Matchings in 3-uniform hypergraphs

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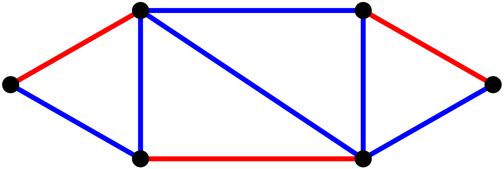
Joint work with Daniela Kühn and Deryk Osthus (University of Birmingham)

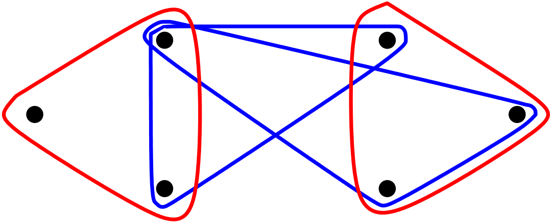
Characterising graphs with perfect matchings

- Hall's Theorem characterises all those bipartite graphs with perfect matchings.
- Tutte's Theorem characterises all those graphs with perfect matchings.



Perfect matchings in r-uniform hypergraphs

- for $r \geq 3$ decision problem NP-complete (Garey, Johnson '79)
- Natural to look for simple sufficient conditions



minimum ℓ-degree conditions

- H r-uniform, $1 \leq \ell < r$
- $d_H(v_1, \ldots, v_\ell) = \#$ edges containing $v_1, \ldots, v_\ell$
- minimum ℓ-degree $\delta_\ell(H) =$ minimum over all $d_H(v_1, \ldots, v_\ell)$
- $\delta_1(H) =$ minimum vertex degree
- $\delta_{r-1}(H) =$ minimum codegree

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Some previous results

- Rödl, Ruciński and Szemerédi '09 characterised the minimum codegree that ensures a perfect matching ($\delta_{r-1}(H) \approx |H|/2 \implies$ p.m.)

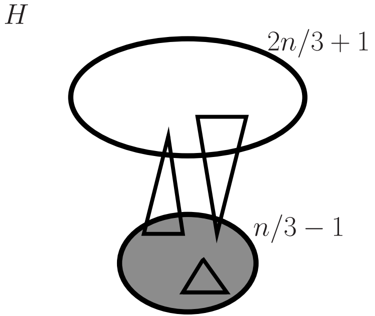
Theorem (Hán, Person and Schacht '09)

$\forall\ \varepsilon > 0\ \exists\ n_0 \in \mathbb{N}$ s.t if H 3-uniform, $n := |H| \geq n_0$ and

$$\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2} + \varepsilon n^2$$

then H contains a perfect matching.

- Result best possible up to error term εn^2



$\delta_1(H) = \binom{n-1}{2} - \binom{2n/3}{2}$

no perfect matching

Theorem (Kühn, Osthus and T.)

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- In fact, we prove a much stronger result...

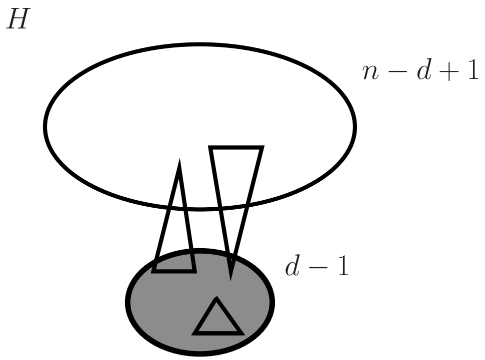
Theorem (Kühn, Osthus and T.)

$\exists\ n_0 \in \mathbb{N}$ s.t if H 3-uniform, $n := |H| \geq n_0$, $1 \leq d \leq n/3$ and

$$\delta_1(H) > \binom{n-1}{2} - \binom{n-d}{2}$$

then H contains a matching of size at least d.

