\}$ is the *dominant degree* of x in G. If $d_G^*(x) = d_G^+(x) = d_G^-(x)$ then we implicitly consider $d_G^*(x)$ to count the number of edges sent out from x in G. In particular, if $d_G^*(x) = d_G^+(x)$ then we define $d_G^*(x, Y) := d_G^+(x, Y)$. Otherwise, we set $d_G^*(x, Y) := d_G^-(x, Y)$.

Let $\mathcal{H}$ be a collection of (di)graphs and G a (di)graph. An $\mathcal{H}$-*packing* in G is a collection of vertex-disjoint copies of elements from $\mathcal{H}$ in G. Given an $\mathcal{H}$-packing $\mathcal{M}$, we write $V(\mathcal{M})$ for the set of vertices covered by $\mathcal{M}$.

Recall that T_r denotes the transitive tournament of r vertices. Given $1 \le i \le r$, we say a vertex $x \in V(T_r)$ is the *ith vertex of T_r* if x has indegree $i - 1$ and outdegree $r - i$ in T_r.

Given a (di)graph G we let $G(t)$ denote the (di)graph obtain from G by replacing each vertex $x \in V(G)$ with a set V_x of t vertices so that, for all $x, y \in V(G)$:

- If $xy \in E(G)$ then there are all possible edges in $G(t)$ with startpoint in V_x and endpoint in V_y;
- If $xy \notin E(G)$ then there are no edges in $G(t)$ with startpoint in V_x and endpoint in V_y.

We set $T_r^t := T_r(t)$ and $K_r^t := K_r(t)$.

Throughout the paper, we write $0 < \alpha \ll \beta \ll \gamma$ to mean that we can choose the constants α, β, γ from right to left. More precisely, there are increasing functions f and g such that, given γ, whenever we choose some $\beta \le f(\gamma)$ and $\alpha \le g(\beta)$, all calculations needed in our proof are valid. Hierarchies of other lengths are defined in the obvious way.

2.2. **Absorbing sets.** Let H be a (di)graph. Given a (di)graph G, a set $S \subseteq V(G)$ is called an H-*absorbing set for $Q \subseteq V(G)$*, if both $G[S]$ and $G[S \cup Q]$ contain perfect H-packings. In this case we say that Q is H-*absorbed by S*. Sometimes we will simply refer to a set $S \subseteq V(G)$ as an H-absorbing set if *there exists* a set $Q \subseteq V(G)$ that is H-absorbed by S.

3. An extremal example for Theorem 1.5

Let $K_{t_1, \ldots, t_r}$ denote the complete r-partite graph with vertex classes of size $t_1, \ldots, t_r$. The following result shows that for $H = K_{t_1, \ldots, t_r}$ with $t_i \ge 2$ (for all $1 \le i \le r$) we cannot replace the term γn in Theorem 1.5 with $\sqrt{n}/3r^2$. The construction given is a generalised version of an extremal graph from the arXiv version of [4].

Proposition 3.1. *Let $r \ge 3$ and $H := K_{t_1, \ldots, t_r}$ with $t_i \ge 2$ (for all $1 \le i \le r$). Let $n \in \mathbb{N}$ be sufficiently large so that $\sqrt{n}$ is an integer that is divisible by $6r^2|H|$. Set $C := \sqrt{n}/3r^2$. Then there*
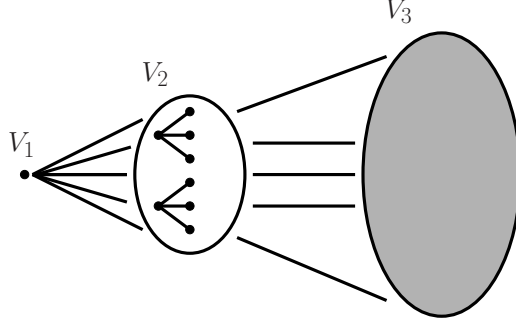
FIGURE 1. An example of a graph G from Proposition 3.1 in the case when $r = 3$.

exists a graph G on n vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies

$$d_i \geq (r-2)n/r + i + C \quad \text{for all } 1 \leq i \leq \frac{n}{r}$$

but such that G does not contain a perfect H-packing.

Proof. Let G denote the graph on n vertices consisting of r vertex classes $V_1, \ldots, V_r$ with $|V_1| = 1$, $|V_2| = n/r + 1 + Cr$, $|V_3| = 2n/r - 2 - 3C$ and $|V_i| = n/r - C$ if $4 \leq i \leq r$ and which contains the following edges:

- All possible edges with an endpoint in V_3 and the other endpoint in $V(G) \backslash V_1$. (In particular, $G[V_3]$ is complete.);
- All edges with an endpoint in V_2 and the other endpoint in $V(G) \setminus V_2$;
- All edges with an endpoint in V_i and the other endpoint in $V(G) \setminus V_i$ for $4 \leq i \leq r$;
- There are $\sqrt{n}/2$ vertex-disjoint stars in V_2, each of size $\lfloor 2|V_2|/\sqrt{n} \rfloor$, $\lceil 2|V_2|/\sqrt{n} \rceil$, which cover all of V_2 (see Figure 1).

In particular, note that the vertex $v \in V_1$ sends all possible edges to $V(G) \setminus V_3$ but no edges to V_3.

Let $d_1 \leq \cdots \leq d_n$ denote the degree sequence of G. Notice that every vertex in V_i for $3 \leq i \leq r-3$ has degree at least $(r-1)n/r + C$. Note that $\lfloor 2|V_2|/\sqrt{n} \rfloor \geq 2\sqrt{n}/r = 6Cr$. Thus, there are $\sqrt{n}/2$ vertices in V_2 of degree at least

$$(r-1)n/r - 1 - Cr + (6Cr - 1) \geq (r-1)n/r + C.$$

The remaining $n/r + 1 + Cr - \sqrt{n}/2 \leq n/r - \sqrt{n}/3 - 1$ vertices in V_2 have degree at least

$$(r-1)n/r - Cr \geq (r-1)n/r - \sqrt{n}/3 + C.$$

Since $d_G(v) \geq (r-2)n/r + 1 + C$ for the vertex $v \in V_1$ we have that $d_i \geq (r-2)n/r + i + C$ for all $1 \leq i \leq n/r$.

Suppose that $v \in V_1$ lies in a copy H' of H in G. Then by construction of G, two of the vertex classes U_1, U_2 of H' must lie entirely in V_2. By definition of H, $H'[U_1 \cup U_2]$ contains a path of length 3. However, $G[V_2]$ does not contain a path of length 3, a contradiction. Thus, v does not lie in a copy of H and so G does not contain a perfect H-packing. $\square$

For many graphs H, Kühn and Osthus [22] showed that a minimum degree condition substantially smaller than that in the Alon–Yuster theorem ensures a graph G contains a perfect H-packing. For example, if $H = K_{t,t-1,\ldots,t-1}$ for any $t \geq 3$ then they showed the so-called critical chromatic number of H, not $\chi(H)$, is the parameter that governs the minimum degree threshold that forces a perfect H-packing. Thus, it is interesting that for such graphs H the degree sequence condition in Theorem 1.5 is 'best-possible' up to the term γn. This raises the following natural question.

Question 3.2. *Are there graphs H with $\chi(H) = r$ for which the degree sequence condition in Theorem 1.5 is 'far' from tight?*

4. Deriving Theorems 1.5 and 1.6

4.1. **The auxiliary results for Theorem 1.6.** The proof of Theorem 1.6 splits into two main tasks. Firstly, we construct a 'small' T_r-absorbing set $M \subseteq V(G)$ with the property that both $G[M]$ and $G[M \cup Q]$ contain perfect T_r-packings for *any* 'very small' set $Q \subseteq V(G)$ where $|Q| \in r\mathbb{N}$. Our second aim is to find an 'almost' perfect T_r-packing in $G' := G \setminus M$. Together this ensures that G contains a perfect T_r-packing. Indeed, suppose that G' contains a T_r-packing $\mathcal{M}_1$ covering all but a very small set of vertices Q. Then by definition of M, $G[M \cup Q]$ contains a perfect T_r-packing $\mathcal{M}_2$. Thus, $\mathcal{M}_1 \cup \mathcal{M}_2$ is a perfect T_r-packing in G, as desired.

The following result guarantees our desired absorbing set.

Theorem 4.1. *Let $0 < 1/n \ll \nu \ll \eta \ll 1/r$ where $n, r \in \mathbb{N}$ and $r \geq 2$. Suppose that G is a digraph on n vertices with dominant degree sequence $d_1^* \leq d_2^* \leq \cdots \leq d_n^*$ where*

$$d_i^* \geq (r-2)n/r + i + \eta n \quad \text{for all} \quad i < n/r.$$

Then $V(G)$ contains a set M so that $|M| \leq \nu n$ and M is a T_r-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in r\mathbb{N}$ and $|W| \leq \nu^3 n$.

We prove Theorem 4.1 in Section 8.1. In order to obtain our desired absorbing set M, we first show that G contains many 'connecting structures' of a certain type. Indeed, we show that there is a constant $t \in \mathbb{N}$ such that given any $x, y \in V(G)$ there are 'many' t-sets $X \subseteq V(G)$ so that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect T_r-packings. A result of Lo and Markström [24] then immediately implies the existence of the absorbing set M. A detailed exposition of our absorbing method is given in Section 7.

The next result is used to obtain the almost perfect T_r-packing in G'.

Theorem 4.2. *Let $n, r, h \in \mathbb{N}$ where $r \geq 2$ and $\eta > 0$ such that $0 < 1/n \ll \eta \ll 1/r, 1/h$. Let G be a digraph on n vertices with dominant degree sequence $d_1^* \leq d_2^* \leq \cdots \leq d_n^*$ where*

$$d_i^* \geq (r-2)n/r + i + \eta n \quad \text{for all} \quad i < n/r.$$

Then G contains a T_r^h-packing covering all but at most ηn vertices.

Note that Theorem 4.2 concerns T_r^h-packings rather than simply T_r-packings. (Recall that $T_r^h := T_r(h)$.) In Section 6.3 we use the regularity method to prove Theorem 4.2. We give an outline of the proof in Section 6.1.

