Perfect matchings in hypergraphs

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Including joint work with Daniela Kühn, Deryk Osthus (University of Birmingham) and Yi Zhao (Georgia State)
Graph = collection of points (vertices) joined together by lines (edges)
Suppose $x$ vertex in graph $G$.

- Degree $d(x)$ of $x = \# \text{ of edges incident to } x$
- Minimum degree $\delta(G) = \text{minimum value of } d(x) \text{ amongst all } x \text{ in } G$

$\delta(G) = 2$

$d(x) = 3$
A hypergraph $H$ is a set of vertices $V(H)$ together with a collection $E(H)$ of subsets of $V(H)$ (known as edges).

For example, consider the hypergraph $H$ with

- $V(H) = \{1, 2, 3, 4, 5\}$;
- $E(H) = \{\{1, 2\}, \{1, 2, 3\}, \{2, 4\}, \{3, 4, 5\}\}$.

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Perfect matchings in hypergraphs
A *k*-uniform hypergraph $H$ is hypergraph whose edges contain *precisely* $k$ vertices.

- 2-uniform hypergraphs are graphs.
A matching in a hypergraph $H$ is a collection of vertex-disjoint edges.
A perfect matching is a matching covering all the vertices of $H$. 
A matching in a hypergraph $H$ is a collection of vertex-disjoint edges. A perfect matching is a matching covering all the vertices of $H$. 

\begin{center}
\begin{tikzpicture}
\vertex (a) at (0,0) [draw,shape=circle] ;
\vertex (b) at (1,0) [draw,shape=circle] ;
\vertex (c) at (2,0) [draw,shape=circle] ;
\vertex (d) at (1,-1) [draw,shape=circle] ;
\vertex (e) at (2,-1) [draw,shape=circle] ;
\vertex (f) at (1,-2) [draw,shape=circle] ;
\vertex (g) at (2,-2) [draw,shape=circle] ;
\draw (a) -- (b) -- (c) -- (d) -- (a);
\draw (d) -- (f) -- (g) -- (e) -- (d);
\draw[red] (b) -- (e);
\draw[red] (c) -- (f);
\end{tikzpicture}
\end{center}
A matching in a hypergraph $H$ is a collection of vertex-disjoint edges.
A perfect matching is a matching covering all the vertices of $H$. 

not a matching
A matching in a hypergraph $H$ is a collection of vertex-disjoint edges. A perfect matching is a matching covering all the vertices of $H$. 

![Diagram of a hypergraph with a perfect matching highlighted]
Theorem (Hall’s Marriage theorem)

A bipartite graph with equal size vertex classes $X, Y$

$G$ has perfect matching $\iff \forall S \subseteq X, |N(S)| \geq |S|$

($N(S) =$ set of vertices that receive at least one edge from $S$)
Theorem (Hall’s Marriage theorem)

A bipartite graph with equal size vertex classes $X, Y$

$G$ has perfect matching $\iff \forall S \subseteq X, |N(S)| \geq |S|$  

($N(S) = \text{set of vertices that receive at least one edge from } S$)
Theorem (Hall’s Marriage theorem)

A bipartite graph $G$ with equal size vertex classes $X$, $Y$ has a perfect matching if and only if for all $S \subseteq X$, $|N(S)| \geq |S|$(where $N(S)$ = set of vertices that receive at least one edge from $S$).

**Diagram:**

- $S$ on the left
- $N(S)$ on the right

No perfect matching.
Tutte’s Theorem characterises all those graphs with perfect matchings.
Perfect matchings in $k$-uniform hypergraphs

- for $k \geq 3$ decision problem NP-complete (Garey, Johnson ‘79)
- Natural to look for simple sufficient conditions
minimum $\ell$-degree conditions

- $H$ $k$-uniform hypergraph, $1 \leq \ell < k$
- $d_H(v_1, \ldots, v_\ell) = \# \text{ edges containing } v_1, \ldots, v_\ell$
- minimum $\ell$-degree $\delta_\ell(H) = \min \{d_H(v_1, \ldots, v_\ell)\}$
- $\delta_1(H) = \min \text{ vertex degree}$
- $\delta_{k-1}(H) = \min \text{ codegree}$
minimum $\ell$-degree conditions

- $H$ $k$-uniform hypergraph, $1 \leq \ell < k$
- $d_H(v_1, \ldots, v_\ell) = \#$ edges containing $v_1, \ldots, v_\ell$

