

PROOF OF THE 1-FACTORIZATION AND HAMILTON DECOMPOSITION CONJECTURES III: APPROXIMATE DECOMPOSITIONS

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ABSTRACT. In a sequence of four papers, we prove the following results (via a unified approach) for all sufficiently large n :

- (i) [*1-factorization conjecture*] Suppose that n is even and $D \geq 2\lceil n/4 \rceil - 1$. Then every D -regular graph G on n vertices has a decomposition into perfect matchings. Equivalently, $\chi'(G) = D$.
- (ii) [*Hamilton decomposition conjecture*] Suppose that $D \geq \lfloor n/2 \rfloor$. Then every D -regular graph G on n vertices has a decomposition into Hamilton cycles and at most one perfect matching.
- (iii) We prove an optimal result on the number of edge-disjoint Hamilton cycles in a graph of given minimum degree.

According to Dirac, (i) was first raised in the 1950s. (ii) and (iii) answer questions of Nash-Williams from 1970. The above bounds are best possible. In the current paper, we show the following: suppose that G is close to a complete balanced bipartite graph or to the union of two cliques of equal size. If we are given a suitable set of path systems which cover a set of ‘exceptional’ vertices and edges of G , then we can extend these path systems into an approximate decomposition of G into Hamilton cycles (or perfect matchings if appropriate).

1. INTRODUCTION

1.1. Background and results. In a sequence of four papers, we develop a unified approach to prove the following results on Hamilton decompositions and 1-factorizations. The first of these results confirms the so-called 1-factorization conjecture for all sufficiently large graphs. (A *1-factorization* of a graph G consists of a set of edge-disjoint perfect matchings covering all edges of G .) This conjecture was first stated explicitly by Chetwynd and Hilton [2, 3]. However, they wrote that according to Dirac, it was already discussed in the 1950s.

Theorem 1.1. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, D \in \mathbb{N}$ be such that $n \geq n_0$ is even and $D \geq 2\lceil n/4 \rceil - 1$. Then every D -regular graph G on n vertices has a 1-factorization. Equivalently, $\chi'(G) = D$.*

Date: January 1, 2014.

The research leading to these results was partially supported by the European Research Council under the European Union’s Seventh Framework Programme (FP/2007–2013) / ERC Grant Agreement no. 258345 (B. Csaba, D. Kühn and A. Lo), 306349 (D. Osthus) and 259385 (A. Treglown). The research was also partially supported by the EPSRC, grant no. EP/J008087/1 (D. Kühn and D. Osthus).

The bound on the degree in Theorem 1.1 is best possible. Nash-Williams [13, 14] raised the related problem of finding a Hamilton decomposition in an even-regular graph. Here a decomposition of an (even-regular) graph G into Hamilton cycles consists of a set of edge-disjoint Hamilton cycles covering all edges of G . If G is a regular graph of odd degree, it is natural to ask for a perfect matching in G together with a decomposition of the remaining edges into Hamilton cycles.

Theorem 1.2. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, D \in \mathbb{N}$ be such that $n \geq n_0$ and $D \geq \lfloor n/2 \rfloor$. Then every D -regular graph G on n vertices has a decomposition into Hamilton cycles and at most one perfect matching.*

Again, the bound on the degree in Theorem 1.2 is best possible. In particular, the theorem settles the problem of Nash-Williams for all sufficiently large graphs.

Finally (in combination with [6]), we also prove an optimal result on the number of edge-disjoint Hamilton cycles one can guarantee in a graph of given minimum degree, which (as a special case) answers another question of Nash-Williams. For a detailed discussion of the results and their background we refer to [9].

1.2. Overall structure of the argument. For all of our main results, we split the argument according to the structure of the graph G under consideration:

- (i) G is close to the complete balanced bipartite graph $K_{n/2, n/2}$;
- (ii) G is close to the union of two disjoint copies of a clique $K_{n/2}$;
- (iii) G is a ‘robust expander’.

Roughly speaking, G is a robust expander if for every set S of vertices, the neighbourhood of S is at least a little larger than $|S|$, even if we delete a small proportion of the vertices and edges of G (see Section 4.3). The main result of [10] states that every dense regular robust expander has a Hamilton decomposition. This immediately implies Theorems 1.1 and 1.2 in Case (iii). Case (i) is proved in [4]. Case (ii) is proved in [7, 9]. Moreover, [9] also includes a more detailed discussion of the overall structure of the proof.

1.3. Contribution of the current paper. The arguments in [4, 9] make use of an ‘approximate decomposition’ result, which is proved in the current paper: suppose that G is as in Case (i) or (ii). Suppose also that we are given a suitable set of path systems which cover a certain set of ‘exceptional’ vertices and edges of G , then we can extend these path systems into an approximate decomposition of G into Hamilton cycles (or perfect matchings if appropriate). The precise statements are given in Lemma 3.2 for Case (i) and Lemma 3.1 for Case (ii).

Roughly speaking, the strategy for the proof of Lemma 3.1 is as follows. By definition, G contains disjoint sets A and B of size about $n/2$ which induce two almost complete graphs. First we reduce the problem of finding an approximate decomposition of G to that of finding approximate Hamilton decompositions of suitable graphs G_A^* and G_B^* . (Here G_A^* is an almost complete multigraph with vertex set A which contains $G[A]$, and G_B^* is defined similarly.) Moreover, each Hamilton cycle C in this approximate decomposition is required to contain a certain path system P . We find these Hamilton cycles by first extending P into a path system which is ‘balanced’

with respect to a given blown-up cycle. This balanced path system is extended into a 1-factor F using edges which wind around the blown-up cycle. Finally, F is transformed into a Hamilton cycle C using some edges which were set aside earlier. (In particular, most edges of each Hamilton cycle in our approximate decomposition wind around a blown-up cycle.) A more detailed sketch is given in Section 5.1. The argument for the bipartite case (i) is more involved but uses similar ideas (in fact, some intermediate results are used in both cases).

This paper is organized as follows. After introducing some notation, we state the main results (Lemmas 3.1 and 3.2) in Section 3. After introducing some tools in Section 4, we prove Lemma 3.1 in Sections 5–7. We then prove Lemma 3.2 using a similar approach in the final section.

2. NOTATION

The digraphs considered in this paper do not contain loops and we allow at most two edges between any pair of distinct vertices, at most one edge in each direction. Given a graph or digraph G , we write $V(G)$ for its vertex set, $E(G)$ for its edge set, $e(G) := |E(G)|$ for the number of edges in G and $|G| := |V(G)|$ for the number of vertices in G .

If G is a graph and v is a vertex of G , we write $N_G(v)$ for the set of all neighbours of v in G . If $A \subseteq V(G)$, we write $d_G(v, A)$ for the number of neighbours of v in G which lie in A . We write $\delta(G)$ for the minimum degree of G , $\Delta(G)$ for its maximum degree and $\chi'(G)$ for the edge-chromatic number of G . Given $A, B \subseteq V(G)$, we write $e_G(A)$ for the number of edges of G which have both endvertices in A and $e_G(A, B)$ for the number of AB -edges of G , i.e. for the number of edges of G which have one endvertex in A and its other endvertex in B . If $A \cap B = \emptyset$, we denote by $G[A, B]$ the bipartite subgraph of G whose vertex classes are A and B and whose edges are all AB -edges of G . We often omit the index G if the graph G is clear from the context. An AB -path is a path having one endvertex in A and its other endvertex in B . We often view a matching M as a graph (in which every vertex has degree precisely one).