We now deduce Theorem 1.6 from Theorems 4.1 and 4.2.

Proof of Theorem 1.6. Define constants η', ν, η and $n_0 \in \mathbb{N}$ such that $0 < 1/n_0 \ll \eta' \ll \nu \ll \eta \ll \gamma, 1/r$. Let G be a digraph on $n \geq n_0$ vertices as in the statement of the theorem. Apply Theorem 4.1 to G with parameters ν, η to obtain a set M so that $|M| \leq \nu n$ and M is a T_r-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in r\mathbb{N}$ and $|W| \leq \nu^3 n$.

Set $G' := G \setminus M$ and $n' := |G'|$. It is easy to see that G' has dominant degree sequence $d_{G',1}^* \leq d_{G',2}^* \leq \cdots \leq d_{G',n'}^*$ where

$$d_{G',i}^* \geq (r-2)n'/r + i + \eta' n' \quad \text{for all} \quad i < n'/r.$$

Apply Theorem 4.2 to G' with η' playing the role of η. Then G' contains a T_r-packing $\mathcal{M}_1$ covering all but a set W of at most $\eta' n' \leq \nu^3 n$ vertices. Since n is divisible by r, $|W|$ is divisible by r. Hence, by definition of M, $G[M \cup W]$ contains a perfect T_r-packing $\mathcal{M}_2$. Then $\mathcal{M}_1 \cup \mathcal{M}_2$ is a perfect T_r-packing in G, as desired. □

4.2. **The auxiliary results for Theorem 1.5.** The proof of Theorem 1.5 follows the same general strategy as the proof of Theorem 1.6. The next result guarantees our desired absorbing set.

Theorem 4.3. *Let $n, r, h \in \mathbb{N}$ where $r \geq 2$ and $\nu, \eta > 0$ such that $0 < 1/n \ll \nu \ll \eta \ll 1/r, 1/h$. Suppose that H is a graph on h vertices with $\chi(H) = r$. Let G be a graph on n vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ where*

$$d_i \geq (r-2)n/r + i + \eta n \quad \text{for all} \quad i < n/r.$$

Then $V(G)$ contains a set M so that $|M| \leq \nu n$ and M is an H-absorbing set for any $W \subseteq V(G) \backslash M$ such that $|W| \in h\mathbb{N}$ and $|W| \leq \nu^3 n$.

Theorem 4.3 is proved in Section 8.2. The next result is a simple consequence of Theorem 4.2.

Theorem 4.4. *Let $n, r, h \in \mathbb{N}$ where $r \geq 2$ and $\eta > 0$ such that $0 < 1/n \ll \eta \ll 1/r, 1/h$. Suppose that H is a graph on h vertices with $\chi(H) = r$. Let G be a graph on n vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ where*

$$d_i \geq (r-2)n/r + i + \eta n \quad \text{for all} \quad i < n/r.$$

Then G contains an H-packing covering all but at most ηn vertices.

Proof. Let G' denote the digraph obtained from G by replacing each edge $xy \in E(G)$ with directed edges xy, yx. So G' has dominant degree sequence $d_1^* \leq d_2^* \leq \cdots \leq d_n^*$ where

$$d_i^* \geq (r-2)n/r + i + \eta n \quad \text{for all} \quad i < n/r.$$

Thus, Theorem 4.2 implies that G' contains a T_r^h-packing $\mathcal{M}$ covering all but at most ηn vertices. By construction of G', $\mathcal{M}$ corresponds to a K_r^h-packing in G. Notice that K_r^h contains a perfect H-packing. Therefore, G contains an H-packing covering all but at most ηn vertices. $\square$

Similarly to the proof of Theorem 1.6, we now deduce Theorem 1.5 from Theorems 4.3 and 4.4.
Proof of Theorem 1.5. Define constants η', ν, η and $n_0 \in \mathbb{N}$ such that $0 < 1/n_0 \ll \eta' \ll \nu \ll \eta \ll \gamma, 1/r, 1/|H|$. Set $h := |H|$. Let G be a graph on $n \geq n_0$ vertices as in the statement of the theorem. Apply Theorem 4.3 to G with parameters ν, η to obtain a set M so that $|M| \leq \nu n$ and M is an H-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in h\mathbb{N}$ and $|W| \leq \nu^3 n$.

Set $G' := G \setminus M$ and $n' := |G'|$. It is easy to see that G' has degree sequence $d_{G',1} \leq d_{G',2} \leq \cdots \leq d_{G',n'}$ where

$$d_{G',i} \geq (r-2)n'/r + i + \eta' n' \quad \text{for all} \quad i < n'/r.$$

Apply Theorem 4.4 to G' with η' playing the role of η. Then G' contains an H-packing $\mathcal{M}_1$ covering all but a set W of at most $\eta' n' \leq \nu^3 n$ vertices. Since n is divisible by h, $|W|$ is divisible by h. Hence, by definition of M, $G[M \cup W]$ contains a perfect H-packing $\mathcal{M}_2$. Then $\mathcal{M}_1 \cup \mathcal{M}_2$ is a perfect H-packing in G, as desired. $\square$

5. The Regularity lemma and related tools

In the proofs of Theorems 4.2 and 4.3 we will apply Szemerédi's Regularity lemma [27]. Before we state it we need some more definitions. The *density* of a bipartite graph $G = (A, B)$ with vertex classes A and B is defined to be

$$d_G(A, B) := \frac{e_G(A, B)}{|A||B|}.$$

We will write $d(A, B)$ if this is unambiguous. Given any $\varepsilon > 0$ we say that G is *ε-regular* if for all $X \subseteq A$ and $Y \subseteq B$ with $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$ we have that $|d(X, Y) - d(A, B)| < \varepsilon$.

Given disjoint vertex sets A and B in a graph G, we write $(A, B)_G$ for the induced bipartite subgraph of G whose vertex classes are A and B. If G is a digraph, we write $(A, B)_G$ for the

oriented bipartite subgraph of G whose vertex classes are A and B and whose edges are all the edges from A to B in G. We say $(A, B)_G$ is ε-regular and has density d if the underlying bipartite graph of $(A, B)_G$ is ε-regular and has density d. (Note that the ordering of the pair (A, B) is important.)

The Diregularity lemma is a variant of the Regularity lemma for digraphs due to Alon and Shapira [1]. Its proof is similar to the undirected version. We will use the following degree form of the Regularity lemma which is stated in terms of both graphs and digraphs. It is derived from the Diregularity lemma. (See for example [30] for a proof of the directed version and [21] for a sketch of the undirected version.)

Lemma 5.1 (Degree form of the Regularity lemma). *For every $\varepsilon \in (0, 1)$ and every integer M' there are integers M and n_0 such that if G is a graph or digraph on $n \geq n_0$ vertices and $d \in [0, 1]$ is any real number, then there is a partition of the vertices of G into $V_0, V_1, \ldots, V_k$ and a spanning sub(di)graph G' of G such that the following holds:*

- *$M' \leq k \leq M$;*
- *$|V_0| \leq \varepsilon n$;*
- *$|V_1| = \cdots = |V_k| =: m$;*
- *If G is a graph then $d_{G'}(x) > d_G(x) - (d + \varepsilon)n$ for all vertices $x \in V(G)$;*
- *If G is a digraph then $d_{G'}^+(x) > d_G^+(x) - (d + \varepsilon)n$ and $d_{G'}^-(x) > d_G^-(x) - (d + \varepsilon)n$ for all vertices $x \in V(G)$;*
- *$e(G'[V_i]) = 0$ for all $i \geq 1$;*
- *For all $1 \leq i, j \leq k$ with $i \neq j$ the pair $(V_i, V_j)_{G'}$ is ε-regular and has density either 0 or density at least d.*

We call $V_1, \ldots, V_k$ *clusters*, V_0 the *exceptional set* and the vertices in V_0 *exceptional vertices*. We refer to G' as the *pure (di)graph*. The *reduced (di)graph R of G with parameters ε, d and M'* is the (di)graph whose vertex set is $V_1, \ldots, V_k$ and which $V_i V_j$ is an edge precisely when $(V_i, V_j)_{G'}$ is ε-regular and has density at least d.

Roughly speaking, the next result states that the reduced digraph R of a digraph G as in Theorem 4.2 'inherits' the dominant degree sequence of G.

Lemma 5.2. *Let $M', n_0, r \in \mathbb{N}$ where $r \geq 2$ and let ε, d, η be positive constants such that $1/n_0 \ll 1/M' \ll \varepsilon \ll d \ll \eta < 1$. Let G be a digraph on $n \geq n_0$ vertices with dominant degree sequence $d_1^* \leq d_2^* \leq \cdots \leq d_n^*$ such that*

$$(1) \qquad d_i^* \geq (r-2)n/r + i + \eta n \quad \text{for all} \ i < n/r.$$

Let R be the reduced digraph of G with parameters ε, d and M' and set $k := |R|$. Then R has dominant degree sequence $d_{R,1}^* \leq d_{R,2}^* \leq \cdots \leq d_{R,k}^*$ such that

$$(2) \qquad d_{R,i}^* \geq (r-2)k/r + i + \eta k/2 \quad \text{for all} \ i < k/r.$$

Proof. Let G' denote the pure digraph of G and let $V_1, \ldots, V_k$ denote the clusters of G and V_0 the exceptional set. Set $m := |V_1| = \cdots = |V_k|$. We may assume that $d_R^*(V_1) \leq d_R^*(V_2) \leq \cdots \leq d_R^*(V_k)$. Consider any $i < k/r$. Set $S := \bigcup_{1 \leq j \leq i} V_j$. So $|S| = mi < mk/r \leq n/r$. Thus, by (1) there exists a vertex $x \in S$ such that $d_G^*(x) \geq d_{mi}^* \geq (r-2)n/r + mi + \eta n$. Suppose that $x \in V_j$ where $1 \leq j \leq i$. Since $km \leq n$, Lemma 5.1 implies that

$$d_R^*(V_j) \geq (d_{G'}^*(x) - |V_0|)/m \geq ((r-2)n/r + mi + \eta n - (d + 2\varepsilon)n)/m \geq (r-2)k/r + i + \eta k/2.$$

Since $d_{R,i}^* = d_R^*(V_i) \geq d_R^*(V_j)$ this proves that (2) is satisfied, as desired. □

An analogous proof yields the following version of Lemma 5.2 for graphs.