- Bollobás, Daykin and Erdős '76 proved result in case when $d < n/54$
- Result is tight



$$\delta_1(H) = \binom{n-1}{2} - \binom{n-d}{2}$$

no d-matching

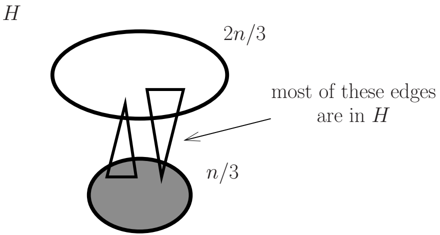
Outline of proof

Theorem

$\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2} \implies$ *perfect matching*

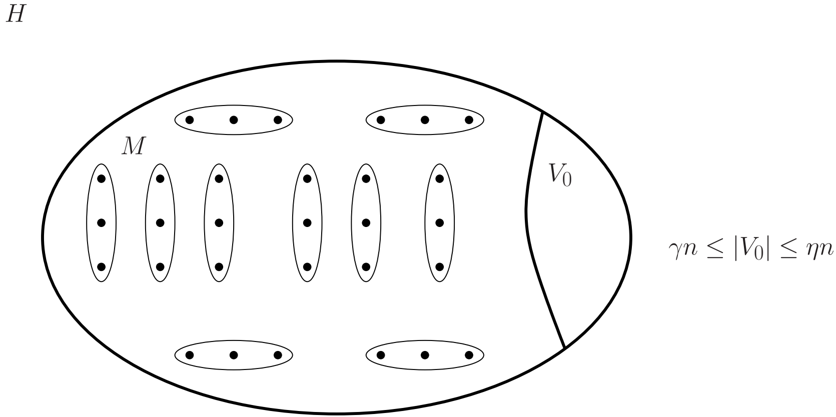
General strategy: show that either

1) H has a perfect matching or;
2) H is 'close' to the extremal example.

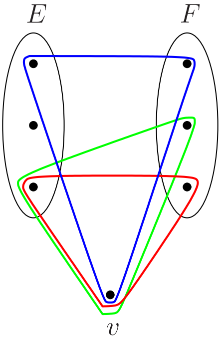


Then one can show that in 2) we must also have a perfect matching.

- M = largest matching in H
- Absorbing lemma (Hán, Person, Schacht) $\implies$ $(1-\eta)n \le |M| \le (1-\gamma)n$ where $0 < \gamma \ll \eta \ll 1$.



- Let $v \in V_0$ and $E, F \in M$
- Consider 'link graph' $L_v(EF)$



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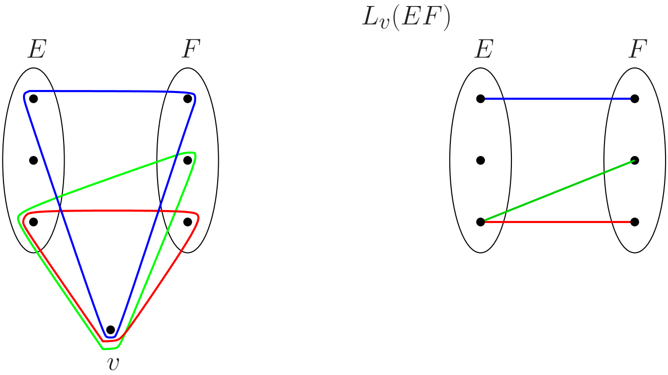

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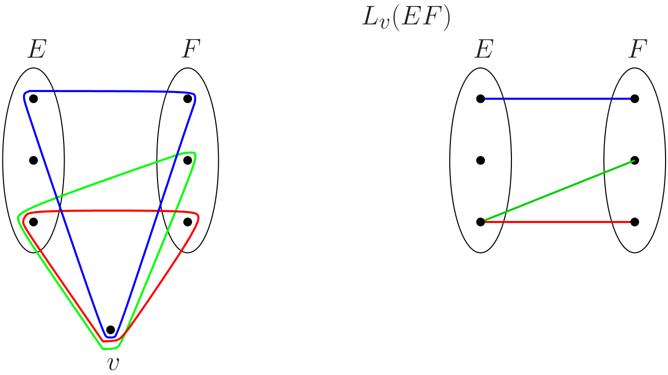


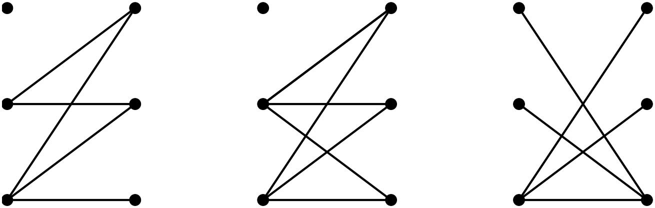
- $\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2} \approx \frac{5}{9}\binom{n}{2} \approx 5\binom{|M|}{2}$
- So 'on average' there are 5 edges in $L_v(EF)$

- We use the link graphs to build a picture as to what H looks like.