If G is a digraph, we write xy for an edge directed from x to y . If $xy \in E(G)$, we say that y is an *outneighbour* of x and x is an *inneighbour* of y . We write $d_G^+(x)$ for the *outdegree* of x (i.e. for the number of outneighbours of x in G) and $d_G^-(x)$ for the *indegree* of x (i.e. for the number of inneighbours of x). We write $d_G^+(x, A)$ for the number of outneighbours of x lying inside A and define $d_G^-(x, A)$ similarly. We denote the minimum outdegree of G by $\delta^+(G)$ and the minimum indegree by $\delta^-(G)$. Given $A, B \subseteq V(G)$, an AB -edge is an edge with initial vertex in A and final vertex in B , and $e_G(A, B)$ denotes the number of these edges in G . If $A \cap B = \emptyset$, we denote by $G[A, B]$ the bipartite subdigraph of G whose vertex classes are A and B and whose edges are all AB -edges of G . By a bipartite digraph $G = G[A, B]$ we mean a digraph which only contains AB -edges. A digraph H is a *1-factor* of another digraph G if $V(H) = V(G)$ and every vertex in H has in- and outdegree precisely one in H .

Given a vertex set V and two multigraphs G and H with $V(G), V(H) \subseteq V$, we write $G + H$ for the multigraph whose vertex set is $V(G) \cup V(H)$ and in which the multiplicity of xy in $G + H$ is the sum of the multiplicities of xy in G and in H (for all $x, y \in V(G) \cup V(H)$). Similarly, if $\mathcal{H} := \{H_1, \dots, H_\ell\}$ is a set of graphs, we define $G + \mathcal{H} := G + H_1 + \dots + H_\ell$. We write $G - H$ for the subgraph of G which is obtained from G by deleting all the edges in $E(G) \cap E(H)$. We say that a graph G has a *decomposition* into H_1, \dots, H_r if $G = H_1 + \dots + H_r$ and the H_i are pairwise edge-disjoint.

A *path system* is a graph Q which is the union of vertex-disjoint paths (some of them might be trivial). We say that P is a *path in* Q if P is a component of Q . A *path sequence* is a digraph which is the union of vertex-disjoint directed paths (some of them might be trivial).

Let V_1, \dots, V_k be pairwise disjoint sets of vertices and let $C = V_1 \dots V_k$ be a directed cycle on these sets. We say that an edge xy of a digraph R *winds around* C if there is some i such that $x \in V_i$ and $y \in V_{i+1}$. In particular, we say that R *winds around* C if all edges of R wind around C .

In order to simplify the presentation, we omit floors and ceilings and treat large numbers as integers whenever this does not affect the argument. The constants in the hierarchies used to state our results have to be chosen from right to left. More precisely, if we claim that a result holds whenever $0 < 1/n \ll a \ll b \ll c \leq 1$ (where n is the order of the graph), then this means that there are non-decreasing functions $f : (0, 1] \rightarrow (0, 1]$, $g : (0, 1] \rightarrow (0, 1]$ and $h : (0, 1] \rightarrow (0, 1]$ such that the result holds for all $0 < a, b, c \leq 1$ and all $n \in \mathbb{N}$ with $b \leq f(c)$, $a \leq g(b)$ and $1/n \leq h(a)$. We will not calculate these functions explicitly. Hierarchies with more constants are defined in a similar way. Given $a, b, c \in \mathbb{R}$, we will write $a = b \pm c$ as shorthand for $b - c \leq a \leq b + c$.

3. STATEMENTS OF THE MAIN RESULTS

3.1. Statement for the two cliques case. In this section, we state our approximate decomposition result in the case when G is a graph which is close to the union of two disjoint copies of $K_{n/2}$. Suppose that A, A_0, B, B_0 forms a partition of a vertex set V of size n such that $|A| = |B|$. Let $V_0 := A_0 \cup B_0$. Let $A' := A_0 \cup A$ and $B' := B_0 \cup B$. V_0 will be a small set which consists of ‘exceptional’ vertices. Below, we will define an ‘exceptional cover’, which is a path system which covers the vertices in V_0 . Our aim is to extend such exceptional covers into edge-disjoint Hamilton cycles (or pairs of edge-disjoint perfect matchings). As an intermediate step, we will consider ‘exceptional systems’, which are exceptional covers that also contain an appropriate number of ‘connections’ between A' and B' .

Our main result (Lemma 3.1) then states the following: suppose that we are given a graph G so that $G[A]$ and $G[B]$ are almost complete and that we are given an appropriate set of edge-disjoint exceptional systems. Then we can extend these exceptional systems into a set of edge-disjoint Hamilton cycles (and possibly perfect matchings) of G covering almost all edges of $G[A]$ and $G[B]$. Moreover, the additional edges are all contained in $G[A] + G[B]$.

More precisely, an *exceptional cover* J is a graph which satisfies the following properties:

- (EC1) J is a path system with $V_0 \subseteq V(J) \subseteq V$.
- (EC2) $d_J(v) = 2$ for every $v \in V_0$ and $d_J(v) \leq 1$ for every $v \in V(J) \setminus V_0$.
- (EC3) $e_J(A), e_J(B) = 0$.

We say that J is an *exceptional system with parameter* ε_0 , or an *ES* for short, if J satisfies the following properties:

- (ES1) J is an exceptional cover.
- (ES2) One of the following is satisfied:
 - (HES) The number of AB -paths in J is even and positive. In this case we say J is a *Hamilton exceptional system*, or *HES* for short.
 - (MES) $e_J(A', B') = 0$. In this case we say J is a *matching exceptional system*, or *MES* for short.
- (ES3) J contains at most $\sqrt{\varepsilon_0}n$ AB -paths.

Note that by (EC2) every AB -path in J must be a maximal path in J . Moreover, the number of AB -paths in J is the number of genuine ‘connections’ between A and B (and thus between A' and B'). If we want to extend J into a Hamilton cycle using only edges induced by A and edges induced by B , this number clearly has to be even and positive. Hamilton exceptional systems will always be extended into Hamilton cycles and matching exceptional systems will always be extended into two disjoint even cycles which together span all vertices (and thus consist of two edge-disjoint perfect matchings).

We will need to consider exceptional systems which are ‘localized’ in the sense that their vertices are contained in a small number of clusters of a given partition. To formalize this, let $K, m \in \mathbb{N}$ and $\varepsilon_0 > 0$. A (K, m, ε_0) -*partition* \mathcal{P} of a set V of vertices is a partition of V into sets A_0, A_1, \dots, A_K and B_0, B_1, \dots, B_K such that $|A_i| = |B_i| = m$ for all $i \geq 1$ and $|A_0 \cup B_0| \leq \varepsilon_0|V|$. The sets A_1, \dots, A_K and B_1, \dots, B_K are called *clusters* of \mathcal{P} and A_0, B_0 are called *exceptional sets*. We often write V_0 for $A_0 \cup B_0$ and think of the vertices in V_0 as ‘exceptional vertices’. Unless stated otherwise, whenever \mathcal{P} is a (K, m, ε_0) -partition, we will denote the clusters by A_1, \dots, A_K and B_1, \dots, B_K and the exceptional sets by A_0 and B_0 . We will also write $A := A_1 \cup \dots \cup A_K$, $B := B_1 \cup \dots \cup B_K$, $A' := A_0 \cup A$ and $B' := B_0 \cup B$.

Given a (K, m, ε_0) -partition \mathcal{P} and $1 \leq i, i' \leq K$, we say that J is an (i, i') -*localized exceptional system* with respect to \mathcal{P} (abbreviated as (i, i') -*ES*) if J is an exceptional system and $V(J) \subseteq V_0 \cup A_i \cup B_{i'}$.