Lemma 5.3. *Let $M', n_0, r \in \mathbb{N}$ where $r \geq 2$ and let ε, d, η be positive constants such that $1/n_0 \ll 1/M' \ll \varepsilon \ll d \ll \eta < 1$. Let G be a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that*

$$d_i \geq (r-2)n/r + i + \eta n \ \ for \ all \ \ i < n/r.$$

Let R be the reduced graph of G with parameters ε, d and M' and set $k := |R|$. Then R has degree sequence $d_{R,1} \leq d_{R,2} \leq \cdots \leq d_{R,k}$ such that

$$d_{R,i} \geq (r-2)k/r + i + \eta k/2 \ \ for \ all \ \ i < k/r.$$

The next well-known observation (see [18] for example) states that a large subgraph of a regular pair is also regular.

Lemma 5.4. *Let $0 < \varepsilon < \alpha$ and $\varepsilon' := \max\{\varepsilon/\alpha, 2\varepsilon\}$. Let (A, B) be an ε-regular pair of density d. Suppose $A' \subseteq A$ and $B' \subseteq B$ where $|A'| \geq \alpha|A|$ and $|B'| \geq \alpha|B|$. Then (A', B') is an ε'-regular pair with density d' where $|d' - d| < \varepsilon$.*

The following result will be applied in the proof of Theorem 4.2 to convert an almost perfect T_r-packing in a reduced digraph into an almost perfect T_r^h-packing in the original digraph G. It is (for example) a special case of Corollary 2.3 in [2].

Lemma 5.5. *Let $\varepsilon, d > 0$ and $m, r, h \in \mathbb{N}$ such that $0 < 1/m \ll \varepsilon \ll d \ll 1/r, 1/h$. Let H be a graph obtained from K_r by replacing every vertex of K_r with m vertices and replacing each edge of K_r with an ε^2-regular pair of density at least d. Then H contains a K_r^h-packing covering all but at most εmr vertices.*

In the proof of Theorem 4.3 we will apply the following well-known counting lemma (often called the Key lemma) from [20].

Lemma 5.6 (Counting lemma [20])**.** *Suppose that $0 < \varepsilon < d$, that $m, t \in \mathbb{N}$ and that R is a graph with $V(R) = \{v_1, \ldots, v_k\}$. Construct a graph G by replacing every vertex $v_i \in V(R)$ by a set V_i of m vertices, and replacing the edges of R with ε-regular pairs of density at least d. For each $v_i \in V(R)$ let U_i denote the set of t vertices in $R(t)$ corresponding to v_i. Let H be a subgraph of $R(t)$ on h vertices and maximum degree $\Delta \in \mathbb{N}$. Set $\delta := d - \varepsilon$ and $\varepsilon_0 := \delta^\Delta/(2 + \Delta)$. If $\varepsilon \leq \varepsilon_0$ and $t - 1 \leq \varepsilon_0 m$ then there are at least*

$$(\varepsilon_0 m)^h \ labelled \ copies \ of \ H \ in \ G$$

so that if $x \in V(H)$ lies in U_i then x is embedded into V_i in G.

6. Almost perfect packings in digraphs

6.1. **Outline of the proof of Theorem 4.2.** In Section 6.3 we prove Theorem 4.2. The idea of the proof is as follows: We apply the Regularity lemma (Lemma 5.1) to obtain a reduced digraph R of G. If R contains an almost perfect T_r-packing then by applying Lemma 5.5 we conclude that G contains an almost perfect T_r^h-packing, as required. Otherwise, suppose that the largest T_r-packing in R covers precisely $d \leq (1 - o(1))|R|$ vertices. We then show that there is a $\{T_r, T_{r+1}\}$-packing in R covering substantially more than d vertices (see Lemma 6.4). Since $T_r(r)$ and $T_{r+1}(r)$ both contain perfect T_r-packings, this implies that the blow-up $R(r)$ of R contains a T_r-packing covering substantially more than dr vertices. Thus, crucially, the largest T_r-packing in $R(r)$ covers a higher proportion of vertices than the largest T_r-packing in R. By repeating this argument, we obtain a blow-up R' of R that contains an almost perfect T_r-packing. Using Lemma 5.5 we then show that this implies that G contains an almost perfect T_r^h-packing, as desired. Arguments of a similar flavour were applied in [18, 11].

6.2. Tools for the proof of Theorem 4.2.

In this section we give a number of tools that will be used in the proof of Theorem 4.2. The following trivial observation will be used in the proof to convert a $\{T_r, T_{r+1}\}$-packing in a (reduced) digraph R into a T_r-packing in the blow-up $R(r)$ of R.

Fact 6.1. *Suppose that $r, t \in \mathbb{N}$ such that r divides t. Then both $T_r(t) = T_r^t$ and $T_{r+1}(t) = T_{r+1}^t$ contain perfect T_r-packings.*

The next result will be used to show that a blow-up of a reduced digraph R 'inherits' the dominant degree sequence of R.

Proposition 6.2. *Let $n, r, t \in \mathbb{N}$ and $\gamma > 0$ where $n > r$. Suppose that H is a digraph on n vertices with dominant degree sequence $d_{H,1}^* \leq \cdots \leq d_{H,n}^*$ such that*

$$(3) \qquad d_{H,i}^* \geq (r-2)n/r + i + \gamma n \quad \text{for all} \ \ i < n/r.$$

Then $H' := H(t)$ has dominant degree sequence $d_{H',1}^* \leq \cdots \leq d_{H',nt}^*$ such that

$$d_{H',i}^* \geq (r-2)nt/r + i + (\gamma n - 1)t \quad \text{for all} \ \ i < nt/r.$$

Proof. Given any $1 \leq j \leq nt$, by definition of H' we have that

$$(4) \qquad d_{H',j}^* = t \times d_{H,\lceil j/t \rceil}^*.$$

Suppose that $j \leq nt/r - t$. Then $\lceil j/t \rceil < n/r$, so (3) and (4) imply that

$$d_{H',j}^* \geq (r-2)nt/r + \lceil j/t \rceil t + \gamma nt \geq (r-2)nt/r + j + \gamma nt.$$

In particular, this implies that for any $i < nt/r$,

$$d_{H',i}^* \geq (r-2)nt/r + (i-t) + \gamma nt = (r-2)nt/r + i + (\gamma n - 1)t,$$

as desired. $\qquad\square$

Proposition 6.3. *Let $n, r \in \mathbb{N}$ such that $n \geq r \geq 2$. Let G be a digraph on n vertices with dominant degree sequence $d_1^* \leq d_2^* \leq \cdots \leq d_n^*$ where*

$$(5) \qquad d_{\lceil n/r \rceil}^* \geq (1 - 1/r)n.$$

Then G contains a copy of T_r.

Proof. Let $r' \in \mathbb{N}$. Suppose that T' is a copy of $T_{r'}$ in G. Let $V(T') = \{x_1, \dots, x_{r'}\}$ where x_i plays the role of the ith vertex of $T_{r'}$. We say that T' is *consistent* if there exists $0 \leq s' \leq r'$ such that

- $d_G^+(x_i) \geq (1 - 1/r)n$ for all $i \leq s'$;
- $d_G^-(x_i) \geq (1 - 1/r)n$ for all $i > s'$.

We call s' a *turning point of T'*. (Note that T' could have more than one turning point.)

(5) implies that there is a copy of T_1 in G that is consistent. Suppose that, for some $1 \leq r' < r$, we have found a consistent copy T' of $T_{r'}$ in G. As before, let $V(T') = \{x_1, \dots, x_{r'}\}$ where x_i plays the role of the ith vertex of $T_{r'}$ and let s' denote a turning point of T'. Set $N' := \bigcap_{i \leq s'} N_G^+(x_i) \cap \bigcap_{i > s'} N_G^-(x_i)$. Since T' is consistent with turning point s' and $r' \leq r - 1$,

$$|N'| \geq \left(1 - \frac{r'}{r}\right) n \geq \frac{n}{r}.$$

In particular, (5) implies there is a vertex $x \in N'$ such that $d_G^*(x) \geq (1 - 1/r)n$. Then $V(T') \cup \{x\}$ spans a consistent copy of $T_{r'+1}$ in G where x plays the role of the $(s'+1)$th vertex in $T_{r'+1}$. (This is true regardless of whether $d_G^+(x) \geq (1 - 1/r)n$ or $d_G^-(x) \geq (1 - 1/r)n$.) This implies that G contains a (consistent) copy of T_r, as desired. $\qquad\square$

The next result is the main tool in the proof of Theorem 4.2. It will be used to convert a T_r-packing in a reduced digraph R into a significantly larger $\{T_r, T_{r+1}\}$-packing.

Lemma 6.4. *Let $\eta, \gamma > 0$ and $n, r \geq 2$ be integers such that $0 < 1/n \ll \gamma \ll \eta \ll 1/r$. Let G be a digraph on n vertices with dominant degree sequence $d_1^* \leq d_2^* \leq \cdots \leq d_n^*$ where*

$$(6) \qquad\qquad d_i^* \geq (r-2)n/r + i + \eta n \quad \text{for all} \ \ i < n/r.$$

Further, suppose that the largest T_r-packing in G covers precisely $n' \leq (1-\eta)n$ vertices. Then there exists a $\{T_r, T_{r+1}\}$-packing in G that covers at least $n' + \gamma n$ vertices.

Proof. By Proposition 6.3, $n' \geq \eta n/2$. Define a bijection $I : V(G) \mapsto [n]$ where $I(x) = i$ implies that $d_G^*(x) = d_i^*$. Let $\mathcal{M}$ denote a T_r-packing in G covering precisely n' vertices so that

$$\sum_{x \in V(G) \setminus V(\mathcal{M})} I(x)$$

is maximised. Set $n'' := n - n'$ and let $G'' := G \setminus V(\mathcal{M})$.