- minimum $\ell$-degree $\delta_\ell(H) = \min$ over all $d_H(v_1, \ldots, v_\ell)$
- $\delta_1(H) =$ minimum vertex degree
- $\delta_{k-1}(H) =$ minimum codegree
minimum $\ell$-degree conditions

- $H$ $k$-uniform hypergraph, $1 \leq \ell < k$
- $d_H(v_1, \ldots, v_\ell) =$ \# edges containing $v_1, \ldots, v_\ell$
- minimum $\ell$-degree $\delta_\ell(H) =$ minimum over all $d_H(v_1, \ldots, v_\ell)$
- $\delta_1(H) =$ minimum vertex degree
- $\delta_{k-1}(H) =$ minimum codegree

$\delta_1(H) = 2$ and $\delta_2(H) = 1$
Theorem (Daykin and Häggkvist 1981)

Suppose $H$ $k$-uniform hypergraph, $|H| = n$ where $k|n$

$$\delta_1(H) \geq (1 - 1/k)\binom{n-1}{k-1} \implies \text{perfect matching}$$

- Condition on $\delta_1(H)$ believed to be far from best possible.

Theorem (Hán, Person and Schacht 2009)

$$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \text{ s.t if } H \text{ 3-uniform, } n := |H| \geq n_0 \text{ and}$$

$$\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2} + \varepsilon n^2$$

$$\implies \text{perfect matching}$$
Theorem (Daykin and Häggkvist 1981)

Suppose $H$ $k$-uniform hypergraph, $|H| = n$ where $k|n$

$$\delta_1(H) \geq (1 - 1/k)\left(\frac{n - 1}{k - 1}\right) \implies \text{perfect matching}$$

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$$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \text{ s.t if } H \text{ 3-uniform, } n := |H| \geq n_0 \text{ and}$$

$$\delta_1(H) > \left(\frac{n - 1}{2}\right) - \left(\frac{2n/3}{2}\right) + \varepsilon n^2$$

$$\implies \text{perfect matching}$$
Result best possible up to error term $\varepsilon n^2$

\[ \delta_1(H) = \left( \frac{n-1}{2} \right) - \left( \frac{2n}{3} \right) \]

no perfect matching
Theorem (Kühn, Osthus and T.)

\[\exists \; n_0 \in \mathbb{N} \; s.t \; if \; H \; 3\text{-uniform,} \; n := |H| \geq n_0 \; and\]

\[\delta_1(H) > \left( \frac{n - 1}{2} \right) - \left( \frac{2n/3}{2} \right)\]

then \(H\) contains a perfect matching.

- Independently, Khan proved this result.
- In fact, we prove a much stronger result\ldots
Theorem (Kühn, Osthus and T.)

\[ \exists \, n_0 \in \mathbb{N} \text{ s.t if } H \text{ 3-uniform, } n := |H| \geq n_0, 1 \leq d \leq n/3 \text{ and } \]

\[ \delta_1(H) > \binom{n - 1}{2} - \binom{n - d}{2} \]

then \( H \) contains a matching of size at least \( d \).

- Bollobás, Daykin and Erdős (1976) proved result in case when \( d < n/54 \)
- Result is tight
\[ \delta_1(H) = \left( \binom{n-1}{2} \right) - \left( \binom{n-d}{2} \right) \]

no \( d \)-matching
More recent developments

- Khan (2011+) determined the exact minimum vertex degree which forces a perfect matching in a 4-uniform hypergraph.
- No other exact vertex degree results are known. (Best known general bounds are due to Markström and Ruciński (2011).)
Theorem (Rödl, Ruciński and Szemerédi 2009)

Let $H$ be a $k$-uniform hypergraph, $|H| = n$ sufficiently large, $k | n$

$$\delta_{k-1}(H) \geq n/2 \implies \text{perfect matching}$$

In fact, they gave exact minimum codegree threshold that forces a perfect matching.
Edges hit $A$ in even no. of vertices

$|A|$ odd

$A$ $|A| \approx |B|$ $B$

$\delta_{k-1}(H) \approx |H|/2$ but no perfect matching
Theorem (Pikhurko 2008)

Suppose \( H \) is a \( k \)-uniform hypergraph on \( n \) vertices and \( k/2 \leq \ell \leq k - 1 \).

\[
\delta_\ell(H) \geq (1/2 + o(1)) \binom{n - \ell}{k - \ell} \implies \text{perfect matching}
\]

- Previous example shows result essentially best-possible.