Lemma 3.1. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll 1/K \ll \rho \ll 1$ and $0 \leq \mu \ll 1$, where $n, K \in \mathbb{N}$ and K is odd. Suppose that G is a graph on n vertices and \mathcal{P} is a (K, m, ε_0) -partition of $V(G)$. Furthermore, suppose that the following conditions hold:*

- (a) $d(v, A_i) = (1 - 4\mu \pm 4/K)m$ and $d(w, B_i) = (1 - 4\mu \pm 4/K)m$ for all $v \in A$, $w \in B$ and $1 \leq i \leq K$.
- (b) *There is a set \mathcal{J} which consists of at most $(1/4 - \mu - \rho)n$ edge-disjoint exceptional systems with parameter ε_0 in G .*

- (c) \mathcal{J} has a partition into K^2 sets $\mathcal{J}_{i,i'}$ (one for all $1 \leq i, i' \leq K$) such that each $\mathcal{J}_{i,i'}$ consists of precisely $|\mathcal{J}|/K^2$ (i, i') -ES with respect to \mathcal{P} .
- (d) If \mathcal{J} contains matching exceptional systems then $|A'| = |B'|$ is even.

Then G contains $|\mathcal{J}|$ edge-disjoint spanning subgraphs $H_1, \dots, H_{|\mathcal{J}|}$ which satisfy the following properties:

- For each H_s there is some $J_s \in \mathcal{J}$ such that $J_s \subseteq H_s$.
- If J_s is a Hamilton exceptional system, then H_s is a Hamilton cycle of G .
If J_s is a matching exceptional system, then H_s is the edge-disjoint union of two perfect matchings in G .

3.2. Statement for the bipartite case. We now state an analogous result for the case when our graph G is close to the complete balanced bipartite graph $K_{n/2, n/2}$. Let $\mathcal{P} := \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$ be a (K, m, ε_0) -partition of a set V of n vertices. Define A, A', B, B' as in the previous subsection. We now define the analogue of an (i, i') -localized exceptional system for the bipartite case as follows.

Given $1 \leq i_1, i_2, i_3, i_4 \leq K$ and $\varepsilon_0 > 0$, an (i_1, i_2, i_3, i_4) -balanced exceptional system with respect to \mathcal{P} and parameter ε_0 , or (i_1, i_2, i_3, i_4) -BES for short, is a path system J with $V(J) \subseteq A_0 \cup B_0 \cup A_{i_1} \cup A_{i_2} \cup B_{i_3} \cup B_{i_4}$ such that the following conditions hold:

- (BES1) $d_J(v) = 2$ for every vertex $v \in V_0 = A_0 \cup B_0$ and $d_J(v) \leq 1$ for every vertex $v \in A_{i_1} \cup A_{i_2} \cup B_{i_3} \cup B_{i_4}$.
- (BES2) Every edge of $J[A \cup B]$ is either an $A_{i_1}A_{i_2}$ -edge or a $B_{i_3}B_{i_4}$ -edge.
- (BES3) The edges in J cover precisely the same number of vertices in A as in B .
- (BES4) $e(J) \leq \varepsilon_0 n$.

Note that (BES2) implies that an (i_1, i_2, i_3, i_4) -BES does not contain AB -edges. Furthermore, an (i_1, i_2, i_3, i_4) -BES is also, for example, an (i_2, i_1, i_4, i_3) -BES. We sometimes omit the indices i_1, i_2, i_3, i_4 and just refer to a balanced exceptional system.

We can now state the analogue of Lemma 3.1 for the case when G is close to $K_{n/2, n/2}$.

Lemma 3.2. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll 1/K \ll \rho \ll 1$ and $0 \leq \mu \ll 1$, where $n, K \in \mathbb{N}$ and K is even. Suppose that G is a graph on n vertices and \mathcal{P} is a (K, m, ε_0) -partition of $V(G)$. Furthermore, suppose that the following conditions hold:*

- (a) $d(v, B_i) = (1 - 4\mu \pm 4/K)m$ and $d(w, A_i) = (1 - 4\mu \pm 4/K)m$ for all $v \in A$, $w \in B$ and $1 \leq i \leq K$.
- (b) There is a set \mathcal{J} which consists of at most $(1/4 - \mu - \rho)n$ edge-disjoint balanced exceptional systems with parameter ε_0 in G .
- (c) \mathcal{J} has a partition into K^4 sets $\mathcal{J}_{i_1, i_2, i_3, i_4}$ (one for all $1 \leq i_1, i_2, i_3, i_4 \leq K$) such that each $\mathcal{J}_{i_1, i_2, i_3, i_4}$ consists of precisely $|\mathcal{J}|/K^4$ (i_1, i_2, i_3, i_4) -BES with respect to \mathcal{P} .
- (d) For each $v \in A \cup B$ the number of $J \in \mathcal{J}$ such that v is incident with an edge of J is at most $2\varepsilon_0 n$.

Then G contains $|\mathcal{J}|$ edge-disjoint Hamilton cycles such that each of these Hamilton cycles contains some $J \in \mathcal{J}$.

4. USEFUL RESULTS

4.1. A Chernoff estimate. We will use the following Chernoff bound for the binomial distribution (see e.g. [5, Corollary 2.3]). Recall that a binomial random variable with parameters (n, p) is the sum of n independent Bernoulli variables, each taking value 1 with probability p or 0 with probability $1 - p$.

Proposition 4.1. *Suppose X has binomial distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-\frac{a^2}{3}\mathbb{E}X}$.*

4.2. Regular spanning subgraphs. The following lemma implies that any almost complete balanced bipartite graph has an approximate decomposition into perfect matchings. The proof is a straightforward application of the MaxFlowMinCut theorem.

Lemma 4.2. *Suppose that $0 < 1/m \ll \varepsilon \ll \rho \ll 1$, that $0 \leq \mu \leq 1/4$ and that $m, \mu m, \rho m \in \mathbb{N}$. Suppose that Γ is a bipartite graph with vertex classes U and V of size m and with $(1 - \mu - \varepsilon)m \leq \delta(\Gamma) \leq \Delta(\Gamma) \leq (1 - \mu + \varepsilon)m$. Then Γ contains a spanning $(1 - \mu - \rho)m$ -regular subgraph Γ' . In particular, Γ contains at least $(1 - \mu - \rho)m$ edge-disjoint perfect matchings.*

Proof. We first obtain a directed network N from Γ by adding a source s and a sink t . We add a directed edge su of capacity $(1 - \mu - \rho)m$ for each $u \in U$ and a directed edge vt of capacity $(1 - \mu - \rho)m$ for each $v \in V$. We give all the edges in Γ capacity 1 and direct them from U to V .

Our aim is to show that the capacity of any (s, t) -cut is at least $(1 - \mu - \rho)m^2$. By the MaxFlowMinCut theorem this would imply that N admits an integer-valued flow of value $(1 - \mu - \rho)m^2$ which by construction of N implies the existence of our desired subgraph Γ' .