Claim 6.5. *There are at least γn vertices $x \in V(G'')$ such that $d_G^*(x, V(\mathcal{M})) \geq (r-1)n'/r + \gamma n$.*

Suppose for a contradiction that the claim is false. Let $V(G'') = \{x_1, \ldots, x_{n''}\}$ where $I(x_1) < I(x_2) < \cdots < I(x_{n''})$. For each $1 \leq i \leq n''$, set $s_i := I(x_i)$. So for each $1 \leq i \leq n''$, $d_G^*(x_i) = d_{s_i}^*$ where $s_1 < s_2 < \cdots < s_{n''}$. Let j be the largest integer such that $s_j < n/r$ (so $0 \leq j \leq n''$). We call $x_1, \ldots, x_j$ *small vertices* and $x_{j+1}, \ldots, x_{n''}$ *big vertices*.

Notice that (6) implies that every big vertex $x \in V(G'')$ satisfies $d_G^*(x) \geq (r-1)n/r + 3\eta n/4$. We say a big vertex $x \in V(G'')$ is *bad* if $d_G^*(x, V(G'')) \leq (r-1)n''/r + \eta n/2$. At most γn big vertices $x \in V(G'')$ are bad. Indeed, otherwise there are at least γn vertices $x \in V(G'')$ such that

$$d_G^*(x, V(\mathcal{M})) \geq (r-1)n/r + 3\eta n/4 - (r-1)n''/r - \eta n/2 \geq (r-1)n'/r + \gamma n,$$

a contradiction to our assumption.

Consider any small vertex $x_i \in V(G'')$. Then $d_G^*(x_i) = d_{s_i}^* \geq (r-2)n/r + s_i + \eta n$ by (6) since $s_i < n/r$. Suppose for a contradiction that $d_G^*(x_i, V(G'')) \leq (r-2)n''/r + i + 2\gamma n$. Then

$$(7)$$
$$d_G^*(x_i, V(\mathcal{M})) \geq (r-2)n/r + s_i + \eta n - (r-2)n''/r - i - 2\gamma n \geq (r-2)n'/r + (s_i - i) + \eta n/2.$$

Without loss of generality assume that $d_G^*(x_i) = d_G^+(x_i)$. Then (7) implies that there are at least $(s_i - i) + \eta n/2$ vertices $y \in V(\mathcal{M})$ with the property that if y lies in a copy T_r' of T_r in $\mathcal{M}$ then x_i sends an edge to every vertex in $V(T_r') \setminus \{y\}$. In particular, $(V(T_r') \cup \{x_i\}) \setminus \{y\}$ spans a copy of T_r in G. Since $I(x_{i'}) < I(x_i)$ for all $i' < i$, this implies that there is one such vertex y with $s_i = I(x_i) < I(y)$. Let $\mathcal{M}'$ denote the T_r-packing in G obtained from $\mathcal{M}$ by replacing the copy T_r' of T_r covering y with the copy of T_r spanning $(V(T_r') \cup \{x_i\}) \setminus \{y\}$. Then $\mathcal{M}'$ is a T_r-packing in G covering precisely n' vertices and with

$$\sum_{x \in V(G) \setminus V(\mathcal{M})} I(x) < \sum_{x \in V(G) \setminus V(\mathcal{M}')} I(x),$$

a contradiction to the choice of $\mathcal{M}$. So $d_G^*(x_i, V(G'')) \geq (r-2)n''/r + i + 2\gamma n$.

In summary, $d_G^*(x_i, V(G'')) \geq (r-2)n''/r + i + 2\gamma n$ for each small vertex $x_i \in V(G'')$. Further, at most γn big vertices $x \in V(G'')$ are bad. By deleting the bad vertices from G'' we obtain a subdigraph G^* of G'' with dominant degree sequence $d_{G^*,1}^* \leq d_{G^*,2}^* \leq \cdots \leq d_{G^*,n^*}^*$ where $n^* := |G^*|$ so that

$$d_{G^*,i}^* \geq (r-2)n^*/r + i + \gamma n \quad \text{for all} \ \ i < n^*/r.$$

Hence, Proposition 6.3 implies that G^* contains a copy T_r^* of T_r. Then $\mathcal{M} \cup \{T_r^*\}$ is a T_r-packing in G covering more than n' vertices, a contradiction. This completes the proof of the claim.

Given any $x \in V(G'')$ such that $d_G^*(x, V(\mathcal{M})) \geq (r-1)n'/r + \gamma n$ there are at least γn copies T_r' of T_r in $\mathcal{M}$ so that x sends all possible edges to $V(T_r')$ in G or receives all possible edges from $V(T_r')$ in G. In particular, $V(T_r') \cup \{x\}$ spans a copy of T_{r+1} in G. Thus, Claim 6.5 implies that there exists a $\{T_r, T_{r+1}\}$-packing in G that covers at least $n' + \gamma n$ vertices, as desired. $\qquad\square$

6.3. **Proof of Theorem 4.2.** Define additional constants ε, d, γ and $M' \in \mathbb{N}$ so that $0 < 1/n \ll 1/M' \ll \varepsilon \ll d \ll \gamma \ll \eta \ll 1/r, 1/h$. Set $z := \lceil 1/\gamma \rceil$. Apply Lemma 5.1 with parameters ε, d and M' to G to obtain clusters $V_1, \ldots, V_k$, an exceptional set V_0 and a pure digraph G'. Set $m := |V_1| = \cdots = |V_k|$. Let R be the reduced digraph of G with parameters ε, d and M'. Lemma 5.2 implies that R has dominant degree sequence $d_{R,1}^* \leq d_{R,2}^* \leq \cdots \leq d_{R,k}^*$ such that

$$(8) \qquad d_{R,i}^* \geq (r-2)k/r + i + \eta k/2 \quad \text{for all} \ \ i < k/r.$$

Claim 6.6. $R' := R(r^z)$ *contains a T_r-packing covering at least* $(1 - \eta/2)kr^z = (1 - \eta/2)|R'|$ *vertices.*

If R contains a T_r-packing covering at least $(1-\eta/2)k$ vertices then Fact 6.1 implies that Claim 6.6 holds. So suppose that the largest T_r-packing in R covers precisely $d \leq (1-\eta/2)k$ vertices. Then by Lemma 6.4, R contains a $\{T_r, T_{r+1}\}$-packing that covers at least $d + \gamma k$ vertices. Thus, by Fact 6.1, $R(r)$ contains a T_r-packing covering at least $(d + \gamma k)r$ vertices. (So at least a γ-proportion of the vertices in $R(r)$ are covered.) Further, Proposition 6.2 and (8) imply that $R(r)$ has dominant degree sequence $d_{R(r),1}^* \leq \cdots \leq d_{R(r),kr}^*$ such that

$$d_{R(r),i}^* \geq (r-2)k + i + \eta kr/4 \quad \text{for all} \ \ i < k.$$

If $R(r)$ contains a T_r-packing covering at least $(1-\eta/2)kr$ vertices then again Fact 6.1 implies that the claim holds. So suppose that the largest T_r-packing in $R(r)$ covers precisely $d' \leq (1-\eta/2)kr$ vertices. Recall that $d' \geq (d+\gamma k)r$. By Lemma 6.4, $R(r)$ contains a $\{T_r, T_{r+1}\}$-packing that covers at least $d' + \gamma kr \geq (d + 2\gamma k)r$ vertices. Thus, by Fact 6.1, $R(r^2)$ contains a T_r-packing covering at least $(d + 2\gamma k)r^2$ vertices. (So at least a 2γ-proportion of the vertices in $R(r^2)$ are covered.) Repeating this argument at most z times we see that the claim holds.

For each $1 \leq i \leq k$, partition V_i into classes $V_i^*, V_{i,1}, \ldots, V_{i,r^z}$ where $m' := |V_{i,j}| = \lfloor m/r^z \rfloor \geq m/(2r^z)$ for all $1 \leq j \leq r^z$. Since $mk \geq (1-\varepsilon)n$ by Lemma 5.1,

$$(9) \qquad m'|R'| = \lfloor m/r^z \rfloor kr^z \geq mk - kr^z \geq (1 - 2\varepsilon)n.$$

Lemma 5.4 implies that if $(V_{i_1}, V_{i_2})_{G'}$ is ε-regular with density at least d then $(V_{i_1,j_1}, V_{i_2,j_2})_{G'}$ is $2\varepsilon r^z$-regular with density at least $d - \varepsilon \geq d/2$ (for all $1 \leq j_1, j_2 \leq r^z$). In particular, we can label the vertex set of R' so that $V(R') = \{V_{i,j} : 1 \leq i \leq k, \ 1 \leq j \leq r^z\}$ where $V_{i_1,j_1}V_{i_2,j_2} \in E(R')$ implies that $(V_{i_1,j_1}, V_{i_2,j_2})_{G'}$ is $2\varepsilon r^z$-regular with density at least $d/2$.

By Claim 6.6, R' has a T_r-packing $\mathcal{M}$ that contains at least $(1-\eta/2)|R'|$ vertices. Consider any copy T_r' of T_r in $\mathcal{M}$ and let $V(T_r') = \{V_{i_1,j_1}, V_{i_2,j_2}, \ldots, V_{i_r,j_r}\}$. Set V' to be the union of $V_{i_1,j_1}, V_{i_2,j_2}, \ldots, V_{i_r,j_r}$. Note that $0 < 1/m' \ll 2\varepsilon r^z \ll d/2 \ll \gamma \ll 1/r, 1/h$. Thus, Lemma 5.5 implies that $G'[V']$ contains a T_r^h-packing covering all but at most $\sqrt{2\varepsilon r^z}m'r \leq \gamma m'r$ vertices. By considering each copy of T_r in $\mathcal{M}$ we conclude that $G' \subseteq G$ contains a T_r^h-packing covering at least

$$(1-\gamma)m'r \times (1-\eta/2)|R'|/r \overset{(9)}{\geq} (1-\gamma)(1-\eta/2)(1-2\varepsilon)n \geq (1-\eta)n$$

vertices, as desired. $\qquad\square$

7. Overview of our absorbing approach

The 'absorbing method' was first used in [26] and has subsequently been applied to numerous embedding problems in extremal graph theory. In this section we give an overview of the absorbing method that we will apply to prove Theorems 4.1 and 4.3.

