Hamilton cycles in directed graphs

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Joint work with Daniela Kühn and Deryk Osthus (University of Birmingham)

Theorem (Dirac, 1952)

Graph G of order $n \geq 3$ and $\delta(G) \geq n/2 \implies G$ Hamiltonian.

Theorem (Ghouila-Houri, 1966)

Digraph G of order $n \geq 2$ with $\delta^+(G), \delta^-(G) \geq n/2 \implies G$ Hamiltonian.

Theorem (Chvátal, 1972)

Let G be a graph with degree sequence $d_1 \leq \cdots \leq d_n$. G has a Hamilton cycle if

$$d_i \geq i+1 \ \text{ or } \ d_{n-i} \geq n-i \quad \forall\, i < n/2.$$

- The bound on the degrees in Chvátal's theorem is best possible.

degree sequence: $\underbrace{k,\dots,k}_{k \text{ times}}, \underbrace{n-k-1,\dots,n-k-1}_{n-2k \text{ times}}, \underbrace{n-1,\dots,n-1}_{k \text{ times}}$

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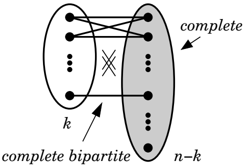

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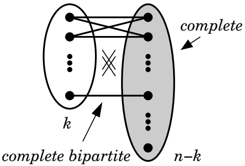

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- Nash-Williams raised the question of a digraph analogue of Chvátal's theorem.

Conjecture (Nash-Williams, 1975)

G strongly connected digraph whose out- and indegree sequences $d_1^+ \leq \cdots \leq d_n^+$ and $d_1^- \leq \cdots \leq d_n^-$ satisfy

(i) $d_i^+ \geq i+1 \quad or \quad d_{n-i}^- \geq n-i \quad \forall i < n/2,$

(ii) $d_i^- \geq i+1 \quad or \quad d_{n-i}^+ \geq n-i \quad \forall i < n/2.$

Then G contains a Hamilton cycle.

- If true, the conjecture is much stronger than Ghouila-Houri's theorem.

- If the Nash-Williams conjecture is true then it is best possible.

N-W conjecture

(i) $d_i^+ \geq i+1$ *or* $d_{n-i}^- \geq n-i$

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$|K'| = n-k-2$ and $|K| = k-1$

outdegree sequence: $\underbrace{k-1,\dots,k-1}_{k-1 \text{ times}}, k, k, \underbrace{n-1,\dots,n-1}_{n-k-1 \text{ times}}$

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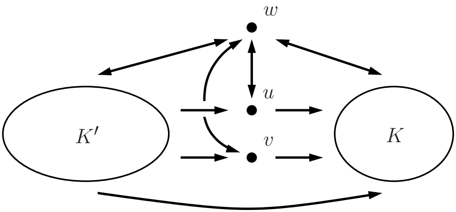

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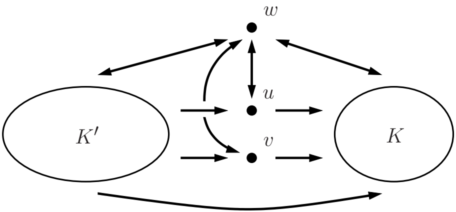

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Theorem (Kühn, Osthus, T.)

$\forall\ \eta > 0\ \exists\, n_0 = n_0(\eta)$ s.t. if G is a digraph on $n \geq n_0$ vertices s.t.

- $d_i^+ \geq i + \eta n$ *or* $d^-_{n-i-\eta n} \geq n - i \quad \forall i < n/2$,
- $d_i^- \geq i + \eta n$ *or* $d^+_{n-i-\eta n} \geq n - i \quad \forall i < n/2$,

then G contains a Hamilton cycle.

Corollary

The conditions in the above theorem imply G is pancyclic. That is, G contains a cycle of length $i \ \ \forall\, 2 \leq i \leq |G|$.

- The following result is an immediate corollary of Chvátal's theorem.

Theorem (Pósa, 1962)

Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq \cdots \leq d_n$. G has a Hamilton cycle if

- $d_i \geq i + 1 \ \ \forall\, i < (n-1)/2$

and if additionally $d_{\lceil n/2 \rceil} \geq \lceil n/2 \rceil$ when n is odd.