Consider any (s, t) -cut (S, \bar{S}) where $S = \{s\} \cup S_1 \cup S_2$ with $S_1 \subseteq U$ and $S_2 \subseteq V$. Let $\bar{S}_1 := U \setminus S_1$ and $\bar{S}_2 := V \setminus S_2$. The capacity of this cut is

$$(1 - \mu - \rho)m(m - |S_1|) + e(S_1, \bar{S}_2) + (1 - \mu - \rho)m|S_2|$$

and therefore our aim is to show that

$$(4.1) \quad e(S_1, \bar{S}_2) \geq (1 - \mu - \rho)m(|S_1| - |S_2|).$$

If $|S_1| \leq (1 - \mu - \rho)m$, then

$$\begin{aligned} e(S_1, \bar{S}_2) &\geq ((1 - \mu - \varepsilon)m - |S_2|)|S_1| \\ &= (1 - \mu - \rho)m(|S_1| - |S_2|) + (\rho - \varepsilon)m|S_1| + |S_2|((1 - \mu - \rho)m - |S_1|) \\ &\geq (1 - \mu - \rho)m(|S_1| - |S_2|). \end{aligned}$$

Thus, we may assume that $|S_1| > (1 - \mu - \rho)m$. Note that $|S_1| - |S_2| = |\bar{S}_2| - |\bar{S}_1|$. Therefore, by a similar argument, we may also assume that $|\bar{S}_2| > (1 - \mu - \rho)m$ and

so $|S_2| \leq (\mu + \rho)m$. This implies that

$$\begin{aligned} e(S_1, \bar{S}_2) &\geq \sum_{x \in S_1} d_\Gamma(x) - \sum_{y \in S_2} d_\Gamma(y) \geq (1 - \mu - \varepsilon)m|S_1| - (1 - \mu + \varepsilon)m|S_2| \\ &= (1 - \mu - \rho)m(|S_1| - |S_2|) + \rho m(|S_1| - |S_2|) - \varepsilon m(|S_1| + |S_2|) \\ &> (1 - \mu - \rho)m(|S_1| - |S_2|) + (1 - 2\mu - 2\rho)\rho m^2 - (1 + \mu + \rho)\varepsilon m^2 \\ &\geq (1 - \mu - \rho)m(|S_1| - |S_2|). \end{aligned}$$

(Note that the last inequality follows as $\varepsilon \ll \rho \ll 1$ and $\mu \leq 1/4$.) So indeed (4.1) is satisfied, as desired. \square

4.3. Robust expansion. Given $0 < \nu \leq \tau < 1$, we say that a digraph G on n vertices is a *robust (ν, τ) -outexpander*, if for all $S \subseteq V(G)$ with $\tau n \leq |S| \leq (1 - \tau)n$ the number of vertices that have at least νn inneighbours in S is at least $|S| + \nu n$. The following result was derived in [8] as a straightforward consequence of the result from [11] that every robust outexpander of linear minimum degree has a Hamilton cycle.

Theorem 4.3. *Suppose that $0 < 1/n \ll \gamma \ll \nu \ll \tau \ll \eta < 1$. Let G be a digraph on n vertices with $\delta^+(G), \delta^-(G) \geq \eta n$ which is a robust (ν, τ) -outexpander. Let y_1, \dots, y_p be distinct vertices in $V(G)$ with $p \leq \gamma n$. Then G contains a directed Hamilton cycle visiting y_1, \dots, y_p in this order.*

4.4. A regularity concept for sparse graphs. We now formulate a concept of ε -superregularity which is suitable for ‘sparse’ graphs. Let G be a bipartite graph with vertex classes U and V , both of size m . Given $A \subseteq U$ and $B \subseteq V$, we write $d(A, B) := e(A, B)/|A||B|$ for the density of G between A and B . Given $0 < \varepsilon, d, d^*, c < 1$, we say that G is (ε, d, d^*, c) -*superregular* if the following conditions are satisfied:

- (Reg1) Whenever $A \subseteq U$ and $B \subseteq V$ are sets of size at least εm , then $d(A, B) = (1 \pm \varepsilon)d$.
- (Reg2) For all $u, u' \in V(G)$ we have $|N(u) \cap N(u')| \leq c^2 m$.
- (Reg3) $\Delta(G) \leq cm$.
- (Reg4) $\delta(G) \geq d^* m$.

Note that the above definitions also make sense if G is ‘sparse’ in the sense that $d < \varepsilon$ (which will be the case in our proofs). A bipartite digraph $G = G[U, V]$ is (ε, d, d^*, c) -*superregular* if this holds for the underlying undirected graph of G .

The following observation follows immediately from the definition.

Proposition 4.4. *Suppose that $0 < 1/m \ll d^*, d, \varepsilon, \varepsilon', c \ll 1$ and $2\varepsilon' \leq d^*$. Let G be an (ε, d, d^*, c) -superregular bipartite graph with vertex classes U and V of size m . Let $U' \subseteq U$ and $V' \subseteq V$ with $|U'| = |V'| \geq (1 - \varepsilon')m$. Then $G[U', V']$ is $(2\varepsilon, d, d^*/2, 2c)$ -superregular.*

The following two simple observations were made in [10].

Proposition 4.5. *Suppose that $0 < 1/m \ll d^*, d, \varepsilon, c \ll 1$. Let G be an (ε, d, d^*, c) -superregular bipartite graph with vertex classes U and V of size m . Suppose that G' is obtained from G by removing at most $\varepsilon^2 dm$ edges incident to each vertex from G . Then G' is $(2\varepsilon, d, d^* - \varepsilon^2 d, c)$ -superregular.*

Lemma 4.6. *Let $0 < 1/m \ll \nu \ll \tau \ll d \leq \varepsilon \ll \mu, \zeta \leq 1/2$ and let G be an $(\varepsilon, d, \zeta d, d/\mu)$ -superregular bipartite graph with vertex classes U and V of size m . Let $A \subseteq U$ be such that $\tau m \leq |A| \leq (1 - \tau)m$. Let $B \subseteq V$ be the set of all those vertices in V which have at least νm neighbours in A . Then $|B| \geq |A| + \nu m$.*

5. SYSTEMS AND BALANCED EXTENSIONS

5.1. Sketch proof of Lemma 3.1. Roughly speaking, the Hamilton cycles we find will have the following structure: let $A_1, \dots, A_K \subseteq A$ and $B_1, \dots, B_K \subseteq B$ be the clusters of the (K, m, ε_0) -partition \mathcal{P} of $V(G)$ given in Lemma 3.1. So K is odd. Let \mathcal{R}_A be the complete graph on A_1, \dots, A_K and \mathcal{R}_B be the complete graph on B_1, \dots, B_K . Since K is odd, Walecki's theorem [12] implies that \mathcal{R}_A has a Hamilton decomposition $C_{A,1}, \dots, C_{A,(K-1)/2}$, and similarly \mathcal{R}_B has a Hamilton decomposition $C_{B,1}, \dots, C_{B,(K-1)/2}$. Every Hamilton cycle C we construct in G will have the property that there is a j so that almost all edges of $C[A]$ wind around $C_{A,j}$ and almost all edges of $C[B]$ wind around $C_{B,j}$. Below, we describe the main ideas involved in the construction of the Hamilton cycles in more detail.

As indicated above, the first idea is that we can reduce the problem of finding the required edge-disjoint Hamilton cycles (and possibly perfect matchings) in G to that of finding appropriate Hamilton cycles on each of A and B separately. We achieve this by introducing suitable path systems J_A^* and J_B^* .

More precisely, let \mathcal{J} be a set of edge-disjoint exceptional systems as given in Lemma 3.1. By deleting some edges if necessary, we may further assume that \mathcal{J} is an edge-decomposition of $G - G[A] - G[B]$. Thus, in order to prove Lemma 3.1, we have to find $|\mathcal{J}|$ suitable edge-disjoint subgraphs $H_{A,1}, \dots, H_{A,|\mathcal{J}|}$ of $G[A]$ and $|\mathcal{J}|$ suitable edge-disjoint subgraphs $H_{B,1}, \dots, H_{B,|\mathcal{J}|}$ of $G[B]$ such that $H_s := H_{A,s} + H_{B,s} + J_s$ are the desired spanning subgraphs of G . To prove this, for each $J \in \mathcal{J}$, we define two auxiliary subgraphs J_A^* and J_B^* with the following crucial properties:

- (α_1) J_A^* and J_B^* are matchings whose vertices are contained in A and B , respectively;
- (α_2) the union of any Hamilton cycle C_A^* in $G[A] + J_A^*$ containing J_A^* (in some suitable order) and any Hamilton cycle C_B^* in $G[B] + J_B^*$ containing J_B^* (in some suitable order) corresponds to either a Hamilton cycle of G containing J or to the union of two edge-disjoint perfect matchings of G containing J .