Suppose that in some (di)graph (or hypergraph) G we wish to find a 'small' H-absorbing set M in G that absorbs any 'very small' set of vertices in G. To prove that such an H-absorbing set M exists it is sufficient to show that G has many 'connecting structures' of a certain type. That is, it suffices to show that there is a constant $t \in \mathbb{N}$ such that, given any $x, y \in V(G)$, there are 'many' t-sets $X \subseteq V(G)$ so that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-packings. The following lemma of Lo and Markström [24] makes this observation precise.

Lemma 7.1 (Lo and Markström [24]). *Let $h, t \in \mathbb{N}$ and let $\gamma > 0$. Suppose that H is a hypergraph on h vertices. Then there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is a hypergraph on $n \geq n_0$ vertices so that, for any $x, y \in V(G)$, there are at least γn^{th-1} $(th-1)$-sets $X \subseteq V(H)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-packings. Then $V(G)$ contains a set M so that*

- *$|M| \leq (\gamma/2)^h n/4$;*
- *M is an H-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in h\mathbb{N}$ and $|W| \leq (\gamma/2)^{2h} hn/32$.*

Moreover, the analogous result holds for digraphs H and G.

(Lo and Markström [24] only proved Lemma 7.1 in the case of hypergraphs. However, the proof of the digraph case is identical.)

In view of Lemma 7.1 it is natural to seek sufficient conditions that guarantee our desired connecting structures X for each $x, y \in V(G)$. For this, we introduce the definition of an H-*path* below. In what follows we will be working in the (di)graph setting, although everything naturally extends to hypergraphs.

Suppose that H is a (di)graph on h vertices. An H-*path* P is a (di)graph with the following properties:

   (i) $|P| = th + 1$ for some $t \in \mathbb{N}$;
   (ii) $V(P) = X_1 \cup \cdots \cup X_t \cup \{y_1, \ldots, y_{t+1}\}$ where $|X_i| = h - 1$ for all $1 \leq i \leq t$;
   (iii) $P[X_i \cup \{y_i\}] = P[X_i \cup \{y_{i+1}\}] = H$ for all $1 \leq i \leq t$.

We call y_1 and y_{t+1} the *endpoints of P* and t the *length of P*. An example of a T_3-path of length 3 is given in Figure 2. A *truncated H-path* Q is a (di)graph with the following properties:

   (i) $|Q| = th - 1$ for some $t \in \mathbb{N}$;
   (ii) $V(Q) = X_1 \cup \cdots \cup X_t \cup \{y_2, \ldots, y_t\}$ where $|X_i| = h - 1$ for all $1 \leq i \leq t$;
   (iii) $Q[X_i \cup \{y_i\}] = Q[X_i \cup \{y_{i+1}\}] = H$ for all $2 \leq i \leq t-1$ and $Q[X_1 \cup \{y_2\}] = Q[X_t \cup \{y_t\}] = H$.

We call X_1 and X_t the *endsets of Q* and t the *length of Q*. Notice that given any H-path P of length t and endpoints x and y, $Q := P \setminus \{x, y\}$ is a truncated H-path of length t. In this case we say that Q is *the truncated H-path of P*.

The following simple observation follows immediately from the definition of an H-path.

Fact 7.2. *Let H be a (di)graph. Let P_1 be an H-path of length t_1 with endpoints x, y and let P_2 be an H-path of length t_2 with endpoints y, z. Suppose that $P_1 \setminus \{y\}$ and $P_2 \setminus \{y\}$ are vertex-disjoint. Then $P_1 \cup P_2$ is an H-path of length $t_1 + t_2$ with endpoints x, z.*

Suppose that P is an H-path in a (di)graph G with endpoints x and y. Set $X := V(P) \setminus \{x, y\}$. Then by condition (iii) in the definition of an H-path, both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-packings. Thus, to show that the hypothesis of Lemma 7.1 holds it suffices to prove that, given any $x, y \in V(G)$, there are at least γn^{th-1} $(th-1)$-sets $X \subseteq V(H)$ so that $X \cup \{x, y\}$ spans
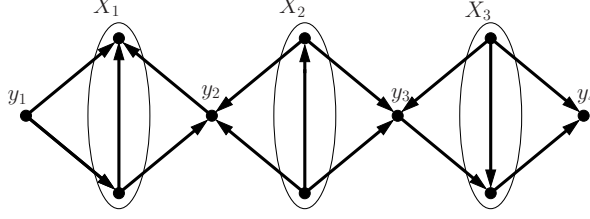
FIGURE 2. An example of a T_3-path of length 3

an H-path in G of length t with endpoints x and y. We will use this approach in the proofs of Theorems 4.1 and 4.3.

Let $0 < \gamma \ll \beta \ll 1/t, 1/h$ where $t, h \in \mathbb{N}$. Suppose that H is a (di)graph on h vertices and G is a sufficiently large (di)graph on n vertices. Define an auxiliary graph $\mathcal{G}$ as follows. $\mathcal{G}$ has vertex set $V(G)$ and given distinct $x, y \in V(G)$, $xy \in E(\mathcal{G})$ precisely when there are at least βn^{h-1} H-paths in G of length one with endpoints x and y. Suppose that for each $x, y \in V(G)$, there are at least βn^{t-1} paths of length t in $\mathcal{G}$ with endpoints x and y. Then Fact 7.2 together with a simple calculation implies that there are at least γn^{th-1} H-paths in G of length t with endpoints x and y. Thus, ultimately to find the connecting structures required to apply Lemma 7.1 it suffices to prove that the auxiliary graph $\mathcal{G}$ is 'well-connected'. Although $\mathcal{G}$ will not be formally required, it will serve as a useful point of reference when describing how we construct our desired T_r-paths in the proof of Theorem 4.1.

8. The absorbing results for Theorems 1.5 and 1.6

8.1. **Proof of Theorem 4.1.** In this section we prove Theorem 4.1. For some fixed $t' \in \mathbb{N}$, we will show that, for any $x, y \in V(G)$, there are 'many' T_r-paths of length t' in G with endpoints x and y. Then by applying Lemma 7.1 this implies that G contains our desired absorbing set M.

Let G be as in Theorem 4.1, $H = T_r$ and recall the definition of the auxiliary graph $\mathcal{G}$ given in Section 7 (where the constant β in the definition of $\mathcal{G}$ is chosen such that $\beta \ll \eta, 1/r$). Roughly speaking, the next result states that given any vertex $x \in V(\mathcal{G})$, there are 'many' vertices y such that $xy \in E(\mathcal{G})$ and y has 'large' dominant degree (relative to x). This 'expansion' property will then be exploited in Lemma 8.3 to conclude that $\mathcal{G}$ is 'well-connected', and thus G contains the desired T_r-paths.

Lemma 8.1. *Let $\gamma, \alpha, \eta > 0$ and $n, r \geq 2$ be integers such that $0 < 1/n \ll \gamma \ll \alpha \ll \eta \ll 1/r$. Let G be a digraph on n vertices with dominant degree sequence $d_1^* \leq d_2^* \leq \cdots \leq d_n^*$ where*

$$(10) \qquad d_i^* \geq (r-2)n/r + i + \eta n \quad \text{for all} \quad i < n/r.$$

Suppose that $x \in V(G)$ such that

$$(11) \qquad d_G^*(x) \geq (r-2)n/r + j + \eta n \quad \text{for some} \quad 1 \leq j < n/r - \alpha n.$$

Then there exist at least γn^r pairs (y, X) where $y \in V(G) \setminus \{x\}$, $X \subseteq V(G) \setminus \{x, y\}$ and the following conditions hold:

(i) $d_G^*(y) \geq (r-2)n/r + j + (\eta + \gamma)n$;
(ii) $X \cup \{x\}$ *and* $X \cup \{y\}$ *both spans copies of T_r in G. In particular, $X \cup \{x, y\}$ spans a copy of a T_r-path of length one in G with endpoints x and y.*

Proof. Without loss of generality assume that $d_G^+(x) \geq (r-2)n/r + j + \eta n$. Suppose first that $r = 2$. Then (10) and (11) imply that there are at least αn vertices $z \in N_G^+(x)$ such that $d_G^*(z) \geq$

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$j + (\eta + \alpha/2)n$. Fix such a vertex z and without loss of generality suppose that $d_G^*(z) = d_G^+(z)$. Then there are at least αn vertices $y \in N_G^+(z) \setminus \{x\}$ such that $d_G^*(y) \geq j + (\eta + \gamma)n$. Set $X := \{z\}$. Then (y, X) satisfies (i) and (ii). Further, there are at least

$$\alpha n \times \alpha n > \gamma n^2$$

choices for (y, X), as desired.

Next suppose that $r \geq 3$. Let $r' \in \mathbb{N}$. Suppose that T' is a copy of $T_{r'}$ in G. Let $V(T') = \{x_1, \ldots, x_{r'}\}$ where x_i plays the role of the ith vertex of $T_{r'}$. We say that T' is *consistent* if there exists $0 \leq s' \leq r'$ such that

- $d_G^+(x_i) \geq (1 - 1/r + 3\eta/4)n$ for all $i \leq s'$;
- $d_G^-(x_i) \geq (1 - 1/r + 3\eta/4)n$ for all $i > s'$.

We call s' a *turning point* of T'.

Claim 8.2. *There are at least $\eta^{r-1}n^{r-2}$ sets $X' \subseteq N_G^+(x)$ such that X' spans a consistent copy of T_{r-2} in G.*

By (10) and (11) there are at least $(r-2)n/r + j + \eta n - n/r \geq \eta n$ vertices $y_1 \in N_G^+(x)$ such that $d_G^*(y_1) \geq (1 - 1/r + 3\eta/4)n$. In particular, $\{y_1\}$ is a consistent copy of T_1. So if $r = 3$ the claim follows. Thus, assume $r \geq 4$.