Fact

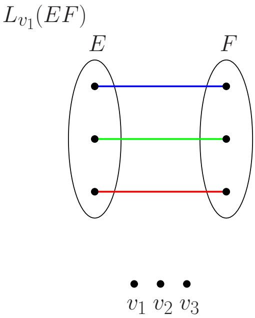
Let B be a balanced bipartite graph on 6 vertices. Then either

- *B contains a perfect matching;*
- *$B \cong B_{023}, B_{033}, B_{113}$ or;*
- *$e(B) \leq 4$.*



B_{023} B_{033} B_{113}

Suppose $\exists\ v_1, v_2, v_3 \in V_0$ and $E, F \in M$ s.t
$L_{v_1}(EF) = L_{v_2}(EF) = L_{v_3}(EF)$ and contains a p.m.



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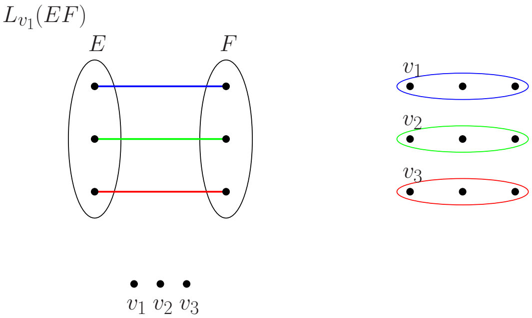

Replace E and F with these edges in M.
We get a larger matching, a contradiction.

So $\nexists$ $v_1, v_2, v_3 \in V_0$ and $E, F \in M$ s.t
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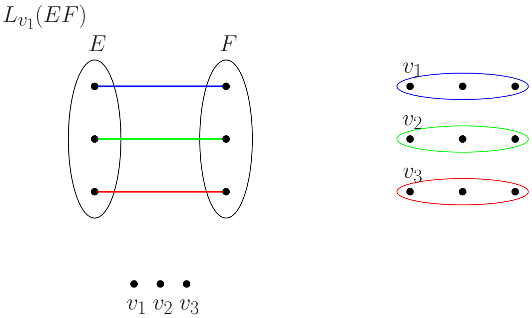

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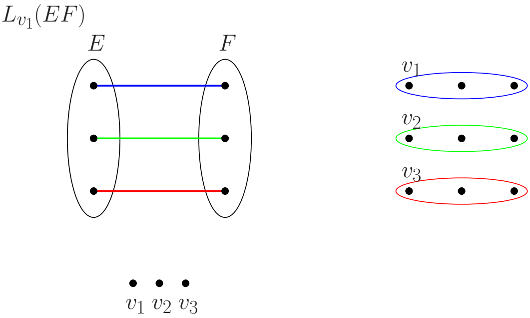
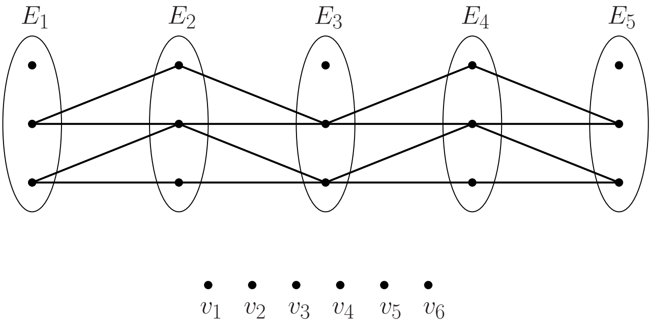


$\implies$ for most $v \in V_0$, most $L_v(EF)$ don't contain a p.m.

