# An improved lower bound for Folkman's theorem

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# Background

**Theorem (Schur 1916)**

$\forall\ r \in \mathbb{N}\ \exists\ n \in \mathbb{N}$ *s.t. whenever* $[n] := \{1, \ldots, n\}$ *is r-coloured* $\implies$ *monochromatic solution to* $x + y = z$.

Given finite $A \subseteq \mathbb{N}$,

$$S(A) := \left\{ \sum_{x \in B} x \ :\ B \subseteq A,\ B \neq \emptyset \right\}.$$

**Theorem (Folkman 1960s)**

$\forall\ r, k \in \mathbb{N}\ \exists\ n = F(k, r)$ *s.t. whenever* $[n]$ *is r-coloured* $\implies\ \exists$ *a set* $A \subset [n]$ *s.t.*

- $|A| = k$;
- $S(A) \subseteq [n]$ *is monochromatic*

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# Upper bound to Folkman's theorem

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Set

$$F(k) := F(k, 2).$$

**Theorem (Taylor 1980)**

$F(k) \leq k^{3^{k^{\cdot^{\cdot^{\cdot^{3}}}}}}$ *(height $2k$)*

# Lower bounds for Folkman's theorem

**Theorem (Erdős and Spencer 1989)**

$F(k) \geq 2^{(ck^2)/\log k}$ *for some absolute constant* $c > 0$.

**Question**

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**Theorem (B.E.N.T.W 2017)**

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# Proof idea

Given any $k$-set $A \subseteq \mathbb{N}$,

$$\frac{k(k+1)}{2} \leq |S(A)| \leq 2^k - 1.$$

For example, if $A = \{1, \ldots, k\}$ then $|S(A)| = \frac{k(k+1)}{2}$.
If $A = \{2^1, \ldots, 2^k\}$ then $|S(A)| = 2^k - 1$.

- If colouring in a 'random-like' way, want to deal with $A$ s.t. $S(A)$ is 'large'.

**Theorem (B.E.N.T.W 2017)**

$F(k) \geq 2^{(2^{k-1})/k}$

**Proof of theorem:**

- Let $n := \lfloor 2^{(2^{k-1})/k} \rfloor$
- 2-colour $[n]$ s.t.:
  (1) **Randomly** red/blue colour the **odds**
  (2) Extend to a 2-colouring for $[n]$ s.t. the colour of $x$ different to $2x$ for all $x$.

e.g. 3 red, 6 blue, 12 red...

- Fix $A \subseteq [n]$ of size $k$ with $S(A) \subseteq [n]$.

**Claim**

$\mathbb{P}(S(A) \ monochromatic) \leq 2^{1-2^{k-1}}$

**Proof:**

**Case 1:** $|S(A)| \leq 2^k - 2$

$\implies \exists\ B_1 \neq B_2 \subseteq A$ s.t. $\sum_{x \in B_1} x = \sum_{x \in B_2} x$

$\implies$ May assume $B_1 \cap B_2 = \emptyset$

$\implies \exists$ 2 elements $y$ and $2y$ in $S(A)$

$\implies S(A)$ **not** monochromatic,

i.e. $\mathbb{P}(S(A)$ monochromatic$) = 0$

**Case 2:** $|S(A)| = 2^k - 1$

- $\forall$ odd $m \in \mathbb{N}$, let $G_m := \{m, 2m, 4m, \ldots\} \cap [n]$
- Note $[n] = G_1 \cup G_3 \cup G_5 \cup \ldots$

**Subclaim:** $S(A)$ intersects $\geq 2^{k-1}$ of the $G_m$
To prove subclaim, first suppose there is an **odd** $r \in A$. Then

$$|S(A \setminus \{r\})| = 2^{k-1} - 1$$

Further, $\forall\, x \in S(A \setminus \{r\})$,

$$\text{either } x \text{ odd} \quad \text{or} \quad x + r \text{ odd}$$

So at least $(2^{k-1} - 1) + 1 = 2^{k-1}$ odds in $S(A) \implies$ subclaim.

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**Case 2:** $|S(A)| = 2^k - 1$

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**Subclaim:** $S(A)$ intersects $\geq 2^{k-1}$ of the $G_m$

- For each $x \in G_m$, colour of $x$ **independent** of colour of $y \in G_{m^*}$ for $m \neq m^*$.
- So

$$\mathbb{P}(S(A) \text{ is monochromatic}) \leq \left(\frac{1}{2}\right)^{2^{k-1}} \times 2 = 2^{1-2^{k-1}}$$

□

**Claim**

$\mathbb{P}(S(A) \text{ monochromatic}) \leq 2^{1-2^{k-1}}$

Define

$$X := \# \text{ sets s.t. } |A| = k \text{ and } S(A) \text{ monochromatic}$$

$$\mathbb{E}(X) \leq \binom{n}{k} 2^{1-2^{k-1}} < 1$$

$\implies \exists$ a 2-colouring of $[n]$ where $X = 0$.

So

$$F(k) \geq n = \lfloor 2^{(2^{k-1})/k} \rfloor$$

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