- Pósa's theorem is much stronger than Dirac's theorem.

- The following conjecture is a digraph analogue of Pósa's theorem.

Conjecture (Nash-Williams, 1968)

Let G be a digraph on $n \geq 3$ vertices s.t.

- $d_i^+, d_i^- \geq i+1 \ \ \forall\, i < (n-1)/2$

and s.t. $d^+_{\lceil n/2 \rceil}, d^-_{\lceil n/2 \rceil} \geq \lceil n/2 \rceil$ when n is odd. Then G contains a Hamilton cycle.

- If true, this conjecture is much stronger than Ghoulia-Houri's theorem.

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$\forall\ \eta > 0\ \exists\, n_0 = n_0(\eta)$ *s.t. if G is a digraph on $n \geq n_0$ vertices s.t.*

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- This theorem implies an approximate version of the second Nash-Williams conjecture.

Corollary

$\forall\ \eta > 0\ \exists\, n_0 = n_0(\eta)$ *s.t. every digraph G on $n \geq n_0$ vertices with*

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Recent developments...

- Christofides, Keevash, Kühn and Osthus gave a polynomial time algorithm which finds a Hamilton cycle in those digraphs considered in our result.
- They also showed one can relax the condition in our result to
  - $d_i^+ \geq \min\{i + \eta n, n/2\}$ or $d^-_{n-i-\eta n} \geq n - i \quad \forall i < n/2,$
  - $d_i^- \geq \min\{i + \eta n, n/2\}$ or $d^+_{n-i-\eta n} \geq n - i \quad \forall i < n/2,.$

Hamilton cycles in oriented graphs

Theorem (Keevash, Kühn, Osthus, 2009)

$\exists n_0$ s.t. every oriented graph G on $n \geq n_0$ vertices with

$$\delta^+(G), \delta^-(G) \geq \frac{3n-4}{8}$$

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Can we strengthen this theorem in the same way as Pósa's theorem strengthens Dirac's theorem?

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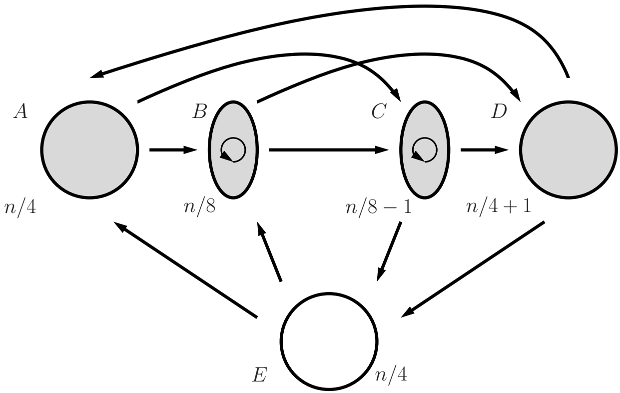

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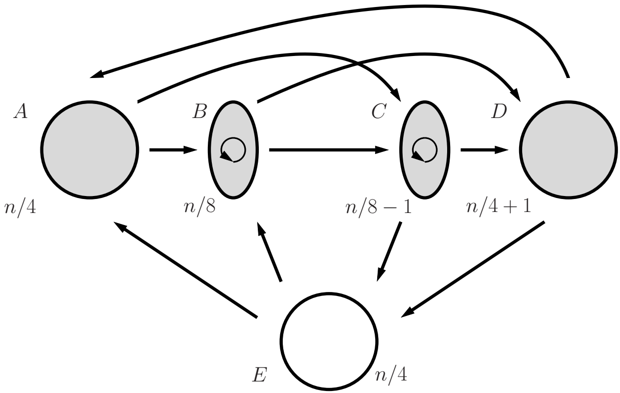

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Robustly expanding digraphs

- To prove the approximate version of the Nash-Williams conjecture we in fact showed that...