Suppose $\exists\ v_1, \ldots, v_6 \in V_0$ and $E_1, \ldots, E_5 \in M$ s.t:



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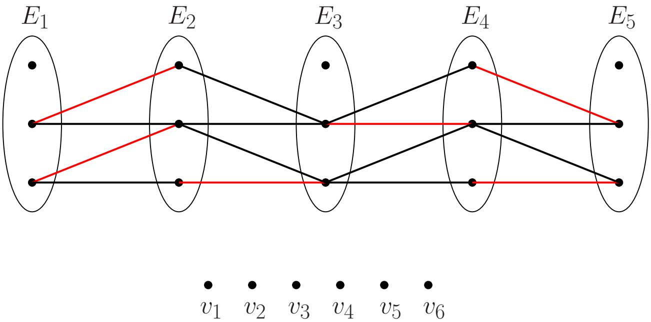

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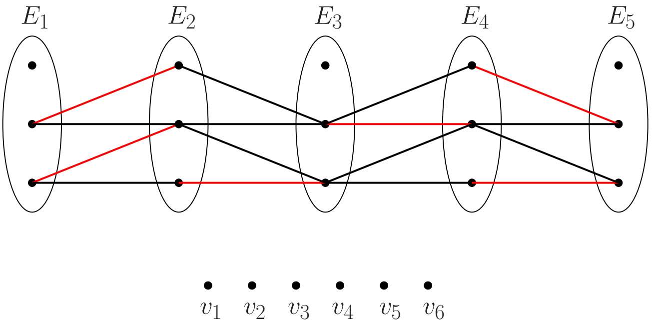
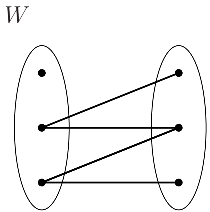
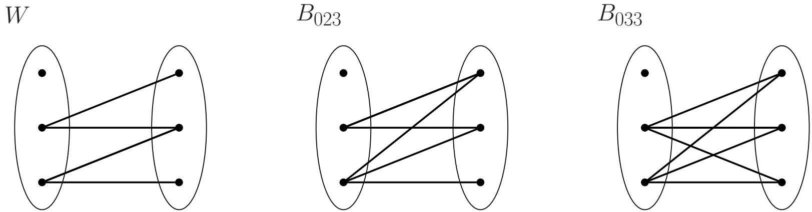


This 6-matching corresponds to a 6-matching in H.
Can extend M, a contradiction.

Each of the link graphs in the previous configuration were of the form:



Both B_{023} and B_{033} contain W.



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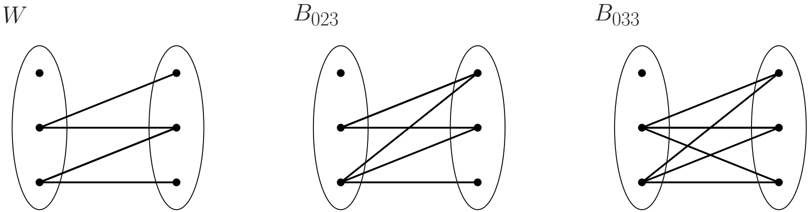

A 'bad' configuration occurs unless for most $v \in V_0$, most link graphs $L_v(EF) \ncong B_{023}, B_{033}$.

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So for most $v \in V_0$, most of the link graphs $L_v(EF)$ are s.t

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  - But recall 'typically' $L_v(EF)$ contains 5 edges.
  - So if 'many' $L_v(EF)$ contain ≤ 4 edges, 'many' contain ≥ 6 edges, a contradiction.