Suppose for some $1 \leq r' \leq r - 3$, T' is a consistent copy of $T_{r'}$ in G such that $V(T') \subseteq N_G^+(x)$. Let $V(T') = \{y_1, \ldots, y_{r'}\}$ where y_i plays the role of the ith vertex of $T_{r'}$ and let s' denote a turning point of T'. Set $N' := N_G^+(x) \cap \bigcap_{i \leq s'} N_G^+(y_i) \cap \bigcap_{i > s'} N_G^-(y_i)$. Since T' is consistent with turning point s' and $r' \leq r - 3$, (11) implies that

$$|N'| \geq (r-2)n/r + j + \eta n - r'n/r \geq n/r + \eta n.$$

In particular, (10) implies that there are at least ηn vertices $w \in N'$ such that $d_G^*(w) \geq (1 - 1/r + 3\eta/4)n$. Notice that $V(T') \cup \{w\} \subseteq N_G^+(x)$ spans a consistent copy of $T_{r'+1}$ in G where w plays the role of the $(s'+1)$th vertex in $T_{r'+1}$. (This is true regardless of whether $d_G^+(w) \geq (1 - 1/r + 3\eta/4)n$ or $d_G^-(w) \geq (1 - 1/r + 3\eta/4)n$.)

This argument shows that there are at least

$$(\eta n)^{r-2} \times \frac{1}{(r-2)!} \geq \eta^{r-1}n^{r-2}$$

sets $X' \subseteq N_G^+(x)$ such that X' spans a consistent copy of T_{r-2} in G, thus proving the claim.

Fix a set X' as in Claim 8.2. Let $X' = \{x_1, \ldots, x_{r-2}\}$ where x_i plays the role of the ith vertex of a consistent copy T'_{r-2} of T_{r-2} in G. Let s denote a turning point of T'_{r-2}. Set $N := N_G^+(x) \cap \bigcap_{i \leq s} N_G^+(x_i) \cap \bigcap_{i > s} N_G^-(x_i)$. Then

$$|N| \stackrel{(11)}{\geq} (r-2)n/r + j + \eta n - (r-2)n/r = j + \eta n.$$

In particular, (10) and (11) imply that there are at least αn vertices $z \in N$ such that $d_G^*(z) \geq (r-2)n/r + j + (\eta + \alpha/2)n$.

Fix such a vertex z and let $X := X' \cup \{z\}$. Note that $X \cup \{x\}$ spans a copy of T_r in G. Without loss of generality suppose that $d_G^+(z) \geq (r-2)n/r + j + (\eta + \alpha/2)n$. Set $N^* := N_G^+(z) \cap \bigcap_{i \leq s} N_G^+(x_i) \cap \bigcap_{i > s} N_G^-(x_i)$. Then $|N^*| \geq (r-2)n/r + j + (\eta + \alpha/2)n - (r-2)n/r = j + (\eta + \alpha/2)n$. Thus, by (10) there are at least αn vertices $y \in N^* \setminus \{x\}$ such that $d_G^*(y) \geq (r-2)n/r + j + (\eta + \gamma)n$. Further, $X \cup \{y\}$ spans a copy of T_r in G (this follows by the choice of z and y). Fix such a vertex y. So (i) and (ii) hold.

Note that there were at least $\eta^{r-1}n^{r-2}$ choices for X', αn choices for z and αn choices for y. So in total there are at least

$$\eta^{r-1}n^{r-2} \times \alpha n \times \alpha n \times \frac{1}{r-1} \geq \gamma n^r$$

choices for (y, X), as desired. $\square$

The next result guarantees our desired T_r-paths in G (implicitly implying that our auxiliary graph $\mathcal{G}$ is 'well-connected').

Lemma 8.3. *Let $\xi, \gamma, \alpha, \eta > 0$ and $n, r \geq 2$ be integers such that $0 < 1/n \ll \xi \ll \gamma \ll \alpha \ll \eta \ll 1/r$. Let G be a digraph on n vertices with dominant degree sequence $d_1^* \leq d_2^* \leq \cdots \leq d_n^*$ where*

$$(12) \qquad\qquad d_i^* \geq (r-2)n/r + i + \eta n \quad \text{for all} \ \ i < n/r.$$

Set $t := (2\lfloor 1/\gamma r - \alpha/\gamma \rfloor + 1)r - 1$ and $t' := 2\lfloor 1/\gamma r - \alpha/\gamma \rfloor + 1$. Given any distinct $x, y \in V(G)$ there are at least ξn^t t-sets $A \subseteq V(G) \setminus \{x, y\}$ so that $A \cup \{x, y\}$ spans a T_r-path of length t' in G with endpoints x and y. In particular, for each such set A, $G[A \cup \{x\}]$ and $G[A \cup \{y\}]$ contain perfect T_r-packings.

Proof. Define an additional constant β so that $\xi \ll \beta \ll \gamma$. Set $t'' := \lfloor 1/\gamma r - \alpha/\gamma \rfloor$.

Claim 8.4. *Let $1 \leq i \leq t''$. There are at least $\beta^i n^{ir}$ pairs (w, X) where $w \in V(G) \setminus \{x, y\}$, $X \subseteq V(G) \setminus \{x, y, w\}$ and $|X| = ir - 1$ with the following properties:*

- *$d_G^*(w) \geq (r-2)n/r + (\eta + i\gamma)n$;*
- *$X \cup \{x, w\}$ spans a T_r-path of length i in G with endpoints x and w. In particular, $G[X \cup \{x\}]$ and $G[X \cup \{w\}]$ both contain perfect T_r-packings.*

We prove the claim inductively. By (12), $d_G^*(x) \geq (r-2)n/r + 1 + \eta n$. Thus, by Lemma 8.1, Claim 8.4 holds for $i = 1$. Suppose that for some $1 \leq i < t''$, Claim 8.4 holds. Let (w, X) be as in Claim 8.4 with $|X| = ir - 1$. Set $j := i\gamma n < n/r - \alpha n$. Note that

$$d_G^*(w) \geq (r-2)n/r + (\eta + i\gamma)n = (r-2)n/r + j + \eta n.$$

By Lemma 8.1 there exist at least γn^r pairs (w', X') where $w' \in V(G) \setminus \{w\}$, $X' \subseteq V(G) \setminus \{w, w'\}$ and the following conditions hold:

- *$d_G^*(w') \geq (r-2)n/r + j + (\eta + \gamma)n = (r-2)n/r + (\eta + (i+1)\gamma)n$;*
- *$X' \cup \{w, w'\}$ spans a T_r-path of length one in G with endpoints w and w'. That is, $X' \cup \{w\}$ and $X' \cup \{w'\}$ both spans copies of T_r in G.*

Fix such a pair (w', X') so that $X' \cup \{w'\}$ is disjoint from $X \cup \{x, y\}$; There are at least

$$\gamma n^r - r(ir+1)\binom{n}{r-1} \geq \gamma n^r/2$$

choices for (w', X').

Set $X'' := X \cup \{w\} \cup X'$. Then by Fact 7.2, (w', X'') is such that $w' \in V(G) \setminus \{x, y\}$, $X'' \subseteq V(G) \setminus \{x, y, w'\}$ and $|X''| = (i+1)r - 1$ with the following properties:

- *$d_G^*(w') \geq (r-2)n/r + (\eta + (i+1)\gamma)n$;*
- *$X'' \cup \{x, w'\}$ spans a T_r-path of length $i+1$ in G with endpoints x and w'. In particular, $G[X'' \cup \{x\}]$ and $G[X'' \cup \{w'\}]$ both contain perfect T_r-packings.*

There are at least $\beta^i n^{ir}$ choices for (w, X) and at least $\gamma n^r/2$ choices for (w', X'). Thus, in total there are at least

$$\beta^i n^{ir} \times \frac{\gamma n^r}{2} \times \frac{1}{((i+1)r)!} \geq \beta^{i+1} n^{(i+1)r}$$

such pairs (w', X''). Thus, Claim 8.4 holds with respect to $i+1$, as desired.

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Fix a pair (w, X) as in Claim 8.4 with $i := t''$. The proof of the next claim is analogous to that of Claim 8.4 (so we omit it).

Claim 8.5. *Let $1 \leq i \leq t''$. There are at least $\beta^i n^{ir}$ pairs (z, Y) where $z \in V(G)$, $Y \subseteq V(G) \setminus \{z\}$ and $|Y| = ir - 1$ with the following properties:*

- *$Y \cup \{z\}$ is disjoint from $X \cup \{w, x, y\}$;*
- *$d_G^*(z) \geq (r-2)n/r + (\eta + i\gamma)n$;*
- *$Y \cup \{y, z\}$ spans a T_r-path of length i in G with endpoints y and z. In particular, $G[Y \cup \{y\}]$ and $G[Y \cup \{z\}]$ both contain perfect T_r-packings.*

Fix a pair (z, Y) as in Claim 8.5 with $i := t''$. Note that

(13) $\quad d_G^*(w), d_G^*(z) \geq (r-2)n/r + (\eta + t''\gamma)n \geq (r-1)n/r + (\eta - \alpha - \gamma)n \geq (r-1)n/r + 3\eta n/4.$

Without loss of generality assume that $d_G^+(w) = d_G^*(w)$ and $d_G^+(z) = d_G^*(z)$ (the other cases follow analogously). Recall the definition of a consistent copy of $T_{r'}$ in G given in Lemma 8.1. By applying (13) and arguing in a similar way to the proof of Claim 8.2 we obtain the following claim.

Claim 8.6. *Suppose that $r \geq 3$. Then there are at least $\eta^{r-1} n^{r-2}$ sets $Z' \subseteq N_G^+(w) \cap N_G^+(z)$ such that Z' spans a consistent copy of T_{r-2} in G.*

Suppose that $r \geq 3$. Fix a set Z' as in Claim 8.6 so that Z' is disjoint from $X \cup Y \cup \{x, y\}$; There are at least

$$\eta^{r-1} n^{r-2} - 2t'' r \binom{n}{r-3} \geq \eta^{r-1} n^{r-2}/2$$

choices for Z'. Let $Z' = \{z_1, \ldots, z_{r-2}\}$ where z_i plays the role of the ith vertex of a consistent copy T'_{r-2} of T_{r-2} in G. Let s denote a turning point of T'_{r-2}. Set $N := N_G^+(w) \cap N_G^+(z) \cap \bigcap_{i \leq s} N_G^+(z_i) \cap \bigcap_{i > s} N_G^-(z_i)$. Then

$$|N| \overset{(13)}{\geq} (r-2)n/r + 3\eta n/2 - (r-2)n/r = 3\eta n/2.$$

Fix some $z' \in N \setminus (X \cup Y \cup \{x, y\})$. Notice that there are at least ηn choices for z'. Set $Z := Z' \cup \{z'\}$. The choice of z' implies that $Z \cup \{w\}$ and $Z \cup \{z\}$ span copies of T_r in G.