Furthermore, J determines which of the cases in (α_2) holds: If J is a Hamilton exceptional system, then (α_2) will give a Hamilton cycle of G , while in the case when J is a matching exceptional system, (α_2) will give the union of two edge-disjoint perfect matchings of G . So roughly speaking, this allows us to work with multigraphs $G_A^* := G[A] + \sum_{J \in \mathcal{J}} J_A^*$ and $G_B^* := G[B] + \sum_{J \in \mathcal{J}} J_B^*$ rather than G in the two steps. Furthermore, the processes of finding Hamilton cycles in G_A^* and in G_B^* are independent (see Section 6.1 for more details).

By symmetry, it suffices to consider G_A^* in what follows. The second idea of the proof is that as an intermediate step, we decompose G_A^* into blown-up Hamilton cycles $G_{A,j}^*$. Roughly speaking, we will then find an approximate Hamilton decomposition of each $G_{A,j}^*$ separately.

More precisely, recall that \mathcal{R}_A denotes the complete graph whose vertex set is $\{A_1, \dots, A_K\}$. As mentioned above, \mathcal{R}_A has a Hamilton decomposition $C_{A,1}, \dots, C_{A,(K-1)/2}$. We decompose $G[A]$ into edge-disjoint subgraphs $G_{A,1}, \dots, G_{A,(K-1)/2}$ such that each $G_{A,j}$ corresponds to the ‘blow-up’ of $C_{A,j}$, i.e. $G_{A,j}[U, W] = G[U, W]$ for every edge $UW \in E(C_{A,j})$. (The edges of G lying inside one of the clusters A_1, \dots, A_K are deleted.) We also partition the set $\{J_A^* : J \in \mathcal{J}\}$ into $(K-1)/2$ sets $\mathcal{J}_{A,1}^*, \dots, \mathcal{J}_{A,(K-1)/2}^*$ of roughly equal size. Set $G_{A,j}^* := G_{A,j} + \mathcal{J}_{A,j}^*$. Thus in order to prove Lemma 3.1, we need to find $|\mathcal{J}_{A,j}^*|$ edge-disjoint Hamilton cycles in $G_{A,j}^*$ (for each $j \leq (K-1)/2$). Since $G_{A,j}^*$ is still close to being a blow-up of the cycle $C_{A,j}$, finding such Hamilton cycles seems feasible.

One complication is that in order to satisfy (α_2) , we need to ensure that each Hamilton cycle in $G_{A,j}^*$ contains some $J_A^* \in \mathcal{J}_{A,j}^*$ (and it must traverse the edges of J_A^* in some given order). To achieve this, we will both orient and order the edges of J_A^* . So we will actually consider an ordered directed matching $J_{A,\text{dir}}^*$ instead of J_A^* . (J_A^* itself will still be undirected and unordered). We orient the edges of $G_{A,j}$ such that the resulting oriented graph $G_{A,j,\text{dir}}$ is a blow-up of the directed cycle $C_{A,j}$.

However, $J_{A,\text{dir}}^*$ may not be ‘locally balanced with respect to $C_{A,j}$ ’. This means that it is impossible to extend $J_{A,\text{dir}}^*$ into a directed Hamilton cycle using only edges of $G_{A,j,\text{dir}}$. For example, suppose that $G_{A,j,\text{dir}}$ is a blow-up of the directed cycle $A_1 A_2 \dots A_K$, i.e. each edge of $G_{A,j,\text{dir}}$ joins A_i to A_{i+1} for some $1 \leq i \leq K$. If $J_{A,\text{dir}}^*$ is non-empty and $V(J_{A,\text{dir}}^*) \subseteq A_1$, then $J_{A,\text{dir}}^*$ cannot be extended into a directed Hamilton cycle using edges of $G_{A,j,\text{dir}}$ only. Therefore, each $J_{A,\text{dir}}^*$ will first be extended into a ‘locally balanced path sequence’ PS . PS will have the property that it can be extended to a Hamilton cycle using only edges of $G_{A,j,\text{dir}}$. We will call the set \mathcal{BE}_j consisting of all such PS for all $J_A^* \in \mathcal{J}_{A,j}^*$ a balanced extension of $\mathcal{J}_{A,j}^*$. \mathcal{BE}_j will be constructed in Section 6.3, using edges from a sparse graph H' on A (which is actually removed from $G[A]$ before defining $G_{A,1}, \dots, G_{A,(K-1)/2}$).

Finally, we find the required directed Hamilton cycles in $G_{A,j,\text{dir}} + \mathcal{BE}_j$ in Section 7. We construct these by first extending the path sequences in \mathcal{BE}_j into (directed) 1-factors, using edges which wind around the blow-up of $C_{A,j}$. These are then transformed into Hamilton cycles using a small set of edges set aside earlier (again the set of these edges winds around the blow-up of $C_{A,j}$).

5.2. Systems and balanced extensions. As mentioned above, the proof of Lemma 3.1 requires an edge-decomposition and orientation of $G[A]$ and $G[B]$ into blow-ups of directed cycles as well as ‘balanced extensions’. These are defined in the current subsection.

Let $k, m \in \mathbb{N}$. A (k, m) -equipartition \mathcal{Q} of a set V of vertices is a partition of V into sets V_1, \dots, V_k such that $|V_i| = m$ for all $i \leq k$. The V_i are called *clusters* of \mathcal{Q} . (G, \mathcal{Q}, C) is a (k, m, μ, ε) -cyclic system if the following properties hold:

- (Sys1) G is a digraph and \mathcal{Q} is a (k, m) -equipartition of $V(G)$.
 (Sys2) C is a directed Hamilton cycle on \mathcal{Q} and G winds around C . Moreover, for every edge UW of C , we have $d_G^+(u, W) = (1 - \mu \pm \varepsilon)m$ for every $u \in U$ and $d_G^-(w, U) = (1 - \mu \pm \varepsilon)m$ for every $w \in W$.

So roughly speaking, such a cyclic system is a blown-up Hamilton cycle.

Let \mathcal{Q} be a (k, m) -equipartition of V and let C be a directed Hamilton cycle on \mathcal{Q} . We say that a digraph H with $V(H) \subseteq V$ is *locally balanced with respect to C* if for every edge UW of C , the number of edges of H with initial vertex in U equals the number of edges of H with final vertex in W .

Let M be a directed matching. We say a path sequence PS is a V_i -extension of M with respect to \mathcal{Q} if each edge of M is contained in a distinct directed path in PS having its final vertex in V_i . Let $\mathcal{M} := \{M_1, \dots, M_q\}$ be a set of directed matchings. A set \mathcal{BE} of path sequences is a *balanced extension of \mathcal{M} with respect to (\mathcal{Q}, C) and parameters (ε, ℓ)* if \mathcal{BE} satisfies the following properties:

- (BE1) \mathcal{BE} consists of q path sequences PS_1, \dots, PS_q such that $V(PS_i) \subseteq V$ for each $i \leq q$, each PS_i is locally balanced with respect to C and $PS_1 - M_1, \dots, PS_q - M_q$ are edge-disjoint from each other.
 (BE2) Each PS_s is a V_{i_s} -extension of M_s with respect to \mathcal{Q} for some $i_s \leq k$. Moreover, for each $i \leq k$ there are at most $\ell m/k$ indices $s \leq q$ such that $i_s = i$.
 (BE3) $|V(PS_s) \cap V_i| \leq \varepsilon m$ for all $i \leq k$ and $s \leq q$. Moreover, for each $i \leq k$, there are at most $\ell m/k$ path sequences $PS_s \in \mathcal{BE}$ such that $V(PS_s) \cap V_i \neq \emptyset$.