Theorem (T. and Zhao)

We made Pikhurko’s result exact for \( k \)-uniform hypergraphs where 4 divides \( k \).
Theorem (Pikhurko 2008)

Suppose $H$ a $k$-uniform hypergraph on $n$ vertices and \( \frac{k}{2} \leq \ell \leq k - 1 \).

\[
\delta_{\ell}(H) \geq \left(\frac{1}{2} + o(1)\right) \binom{n - \ell}{k - \ell} \implies \text{perfect matching}
\]

- Previous example shows result essentially best-possible.

Theorem (T. and Zhao)

We have made Pikhurko’s result exact for all $k$.

- Our result implies the theorem of Rödl, Ruciński and Szemerédi.
Theorem (Kühn, Osthus and T.)

\[ \exists \ n_0 \in \mathbb{N} \ s.t \ if \ H \ 3\text{-}uniform, \ n := |H| \geq n_0 \ and \]

\[ \delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2} \]

then \( H \) contains a perfect matching.
Outline of proof

**Theorem**

\[ \delta_1(H) > \left( \frac{n-1}{2} \right) - \left( \frac{2n}{3} \right) \implies \text{perfect matching} \]

General strategy: show that either

1) \( H \) has a perfect matching or;
2) \( H \) is ‘close’ to the extremal example.

Then one can show that in 2) we must also have a perfect matching.
• $M$ = largest matching in $H$

• Absorbing lemma (Hán, Person, Schacht) $\implies$

$(1 - \eta)n \leq |M| \leq (1 - \gamma)n$ where $0 < \gamma \ll \eta \ll 1$. 

\[\gamma n \leq |V_0| \leq \eta n\]
Let \( v \in V_0 \) and \( E, F \in M \). Consider ‘link graph’ \( L_v(EF) \).
Let $v \in V_0$ and $E, F \in M$

Consider ‘link graph’ $L_v(EF)$

\[ \delta_1(H) > \left( n - \frac{1}{2} \right) - \left( \frac{2}{3} n^2 \right) \approx 5 n^2 \approx 5 |M|^2 \]
Let $v \in V_0$ and $E, F \in M$

Consider ‘link graph’ $L_v(EF)$

\[ \delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2} \approx \frac{5}{9} \binom{n}{2} \approx 5 \binom{|M|}{2} \]

So ‘on average’ there are 5 edges in $L_v(EF)$
- We use the link graphs to build a picture as to what $H$ looks like.

**Fact**

Let $B$ be a balanced bipartite graph on 6 vertices. Then either

- $B$ contains a perfect matching;
- $B \cong B_{023}, B_{033}, B_{113}$ or;
- $e(B) \leq 4$.

![Diagram of graphs $B_{023}, B_{033}, B_{113}$](image)
Suppose \( \exists \ v_1, v_2, v_3 \in V_0 \) and \( E, F \in M \) s.t \( L_{v_1}(EF) = L_{v_2}(EF) = L_{v_3}(EF) \) and contains a p.m.

\[
L_{v_1}(EF)
\]

Replace \( E \) and \( F \) with these edges in \( M \).

We get a larger matching, a contradiction.
Suppose \( \exists \ v_1, v_2, v_3 \in V_0 \) and \( E, F \in M \) s.t \( L_{v_1}(EF) = L_{v_2}(EF) = L_{v_3}(EF) \) and contains a p.m.

Replace \( E \) and \( F \) with these edges in \( M \).
We get a larger matching, a contradiction.
So \( \not\exists v_1, v_2, v_3 \in V_0 \) and \( E, F \in M \) s.t. 
\( L_{v_1}(EF) = L_{v_2}(EF) = L_{v_3}(EF) \) and contains a p.m.

Replace \( E \) and \( F \) with these edges in \( M \).
We get a larger matching, a contradiction.
So \( \not\exists v_1, v_2, v_3 \in V_0 \) and \( E, F \in M \) s.t \( L_{v_1}(EF) = L_{v_2}(EF) = L_{v_3}(EF) \) and contains a p.m.

\[
L_{v_1}(EF)
\]

\[
E \quad F
\]

\[
\bullet \quad \bullet \quad \bullet
\]

\[
v_1 \quad v_2 \quad v_3
\]

\( \Rightarrow \) for most \( v \in V_0 \), most \( L_v(EF) \) don’t contain a p.m.
Suppose $\exists v_1, \ldots, v_6 \in V_0$ and $E_1, \ldots, E_5 \in M$ s.t:

This 6-matching corresponds to a 6-matching in $H$. Can extend $M$, a contradiction.
Suppose $\exists v_1, \ldots, v_6 \in V_0$ and $E_1, \ldots, E_5 \in M$ s.t:

This 6-matching corresponds to a 6-matching in $H$. Can extend $M$, a contradiction.
Suppose $\exists v_1, \ldots, v_6 \in V_0$ and $E_1, \ldots, E_5 \in M$ s.t:

This 6-matching corresponds to a 6-matching in $H$. Can extend $M$, a contradiction.
Each of the link graphs in the previous configuration were of the form:

\[ W \]

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]
Both $B_{023}$ and $B_{033}$ contain $W$. 

\begin{align*}
W \\
\begin{tikzpicture}[scale=0.5]
  \node (a) at (-1,0) [circle,fill] {};
  \node (b) at (1,0) [circle,fill] {};
  \node (c) at (0,1) [circle,fill] {};
  \node (d) at (0,-1) [circle,fill] {};
  \draw (a) -- (b);
  \draw (a) -- (c);
  \draw (a) -- (d);
  \draw (b) -- (c);
  \draw (b) -- (d);
  \draw (c) -- (d);
\end{tikzpicture}
\end{align*}

$B_{023}$

$B_{033}$
Both $B_{023}$ and $B_{033}$ contain $W$.

A ‘bad’ configuration occurs unless for most $v \in V_0$, most link graphs $L_v(\text{EF}) \not\cong B_{023}, B_{033}$. 
Fact

Let $B$ be a balanced bipartite graph on 6 vertices. Then either

- $B$ contains a perfect matching;
- $B \cong B_{023}, B_{033}, B_{113}$ or;
- $e(B) \leq 4.$

So for most $v \in V_0$, most of the link graphs $L_v(EF)$ are s.t

- $L_v(EF) \cong B_{113}$ or
- $e(L_v(EF)) \leq 4$
Fact

Let $B$ be a balanced bipartite graph on 6 vertices. Then either

- $B$ contains a perfect matching;
- $B \cong B_{023}, B_{033}, B_{113}$ or;
- $e(B) \leq 4$.

So for most $v \in V_0$, most of the link graphs $L_v(EF)$ are s.t

- $L_v(EF) \cong B_{113}$ or
- $e(L_v(EF)) \leq 4$

- But recall ‘typically’ $L_v(EF)$ contains 5 edges.
- So if ‘many’ $L_v(EF)$ contain $\leq 4$ edges, ‘many’ contain $\geq 6$ edges, a contradiction.
Fact

Let $B$ be a balanced bipartite graph on 6 vertices. Then either

- $B$ contains a perfect matching;
- $B \cong B_{023}, B_{033}, B_{113}$ or;
- $e(B) \leq 4$.

So for most $v \in V_0$, most of the link graphs $L_v(\mathcal{E}F)$ are s.t

- $L_v(\mathcal{E}F) \cong B_{113}$
most of these edges are in $H$
Perfect matchings in hypergraphs
$L_v(EF) \cong B_{113}$
For most $v \in V_0$, most of the link graphs $L_v(EF)$ are s.t

- $L_v(EF) \cong B_{113}$

$B_{113}$

\begin{tikzpicture}[scale=0.8]
  
  % Top vertices
  % Bottom vertices
  % Arrows from top to bottom

  % Diagram

\end{tikzpicture}
For most $v \in V_0$, most of the link graphs $L_v(EF)$ are s.t
- $L_v(EF) \cong B_{113}$

$H \approx 2n/3$

$\approx n/3$

top vertices

bottom vertices
For most $v \in V_0$, most of the link graphs $L_v(EF)$ are s.t
• $L_v(EF) \cong B_{113}$

\[ H \approx 2n/3 \approx n/3 \]

top vertices

don't vertices

Similar arguments imply for each top vertex $x$, $L_x(EF) \cong B_{113}$ for most $E, F \in M$
For most $v \in V_0$, most of the link graphs $L_v(EF)$ are s.t
• $L_v(EF) \cong B_{113}$

Similar arguments imply for each top vertex $x$, $L_x(EF) \cong B_{113}$ for most $E, F \in M \implies H$ ‘close’ to extremal example
Summary of ideas

- Split proof into non-extremal and extremal case analysis.
- Applying the absorbing method so we only need to look for an almost perfect matching.
- Analyse the link graphs to obtain information about the hypergraph.
Open problems

- Characterise the minimum vertex degree that forces a perfect matching in a $k$-uniform hypergraph for $k \geq 5$.
- What about minimum $\ell$-degree conditions for $k$-uniform $H$ where $1 < \ell < k/2$? (Alon, Frankl, Huang, Rödl, Ruciński, Sudakov have some such results.)
- Establish $k$-partite analogues of the known results.