If $r = 2$, then notice that there are at least ηn vertices $z' \in N_G^+(w) \cap N_G^+(z)$ that are not in $X \cup Y \cup \{x, y\}$. Fix such a z' and let $Z := \{z'\}$.

For every $r \geq 2$, set $A := X \cup Y \cup Z \cup \{w, z\}$. So $|A| = (2t'' + 1)r - 1 = t$. By construction of A, $A \cup \{x, y\}$ spans a T_r-path of length t' in G with endpoints x and y. If $r \geq 3$, then recall that there are at least $\beta^{t''} n^{t''r}$ choices for (w, X), at least $\beta^{t''} n^{t''r}$ choices for (z, Y), at least $\eta^{r-1} n^{r-2}/2$ choices for Z' and at least ηn choices for z'. Overall, this implies that there are at least

$$\beta^{t''} n^{t''r} \times \beta^{t''} n^{t''r} \times \eta^{r-1} n^{r-2}/2 \times \eta n \times \frac{1}{t!} \geq \xi n^t$$

choices for A. If $r = 2$, then a similar calculation shows there are at least ξn^t choices for A, as required. $\qquad \square$

We now deduce Theorem 4.1 from Lemmas 7.1 and 8.3.

Proof of Theorem 4.1. Define additional constants ξ, γ, α such that $\nu \ll \xi \ll \gamma \ll \alpha \ll \eta$. Set $t := (2\lfloor 1/\gamma r - \alpha/\gamma \rfloor + 1)r - 1$ and $t' := 2\lfloor 1/\gamma r - \alpha/\gamma \rfloor + 1$ and define $\gamma' > 0$ so that $\nu = (\gamma'/2)^r/4$. Let G be as in the statement of the theorem. Lemma 8.3 implies that, for any $x, y \in V(G)$, there are at least ξn^t t-sets $X \subseteq V(G) \setminus \{x, y\}$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect T_r-packings. Since $\nu \ll \xi \ll 1/r$, and by definition of γ', we have that $\xi n^t \geq \gamma' n^t = \gamma' n^{t'r-1}$. Apply Lemma 7.1 with T_r, r, t', γ' playing the roles of H, h, t, γ. Then $V(G)$ contains a set M so that

- $|M| \leq (\gamma'/2)^r n/4 = \nu n$;

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- M is a T_r-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in r\mathbb{N}$ and $|W| \leq \nu^3 n \leq (\gamma'/2)^{2r} rn/32$.

$\square$

8.2. **Proof of Theorem 4.3.** In this section we prove Theorem 4.3. We will show that there is some fixed $s' \in \mathbb{N}$ such that, for any $x, y \in V(G)$, there are 'many' K_r^h-paths P of length s' in G with endpoints x and y. Consider such a K_r^h-path P and set $X := V(P) \setminus \{x, y\}$. Since K_r^h contains a perfect H-packing (for any H such that $|H| = h$ and $\chi(H) = r$), the definition of a K_r^h-path implies that $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-packings. Then by applying Lemma 7.1 this implies that G contains our desired H-absorbing set M.

The following result guarantees our desired K_r^h-paths in the case when $h = 1$. It is an immediate consequence of Lemma 8.3.

Lemma 8.7. *Let $\xi, \gamma, \alpha, \eta > 0$ and $n, r \geq 2$ be integers such that $0 < 1/n \ll \xi \ll \gamma \ll \alpha \ll \eta \ll 1/r$. Let G be a graph on n vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ where*

$$d_i \geq (r-2)n/r + i + \eta n \ \ for \ all \ \ i < n/r.$$

Set $t := (2\lfloor 1/\gamma r - \alpha/\gamma \rfloor + 1)r - 1$ and $t' := 2\lfloor 1/\gamma r - \alpha/\gamma \rfloor + 1$. Given any distinct $x, y \in V(G)$ there are at least ξn^t t-sets $A \subseteq V(G) \setminus \{x, y\}$ so that $A \cup \{x, y\}$ spans a K_r-path of length t' in G with endpoints x and y. In particular, for each such set A, $G[A \cup \{x\}]$ and $G[A \cup \{y\}]$ contain perfect K_r-packings.

We will apply Lemma 8.7 to a reduced graph R of G in order to conclude that R contains a K_r-path between any two clusters. We then show that this K_r-path in R corresponds to 'many' K_r^h-paths in G. For this, we first show that a special type of 'blow-up' of a K_r-path is itself a K_r^h-path.

Let $h, t, r \in \mathbb{N}$ such that $r+1, t \geq 3$. Consider a K_r-path P of length t. Let $X_1, \ldots, X_t, y_1, \ldots, y_{t+1}$ be as in the definition of P. In particular, P has endpoints y_1 and y_{t+1} and $|X_i| = r - 1$ for all $1 \leq i \leq t$. We define $P^*(h)$ to be the graph obtained from P as follows:

- Replace each vertex set X_i with a set of $h(r-1)$ vertices $X_i(h)$ so that $X_i(h)$ induces a copy of K_{r-1}^h in $P^*(h)$ (for $1 \leq i \leq t$);
- For $2 \leq i \leq t - 1$, we replace y_i with a set Y_i of h vertices; y_t is replaced by a set Y_t of $2h - 1$ vertices. Set $Y_1 := \{y_1\}$, $Y_{t+1} := \{y_{t+1}\}$;
- In $P^*(h)$ there are all possible edges between $Y_i \cup Y_{i+1}$ and $X_i(h)$ (for all $1 \leq i \leq t$).

Given a truncated K_r-path Q of length t, we define $Q^*(h)$ analogously. In particular, if Q is the truncated K_r-path of P then $Q^*(h) = P^*(h) \setminus \{y_1, y_{t+1}\}$.

Note that $P^*(h)$ is a blow-up of P where the vertices in $V(P) \setminus \{y_1, y_t, y_{t+1}\}$ are replaced by h vertices, y_t is replaced by $2h - 1$ vertices and y_1 and y_{t+1} are left 'untouched'. In particular, this immediately implies the following fact.

Fact 8.8. *Let $h, t, r \in \mathbb{N}$ such that $r + 1, t \geq 3$ and let G and H be graphs. Suppose that P is a K_r-path of length t and Q is a truncated K_r-path of length t. If $P \subseteq G$ then $P^*(h) \subseteq G(2h - 1)$. Further, if $Q \subseteq H$ then $Q^*(h) \subseteq H(2h - 1)$.*

We now show that, crucially, $P^*(h)$ is a K_r^h-path of length t.

Lemma 8.9. *Let $h, t, r \in \mathbb{N}$ such that $r + 1, t \geq 3$.*

(a) *Suppose that P is a K_r-path of length t with endpoints y_1 and y_{t+1}. Then $P^*(h)$ is a K_r^h-path of length t with endpoints y_1 and y_{t+1}.*

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(b) *Suppose that Q is a truncated K_r-path of length t with endsets X_1 and X_t. Then $Q^*(h)$ is a truncated K_r^h-path of length t. Further, let Q' denote the graph obtained from $Q^*(h)$ by adding a vertex x that is adjacent to every vertex in $X_1(h)$ and a vertex y that is adjacent to every vertex in $X_t(h)$. Then Q' is a K_r^h-path of length t with endpoints x and y.*

Proof. We first prove (a). Let $X_1, \ldots, X_t, y_1, \ldots, y_{t+1}$ be as in the definition of P and $X_1(h), \ldots,$ $X_t(h), Y_1, \ldots, Y_{t+1}$ be as in the definition of $P^*(h)$. Firstly, note that

$$|P^*(h)| = \left(\sum_{1 \leq i \leq t} |X_i(h)| \right) + 2 + \left(\sum_{2 \leq i \leq t-1} |Y_i| \right) + |Y_t| = h(r-1)t + 2 + h(t-2) + (2h-1) = hrt + 1.$$

So $P^*(h)$ satisfies condition (i) of the definition of a K_r^h-path of length t.

For each $2 \leq i \leq t-1$, define Y'_{i-1}, y'_i so that $|Y'_i| = h-1$ and $Y_i = Y'_{i-1} \cup \{y'_i\}$. Define Y'_{t-1}, Y'_t, y'_t so that $|Y'_{t-1}| = |Y'_t| = h-1$ and $Y_t = Y'_{t-1} \cup Y'_t \cup \{y'_t\}$. Set $X'_i := X_i(h) \cup Y'_i$ for all $1 \leq i \leq t$ and $y'_1 := y_1$, $y'_{t+1} := y_{t+1}$. Thus, $V(P^*(h)) = X'_1 \cup \cdots \cup X'_t \cup \{y'_1, y'_2, \ldots, y'_{t+1}\}$ where $|X'_i| = h(r-1) + h - 1 = hr - 1$ for all $1 \leq i \leq t$. So $P^*(h)$ satisfies condition (ii) of the definition of a K_r^h-path of length t.

Let $1 \leq i \leq t-1$. Recall that $X_i(h)$ induces a copy of K_{r-1}^h in $P^*(h)$. Since $Y'_i \subseteq Y_{i+1}$ and there are all possible edges between Y_{i+1} and $X_i(h)$ in $P^*(h)$, we have that X'_i induces a copy of the complete r-partite graph K^* with $r-1$ vertex classes of size h and one vertex class (namely Y'_i) of size $h-1$. A similar argument implies that X'_t induces a copy of K^* in $P^*(h)$. Recall that $y'_i \in Y_i$ for all $1 \leq i \leq t+1$. So by definition of $P^*(h)$, y'_i and y'_{i+1} send an edge to every vertex in $X_i(h)$ in $P^*(h)$ (for all $1 \leq i \leq t$). Altogether, this implies that $X'_i \cup \{y'_i\}$ and $X'_i \cup \{y'_{i+1}\}$ induce copies of K_r^h in $P^*(h)$. Thus, condition (iii) of the definition of a K_r^h-path of length t is satisfied. So (a) holds. It is easy to see that (b) follows from (a). □

We now apply Lemmas 8.7 and 8.9 to conclude that a graph G as in Theorem 4.3 contains many K_r^h-paths of a given length between any distinct $x, y \in V(G)$.