Note that the ‘moreover part’ of (BE3) implies the ‘moreover part’ of (BE2).

Given an ordered directed matching $M = \{f_1, \dots, f_\ell\}$, we say that a directed cycle C' is *consistent with M* if C' contains M and visits the edges f_1, \dots, f_ℓ in this order. The following observation will be useful: suppose that PS is a V_i -extension of M and let x_j be the final vertex of the path in PS containing f_j . (So x_1, \dots, x_ℓ are distinct vertices of V_i .) Suppose also that C' is a directed cycle which contains PS and visits x_1, \dots, x_ℓ in this order. Then C' is consistent with M .

6. FINDING SYSTEMS AND BALANCED EXTENSIONS FOR THE TWO CLIQUES CASES

Let G be a graph, let \mathcal{P} be a (K, m, ε_0) -partition of $V(G)$ and let \mathcal{J} be a set of exceptional systems as given by Lemma 3.1. The aim of this section is to decompose $G[A] + G[B]$ into (k, m, μ, ε) -cyclic systems and to construct balanced extensions as described in Section 5.1. First we need to define $J_{A,\text{dir}}^*$ and $J_{B,\text{dir}}^*$ for each exceptional system $J \in \mathcal{J}$. Recall from Section 5.1 that these are introduced in order to be able to consider $G[A]$ and $G[B]$ separately (and thus to be able to ignore the exceptional vertices in $V_0 = A_0 \cup B_0$).

6.1. Defining the graphs $J_{A,\text{dir}}^*$ and $J_{B,\text{dir}}^*$. Let A, A_0, B, B_0 be a partition of a vertex set V on n vertices and let J be an exceptional system with parameter ε_0 . Since each maximal path in J has endpoints in $A \cup B$ and internal vertices in V_0 , an exceptional system J naturally induces a matching J_{AB}^* on $A \cup B$. More precisely, if $P_1, \dots, P_{\ell'}$ are the non-trivial paths in J and x_i, y_i are the endpoints of P_i , then we define $J_{AB}^* := \{x_i y_i : i \leq \ell'\}$. Thus $e_{J_{AB}^*}(A, B)$ is equal to the number of AB -paths

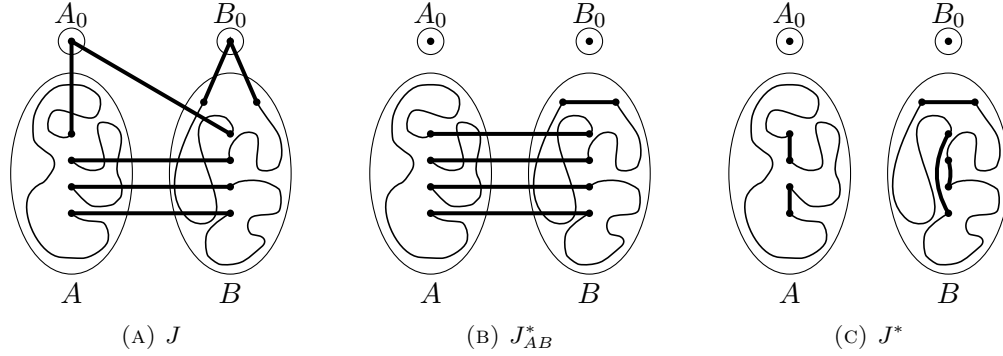


FIGURE 1. The thick lines illustrate the edges of J , J_{AB}^* and J^* , respectively.

in J . In particular, $e_{J_{AB}^*}(A, B)$ is positive and even if J is a Hamilton exceptional system, while $e_{J_{AB}^*}(A, B) = 0$ if J is a matching exceptional system. Without loss of generality we may assume that $x_1y_1, \dots, x_{2\ell}y_{2\ell}$ is an enumeration of the edges of $J_{AB}^*[A, B]$, where $x_i \in A$ and $y_i \in B$. Define

$$J_A^* := J_{AB}^*[A] \cup \{x_{2i-1}x_{2i} : 1 \leq i \leq \ell\} \quad \text{and} \quad J_B^* := J_{AB}^*[B] \cup \{y_{2i}y_{2i+1} : 1 \leq i \leq \ell\}$$

(with indices considered modulo 2ℓ). Let $J^* := J_A^* + J_B^*$. So J^* is a matching and $e(J^*) = e(J_{AB}^*)$. Moreover, by (EC2), (EC3) and (ES3) we have

$$(6.1) \quad e(J^*) = e(J_{AB}^*) \leq |V_0| + \sqrt{\varepsilon_0 n}.$$

We will call the edges in J^* *fictive edges*. Note that if J_1 and J_2 are two edge-disjoint exceptional systems, then J_1^* and J_2^* may not be edge-disjoint.

We say that an (undirected) cycle C is *consistent with J_A^** if C contains J_A^* and (there is an orientation of C which) visits the vertices $x_1, \dots, x_{2\ell}$ in this order. In a similar way we define when a cycle is consistent with J_B^* . The following proposition is proved in [9]. It is illustrated in Figure 1.

Proposition 6.1. *Suppose that A, A_0, B, B_0 forms a partition of a vertex set V . Let J be an exceptional system. Let C_A and C_B be two cycles such that*

- C_A is a Hamilton cycle on A that is consistent with J_A^* ;
- C_B is a Hamilton cycle on B that is consistent with J_B^* .

Then the following assertions hold.

- (i) *If J is a Hamilton exceptional system, then $C_A + C_B - J^* + J$ is a Hamilton cycle on V .*
- (ii) *If J is a matching exceptional system, then $C_A + C_B - J^* + J$ is the union of a Hamilton cycle on A' and a Hamilton cycle on B' . In particular, if both $|A'|$ and $|B'|$ are even, then $C_A + C_B - J^* + J$ is the union of two edge-disjoint perfect matchings on V .*

As mentioned in Section 5.1, we will orient and order the edges of J_A^* and J_B^* in a suitable way to obtain $J_{A,\text{dir}}^*$ and $J_{B,\text{dir}}^*$. Accordingly, we will actually need

an oriented version of Proposition 6.1. For this, we first orient the edges of J_A^* by orienting the edge $x_{2i-1}x_{2i}$ from x_{2i-1} to x_{2i} for all $i \leq \ell$ and the edges of $J_{AB}^*[A]$ arbitrarily. Next we order these directed edges as f_1, \dots, f_{ℓ_A} such that $f_i = x_{2i-1}x_{2i}$ for all $i \leq \ell$, where $\ell_A := e(J_A^*)$. Define $J_{A,\text{dir}}^*$ to be the ordered directed matching $\{f_1, \dots, f_{\ell_A}\}$. Similarly, to define $J_{B,\text{dir}}^*$, we first orient the edges of J_B^* by orienting the edge $y_{2i}y_{2i+1}$ from y_{2i} to y_{2i+1} for all $i \leq \ell$ and the edges of $J_{AB}^*[B]$ arbitrarily. Next we order these directed edges as f'_1, \dots, f'_{ℓ_B} such that $f'_i = y_{2i}y_{2i+1}$ for all $i \leq \ell$, where $\ell_B := e(J_B^*)$. Define $J_{B,\text{dir}}^*$ to be the ordered directed matching $\{f'_1, \dots, f'_{\ell_B}\}$. Note that if J is an (i, i') -ES, then $V(J_{A,\text{dir}}^*) \subseteq A_i$ and $V(J_{B,\text{dir}}^*) \subseteq B_{i'}$. Recall from Section 5.2 that a directed cycle $C_{A,\text{dir}}$ is *consistent with $J_{A,\text{dir}}^*$* if $C_{A,\text{dir}}$ contains $J_{A,\text{dir}}^*$ and visits the edges f_1, \dots, f_{ℓ_A} in this order. The following proposition follows easily from Proposition 6.1.