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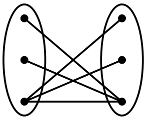
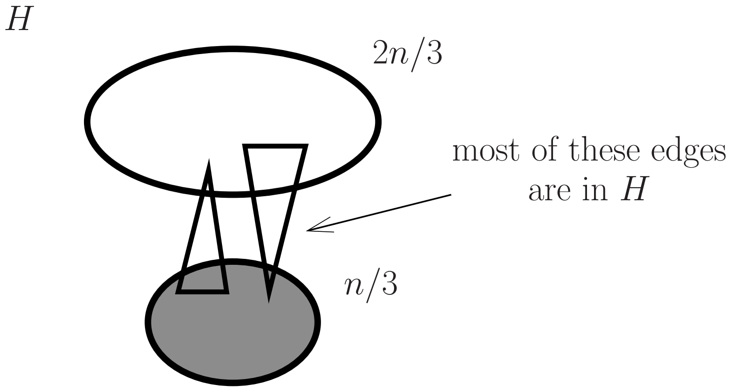
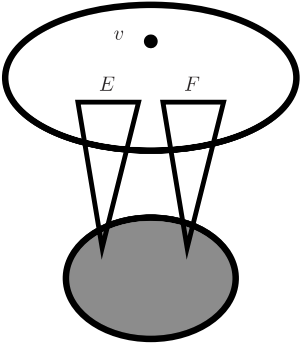
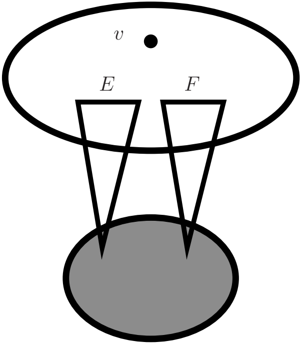
B_{113}



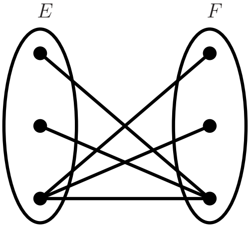


H
$2n/3$
most of these edges
are in H
$n/3$


v
E
F



$L_v(EF) \cong B_{113}$



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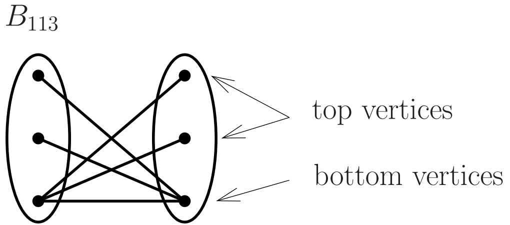

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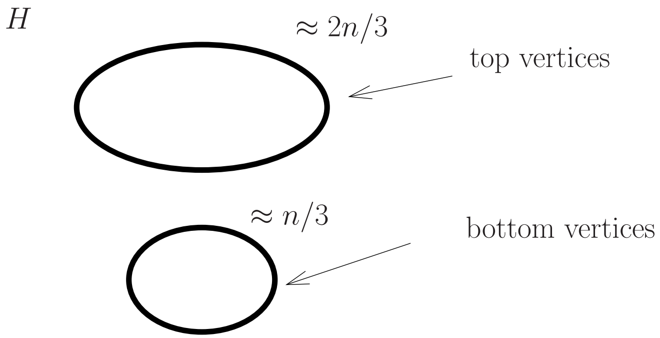

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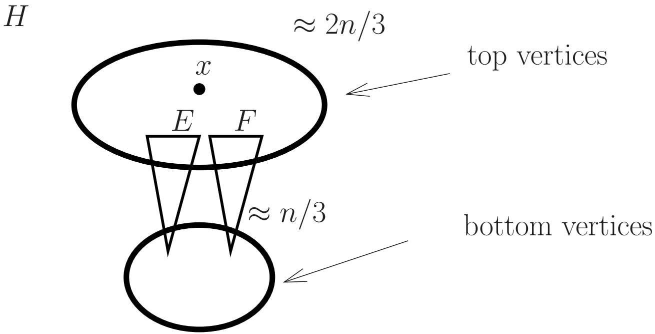

Similar arguments imply for each top vertex x, $L_x(EF) \cong B_{113}$ for most $E, F \in M$

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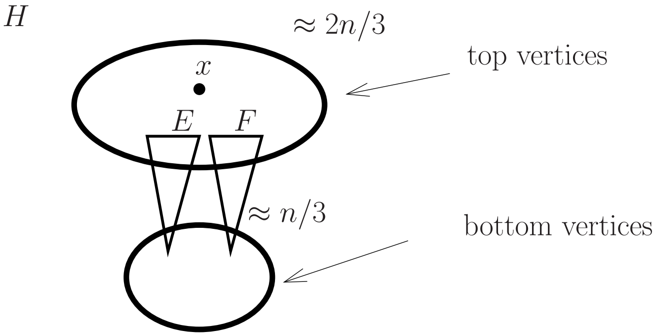

Similar arguments imply for each top vertex x, $L_x(EF) \cong B_{113}$ for most $E, F \in M \implies H$ 'close' to extremal example