  "Robustly expanding digraphs of large enough minimum degree are Hamiltonian"

- This implies approximate version of the theorem of Keevash, Kühn and Osthus.
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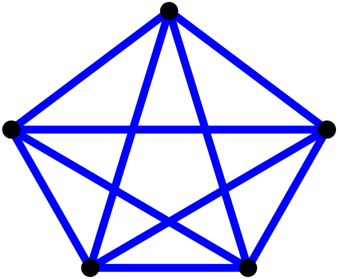
Hamilton decomposition of a graph or digraph G:
set of edge-disjoint Hamilton cycles covering $E(G)$

Theorem (Walecki 1892)

K_n has a Hamilton decomposition $\iff$ n odd



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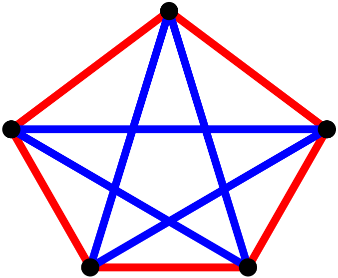

Hamilton decompositions in digraphs

Theorem (Tillson 1980)

Complete digraph on n vertices has Hamilton decomposition $\iff \quad n \neq 4, 6.$

- Tournament: orientation of a complete graph
- Tournament on n vertices is regular if every vertex has equal in- and outdegree (i.e. (n-1)/2)

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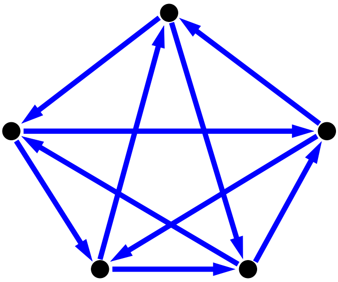

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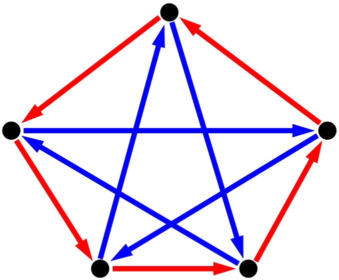

Conjecture (Kelly 1968)

All regular tournaments have Hamilton decompositions.

- There have been several partial results in this direction.

Keevash, Kühn and Osthus: G oriented graph

$$\delta^+(G), \delta^-(G) \geq (3|G|-4)/8 \implies H.C.$$

So regular tournament G contains $\geq |G|/8$ edge-disjoint Hamilton cycles

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Conjecture (Thomassen 1982)

Suppose G regular tournament on n vertices and $A \subseteq E(G)$ s.t $|A| < (n-1)/2$. Then $G - A$ contains a Hamilton cycle.

Theorem (Kühn, Osthus, T.)

Conjecture true for large n

Theorem (Kühn, Osthus, T.)

$\forall\ \eta > 0\ \ \exists\ n_0$ s.t all regular tournaments on $n \geq n_0$ vertices contain $\geq (1/2 - \eta)n$ edge-disjoint Hamilton cycles.

- In fact, result holds for 'almost regular' tournaments.

Naïve approach to theorem

- Remove a γn-regular oriented spanning subgraph H from G ($\gamma \ll 1$).
- Decompose rest of G into 1-factors $F_1, \dots, F_s$.
- Use edges from H to piece together each F_i into Hamilton cycles.
- Need F_i to contain few cycles (a result of Frieze and Krivelevich implies this).
- If H 'quasi-random' could use it to merge cycles using method of 'rotation-extension'.
- Problem: can't necessarily find such H.
- But this approach is a useful starting point.

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Open problems

- Kelly's conjecture!

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'Almost regular' oriented graphs G with $\delta^+(G), \delta^-(G) \geq (3/8 + o(1))|G|$ can be 'almost decomposed' into edge-disjoint Hamilton cycles.

Question

What minimum degree ensures a regular oriented graph has a Hamilton decomposition?

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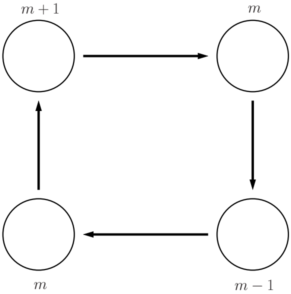

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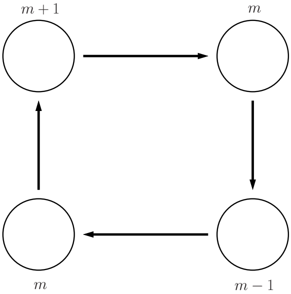

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