Proposition 6.2. *Suppose that A, A_0, B, B_0 forms a partition of a vertex set V . Let J be an exceptional system. Let $C_{A,\text{dir}}$ and $C_{B,\text{dir}}$ be two directed cycles such that*

- $C_{A,\text{dir}}$ is a directed Hamilton cycle on A that is consistent with $J_{A,\text{dir}}^*$;
- $C_{B,\text{dir}}$ is a directed Hamilton cycle on B that is consistent with $J_{B,\text{dir}}^*$.

Then the following assertions hold, where C_A and C_B are the undirected cycles obtained from $C_{A,\text{dir}}$ and $C_{B,\text{dir}}$ by ignoring the directions of all the edges.

- (i) *If J is a Hamilton exceptional system, then $C_A + C_B - J^* + J$ is a Hamilton cycle on V .*
- (ii) *If J is a matching exceptional system, then $C_A + C_B - J^* + J$ is the union of a Hamilton cycle on A' and a Hamilton cycle on B' . In particular, if both $|A'|$ and $|B'|$ are even, then $C_A + C_B - J^* + J$ is the union of two edge-disjoint perfect matchings on V .*

6.2. Finding systems. In this subsection, we will decompose (and orient) $G[A]$ into cyclic systems $(G_{A,j,\text{dir}}, \mathcal{Q}_A, C_{A,j})$, one for each $j \leq (K-1)/2$. Roughly speaking, this corresponds to a decomposition into (oriented) blown-up Hamilton cycles. We will achieve this by considering a Hamilton decomposition of \mathcal{R}_A , where \mathcal{R}_A is the complete graph on $\{A_1, \dots, A_K\}$. So each $C_{A,j}$ corresponds to one of the Hamilton cycles in this Hamilton decomposition. We will split the set $\{J_{A,\text{dir}}^* : J \in \mathcal{J}\}$ into subsets $\mathcal{J}_{A,j}^*$ and assign $\mathcal{J}_{A,j}^*$ to the j th cyclic system. Moreover, for each $j \leq (K-1)/2$, we will also set aside a sparse spanning subgraph $H_{A,j}$ of $G[A]$, which is removed from $G[A]$ before the decomposition into cyclic systems. $H_{A,j}$ will be used later on in order to find a balanced extension of $\mathcal{J}_{A,j}^*$. We proceed similarly for $G[B]$.

Lemma 6.3. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll 1/K \ll \rho \ll 1$ and $0 \leq \mu \ll 1$, where $n, K \in \mathbb{N}$ and K is odd. Suppose that G is a graph on n vertices and \mathcal{P} is a (K, m, ε_0) -partition of $V(G)$. Furthermore, suppose that the following conditions hold:*

- (a) $d(v, A_i) = (1 - 4\mu \pm 4/K)m$ and $d(w, B_i) = (1 - 4\mu \pm 4/K)m$ for all $v \in A$, $w \in B$ and $1 \leq i \leq K$.

- (b) There is a set \mathcal{J} which consists of at most $(1/4 - \mu - \rho)n$ edge-disjoint exceptional systems with parameter ε_0 in G .
- (c) \mathcal{J} has a partition into K^2 sets $\mathcal{J}_{i,i'}$ (one for all $1 \leq i, i' \leq K$) such that each $\mathcal{J}_{i,i'}$ consists of precisely $|\mathcal{J}|/K^2$ (i, i') -ES with respect to \mathcal{P} .
- (d) If \mathcal{J} contains matching exceptional systems then $|A'| = |B'|$ is even.

Then for each $1 \leq j \leq (K-1)/2$, there is a pair of tuples $(G_{A,j}, \mathcal{Q}_A, C_{A,j}, H_{A,j}, \mathcal{J}_{A,j}^*)$ and $(G_{B,j}, \mathcal{Q}_B, C_{B,j}, H_{B,j}, \mathcal{J}_{B,j}^*)$ such that the following assertions hold:

- (a₁) Each of $C_{A,1}, \dots, C_{A,(K-1)/2}$ is a directed Hamilton cycle on $\mathcal{Q}_A := \{A_1, \dots, A_K\}$ such that the undirected versions of these Hamilton cycles form a Hamilton decomposition of the complete graph on \mathcal{Q}_A .
- (a₂) $\mathcal{J}_{A,1}^*, \dots, \mathcal{J}_{A,(K-1)/2}^*$ is a partition of $\{J_{A,\text{dir}}^* : J \in \mathcal{J}\}$.
- (a₃) Each $\mathcal{J}_{A,j}^*$ has a partition into K sets $\mathcal{J}_{A,j,i}^*$ (one for each $1 \leq i \leq K$) such that $|\mathcal{J}_{A,j,i}^*| \leq (1-4\mu-3\rho)m/K$ and each $J_{A,\text{dir}}^* \in \mathcal{J}_{A,j,i}^*$ is an ordered directed matching with $e(J_{A,\text{dir}}^*) \leq 5K\sqrt{\varepsilon_0}m$ and $V(J_{A,\text{dir}}^*) \subseteq A_i$.
- (a₄) $G_{A,1}, \dots, G_{A,(K-1)/2}, H_{A,1}, \dots, H_{A,(K-1)/2}$ are edge-disjoint subgraphs of $G[A]$.
- (a₅) $H_{A,j}[A_i, A_{i'}]$ is a $10K\sqrt{\varepsilon_0}m$ -regular graph for all $j \leq (K-1)/2$ and all $i, i' \leq K$ with $i \neq i'$.
- (a₆) For each $j \leq (K-1)/2$, there exists an orientation $G_{A,j,\text{dir}}$ of $G_{A,j}$ such that $(G_{A,j,\text{dir}}, \mathcal{Q}_A, C_{A,j})$ is a $(K, m, 4\mu, 5/K)$ -cyclic system.
- (a₇) Analogous statements to (a₁)–(a₆) hold for $C_{B,j}, \mathcal{J}_{B,j}^*, G_{B,j}, H_{B,j}$ for all $j \leq (K-1)/2$, with $\mathcal{Q}_B := \{B_1, \dots, B_K\}$.

Proof. Since K is odd, by Walecki's theorem the complete graph on $\{A_1, \dots, A_K\}$ has a Hamilton decomposition. (a₁) follows by orienting the edges of each of these Hamilton cycles to obtain directed Hamilton cycles $C_{A,1}, \dots, C_{A,(K-1)/2}$.