Lemma 8.10. *Let $\gamma', \gamma, \alpha, \eta > 0$ and $n, r, h \in \mathbb{N}$ such that $0 < 1/n \ll \gamma' \ll \gamma \ll \alpha \ll \eta \ll 1/r, 1/h$ where $r \geq 2$. Let G be a graph on n vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ where*

$$(14) \qquad\qquad d_i \geq (r-2)n/r + i + \eta n \quad \text{for all} \ \ i < n/r.$$

Set $s' := 2\lfloor 1/\gamma r - \alpha/\gamma \rfloor + 3$ and $s := hrs' - 1$. Given any distinct $x, y \in V(G)$ there are at least $\gamma' n^s$ s-sets $A \subseteq V(G) \setminus \{x, y\}$ so that $A \cup \{x, y\}$ spans a K_r^h-path of length s' in G with endpoints x and y. In particular, for each such set A, $G[A \cup \{x\}]$ and $G[A \cup \{y\}]$ contain perfect K_r^h-packings.

Proof. Let $\varepsilon, d, \gamma, \alpha, \eta > 0$ and $M' \in \mathbb{N}$ such that

$$0 < 1/M' \ll \varepsilon \ll d \ll \gamma \ll \alpha \ll \eta \ll 1/r, 1/h.$$

Set $t' := 2\lfloor 1/\gamma r - \alpha/\gamma \rfloor + 1$. Let M denote the integer obtained by applying Lemma 5.1 with parameters ε and M'. Let $\gamma' > 0$ and $n \in \mathbb{N}$ be sufficiently large such that

$$0 < 1/n \ll \gamma' \ll 1/M', 1/M.$$

Suppose that G is a graph on n vertices as in the statement of the lemma. Apply Lemma 5.1 with parameters ε, d and M' to G to obtain clusters $V_1, \ldots, V_k$, an exceptional set V_0 and a pure graph G'. Set $m := |V_1| = \cdots = |V_k|$. So $M' \leq k \leq M$ and $(1-\varepsilon)n/M \leq m \leq n/M'$. Let R denote the reduced graph of G with parameters ε, d and M'. Lemma 5.3 implies that R has degree sequence $d_{R,1} \leq d_{R,2} \leq \cdots \leq d_{R,k}$ such that

$$(15) \qquad\qquad d_{R,i} \geq (r-2)k/r + i + \eta k/2 \quad \text{for all} \ \ i < k/r.$$

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Consider distinct $x, y \in V(G)$. We may assume that $x, y \in V_0$. (Otherwise, we delete the at most two clusters containing x or y, and move their vertices into V_0. The same properties of R still hold, just with slightly perturbed parameters k and η.) Let $N_R(x)$ denote the set of clusters $V_i \in V(R)$ such that $d_G(x, V_i) \geq \alpha m$. Set $d_R(x) := |N_R(x)|$. Define $N_R(y)$ and $d_R(y)$ analogously. Note that

$$m d_R(x) + k \alpha m + |V_0| \geq d_G(x) \overset{(14)}{\geq} (r-2)n/r + \eta n.$$

So as $km \leq n$ and $|V_0| \leq \varepsilon n$,

$$(16) \qquad\qquad\qquad\qquad d_R(x) \geq (r-2)k/r + \eta k/2.$$

Similarly, $d_R(y) \geq (r-2)k/r + \eta k/2$.

Using (15) and (16), a simple greedy argument implies that there is a copy K_x of K_{r-1} in R so that:

- $V(K_x) \subseteq N_R(x)$;
- $V(K_x) = \{U_1, \ldots, U_{r-1}\}$ where $d_R(U_i) \geq (r-1)k/r + \eta k/3$ for all $1 \leq i \leq r-2$ and $d_R(U_{r-1}) \geq (r-2)k/r + \eta k/2$.

Note that the latter condition implies that there is a cluster $V_x \in V(R)$ such that $V(K_x) \cup \{V_x\}$ spans a copy of K_r in R. Similarly, there is a copy K_y of K_{r-1} in R and a cluster $V_y \in V(R)$ so that:

- $V(K_y) \subseteq N_R(y)$;
- $V(K_y) = \{W_1, \ldots, W_{r-1}\}$ where $d_R(W_i) \geq (r-1)k/r + \eta k/3$ for all $1 \leq i \leq r-2$ and $d_R(W_{r-1}) \geq (r-2)k/r + \eta k/2$;
- $V(K_y) \cup \{V_y\}$ is vertex-disjoint from $V(K_x) \cup \{V_x\}$ and spans a copy of K_r in R.

Remove the clusters in K_x and K_y from R and denote the resulting graph by R'. Set $k' := |R'|$. (15) implies that R' has degree sequence $d_{R',1} \leq d_{R',2} \leq \cdots \leq d_{R',k'}$ such that

$$(17) \qquad\qquad\qquad d_{R',i} \geq (r-2)k'/r + i + \eta k'/3 \quad \text{for all} \ i < k'/r.$$

Apply Lemma 8.7 with $R', \eta/3$ playing the roles of G, η. So R' contains a K_r-path P of length t' with endpoints V_x and V_y. (Here, we do not need to use the fact that there are 'many' such K_r-paths.) By the choice of V_x and V_y, $\mathcal{V} := V(P) \cup V(K_x) \cup V(K_y)$ spans a truncated K_r-path Q of length $t' + 2 = s'$ in R with endsets $X_1 := V(K_x)$ and $X_{s'} := V(K_y)$.

For each $V \in \mathcal{V}$ let V' denote a subset of V of size αm so that:

- $V' \subseteq N_G(x)$ if $V \in V(K_x)$;
- $V' \subseteq N_G(y)$ if $V \in V(K_y)$.

(So for $V \in V(P)$ the choice of V' is arbitrary.) Notice that we can choose such subsets V' since $V(K_x) \subseteq N_R(x)$ and $V(K_y) \subseteq N_R(y)$. Suppose that $V, W \in \mathcal{V}$ and $VW \in E(R)$. Then since $(V, W)_{G'}$ is ε-regular with density at least d, Lemma 5.4 implies that $(V', W')_{G'}$ is $\varepsilon^{1/2}$-regular with density at least $d/2$.

Set $R'' := R[\mathcal{V}]$. Since $Q \subseteq R''$, Fact 8.8 implies that $Q_1 := Q^*(h)$ is a subgraph of $R''(2h-1)$. Moreover, Lemma 8.9(b) implies that Q_1 is a truncated K_r^h-path of length s'. Let G'' denote the subgraph of G whose vertex set consists of all vertices v such that $v \in V'$ for some $V \in \mathcal{V}$ and so that $G''[V', W'] = G'[V', W']$ for all $V, W \in \mathcal{V}$. Apply Lemma 5.6 with $G'', R'', Q_1, 2h-1, \varepsilon^{1/2}, d/2, \alpha m$ playing the roles of $G, R, H, t, \varepsilon, d, m$. Note that $|Q_1| = s$ and set $\Delta := \Delta(Q_1)$ and $\varepsilon_0 := (d/2 - \varepsilon^{1/2})^\Delta/(2 + \Delta)$. As $\varepsilon \leq \varepsilon_0$ and $2h - 2 \leq \varepsilon_0 \alpha m$, Lemma 5.6 implies that there are at least

$$(\varepsilon_0 \alpha m)^s \geq (d^{\Delta+1} \alpha (1 - \varepsilon) n/M)^s \geq s! \gamma' n^s$$

copies of Q_1 in $G'' \subseteq G$.

Consider any copy Q_1' of Q_1 guaranteed by Lemma 5.6. Recall that Q has endsets $X_1 = V(K_x) \subseteq N_R(x)$ and $X_{s'} = V(K_y) \subseteq N_R(y)$. Lemma 5.6 guarantees that the vertices in the set $X_1(h) \subseteq$

$V(Q_1')$ corresponding to the $r-1$ elements of X_1 are embedded into sets $V' \subseteq N_G(x)$. Similarly, the vertices in the set $X_{s'}(h) \subseteq V(Q_1')$ corresponding to the $r-1$ elements of $X_{s'}$ are embedded into sets $V' \subseteq N_G(y)$. Thus, Lemma 8.9(b) implies that $V(Q_1') \cup \{x, y\}$ spans a K_r^h-path in G of length s' with endpoints x and y.

Since there are at least $s! \gamma' n^s$ such copies Q_1' of Q_1 in G'', there are at least $\gamma' n^s$ s-sets $A \subseteq V(G)$ such that $A = V(Q_1')$ for some such Q_1'. So indeed, there are at least $\gamma' n^s$ s-sets $A \subseteq V(G) \setminus \{x, y\}$ so that $A \cup \{x, y\}$ spans a K_r^h-path of length s' in G with endpoints x and y. □

We now deduce Theorem 4.3 from Lemmas 7.1 and 8.10.

Proof of Theorem 4.3. Define additional constants γ, α such that $\nu \ll \gamma \ll \alpha \ll \eta$. Set $s' := 2\lfloor 1/\gamma r - \alpha/\gamma \rfloor + 3$ and $s := hrs' - 1$ and define $\gamma' > 0$ so that $\nu = (\gamma'/2)^h/4$. In particular, $\gamma' \ll \gamma$.

Let G be as in the statement of the theorem. Lemma 8.10 implies that, for any $x, y \in V(G)$, there are at least $\gamma' n^s$ s-sets $X \subseteq V(G) \setminus \{x, y\}$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect K_r^h-packings and thus, perfect H-packings. Apply Lemma 7.1 with rs', γ' playing the roles of t, γ. Then $V(G)$ contains a set M so that

- $|M| \leq (\gamma'/2)^h n/4 = \nu n$;
- M is an H-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in h\mathbb{N}$ and $|W| \leq \nu^3 n \leq (\gamma'/2)^{2h} rn/32$.

□

Acknowledgements

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