For each $i, i' \leq K$, we partition $\mathcal{J}_{i,i'}$ into $(K-1)/2$ sets $\mathcal{J}_{i,i',j}$ (one for each $j \leq (K-1)/2$) whose sizes are as equal as possible. Note that if $J \in \mathcal{J}_{i,i',j}$, then J is an (i, i') -ES and so $V(J_{A,\text{dir}}^*) \subseteq A_i$. Since \mathcal{P} is a (K, m, ε_0) -partition of $V(G)$, $|V_0| \leq \varepsilon_0 n$ and $(1 - \varepsilon_0)n \leq 2Km$. Hence,

$$e(J_{A,\text{dir}}^*) \leq e(J^*) \stackrel{(6.1)}{\leq} |V_0| + \sqrt{\varepsilon_0}n \leq 5\sqrt{\varepsilon_0}Km.$$

(a₂) is satisfied by setting $\mathcal{J}_{A,j,i}^* := \bigcup_{i' \leq K} \{J_{A,\text{dir}}^* : J \in \mathcal{J}_{i,i',j}\}$ and $\mathcal{J}_{A,j}^* := \bigcup_{i \leq K} \mathcal{J}_{A,j,i}^*$. Note that

$$\begin{aligned} |\mathcal{J}_{A,j,i}^*| &\leq \sum_{i' \leq K} \left(\frac{2|\mathcal{J}_{i,i'}|}{K-1} + 1 \right) \stackrel{(c)}{=} \frac{2|\mathcal{J}|}{K(K-1)} + K \stackrel{(b)}{\leq} \frac{(1/2 - 2\mu - 2\rho)n}{K(K-1)} + K \\ &\leq (1 - 4\mu - 3\rho)m/K \end{aligned}$$

as $2Km \geq (1 - \varepsilon_0)n$ and $1/n \ll \varepsilon_0 \ll 1/K \ll \rho$. Hence (a₃) holds.

For $i, i' \leq K$ with $i \neq i'$, apply Lemma 4.2 with $G[A_i, A_{i'}]$, $4/K, \rho, 4\mu$ playing the roles of $\Gamma, \varepsilon, \rho, \mu$ to obtain a spanning $(1 - 4\mu - \rho)m$ -regular subgraph $H_{i,i'}$ of $G[A_i, A_{i'}]$. Since $H_{i,i'}$ is a regular bipartite graph and $\varepsilon_0 \ll 1/K, \rho \ll 1$ and $0 \leq \mu \ll 1$, there exist $(K-1)/2$ edge-disjoint $10K\sqrt{\varepsilon_0}m$ -regular spanning subgraphs

$H_{i,i',1}, \dots, H_{i,i',(K-1)/2}$ of $H_{i,i'}$. Set $H_{A,j} := \sum_{1 \leq i, i' \leq K} H_{i,i',j}$ for each $j \leq (K-1)/2$. So (a₅) holds.

Define $G_A := G[A] - (H_{A,1} + \dots + H_{A,(K-1)/2})$. Note that, as $\varepsilon_0 \ll 1/K$, (a) implies that $d_{G_A}(v, A_i) = (1 - 4\mu \pm 5/K)m$ for all $v \in A$ and all $i \leq K$. For each $j \leq (K-1)/2$, let $G_{A,j}$ be the graph on A whose edge set is the union of $G_A[A_i, A_{i'}]$ for each edge $A_i A_{i'} \in E(C_{A,j})$. Define $G_{A,j,\text{dir}}$ to be the oriented graph obtained from $G_{A,j}$ by orienting every edge in $G_A[A_i, A_{i'}]$ from A_i to $A_{i'}$ (for each edge $A_i A_{i'} \in E(C_{A,j})$). Note that $(G_{A,j,\text{dir}}, \mathcal{Q}_A, C_{A,j})$ is a $(K, m, 4\mu, 5/K)$ -cyclic system for each $j \leq (K-1)/2$. Therefore, (a₄) and (a₆) hold. (a₇) can be proved by a similar argument. \square

6.3. Constructing balanced extensions. Let $(G_{A,j}, \mathcal{Q}_A, C_{A,j}, H_{A,j}, \mathcal{J}_{A,j}^*)$ be one of the 5-tuples obtained by Lemma 6.3. The next lemma will be applied to find a balanced extension of $\mathcal{J}_{A,j}^*$ with respect to $(\mathcal{Q}_A, C_{A,j})$, using edges of $H_{A,j}$ (after a suitable orientation of these edges). Consider any $J_{A,\text{dir}}^* \in \mathcal{J}_{A,j,i}^*$. Lemma 6.3(a₃) guarantees that $V(J_{A,\text{dir}}^*) \subseteq A_i$, and so $J_{A,\text{dir}}^*$ is an A_i -extension of itself. Therefore, in order to find a balanced extension of $\mathcal{J}_{A,j}^*$, it is enough to extend each $J_{A,\text{dir}}^* \in \mathcal{J}_{A,j}^*$ into a locally balanced path sequence by adding a directed matching which is vertex-disjoint from $J_{A,\text{dir}}^*$ in such a way that (BE3) is satisfied as well.

Lemma 6.4. *Suppose that $0 < 1/m \ll \varepsilon \ll 1$ and that $m, k \in \mathbb{N}$ with $k \geq 3$. Let $\mathcal{Q} = \{V_1, \dots, V_k\}$ be a (k, m) -equipartition of a set V of vertices and let $C = V_1 \dots V_k$ be a directed cycle. Suppose that there exist a set \mathcal{M} and a graph H on V such that the following conditions hold:*

- (i) \mathcal{M} can be partitioned into k sets $\mathcal{M}_1, \dots, \mathcal{M}_k$ such that $|\mathcal{M}_i| \leq m/k$ and each $M \in \mathcal{M}_i$ is an ordered directed matching with $e(M) \leq \varepsilon m$ and $V(M) \subseteq V_i$ (for all $i \leq k$).
- (ii) $H[V_{i-1}, V_{i+1}]$ is a $2\varepsilon m$ -regular graph for all $i \leq k$.

Then there exist an orientation H_{dir} of H and a balanced extension \mathcal{BE} of \mathcal{M} with respect to (\mathcal{Q}, C) and parameters $(2\varepsilon, 3)$ such that each path sequence in \mathcal{BE} is obtained from some $M \in \mathcal{M}$ by adding edges of H_{dir} .

Proof. Fix $i \leq k$ and write $\mathcal{M}_i := \{M_1, \dots, M_{|\mathcal{M}_i|}\}$. We orient each edge in $H[V_{i-1}, V_{i+1}]$ from V_{i-1} to V_{i+1} . By (ii), $H[V_{i-1}, V_{i+1}]$ can be decomposed into $2\varepsilon m$ perfect matchings. Each perfect matching can be split into $1/2\varepsilon$ matchings, each containing at least εm edges. Recall from (i) that $|\mathcal{M}_i| \leq m/k$ and $e(M_j) \leq \varepsilon m$ for all $M_j \in \mathcal{M}_i$. Hence, $H[V_{i-1}, V_{i+1}]$ contains $|\mathcal{M}_i|$ edge-disjoint matchings $M'_1, \dots, M'_{|\mathcal{M}_i|}$ with $e(M'_j) = e(M_j)$ for all $j \leq |\mathcal{M}_i|$. Define $PS_j := M_j \cup M'_j$. Note that PS_j is locally balanced with respect to C . Also, PS_j is a V_i -extension of M_j (as $M_j \in \mathcal{M}_i$ and so $V(M_j) \subseteq V_i$ by (i)). Moreover,

$$(6.2) \quad |V(PS_j) \cap V_{i'}| = \begin{cases} |V(M_j)| = 2e(M_j) \leq 2\varepsilon m & \text{if } i' = i, \\ e(M_j) \leq \varepsilon m & \text{if } i' = i + 1 \text{ or } i' = i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each $i \leq k$, set $\mathcal{PS}_i := \{PS_1, \dots, PS_{|\mathcal{M}_i|}\}$. Therefore, $\mathcal{BE} := \bigcup_{i \leq k} \mathcal{PS}_i$ is a balanced extension of \mathcal{M} with respect to (\mathcal{Q}, C) and parameters $(2\varepsilon, 3)$. Indeed, (BE3) follows from (6.2). As remarked after the definition of a balanced extension, this also implies the ‘moreover part’ of (BE2). Hence the lemma follows (by orienting the remaining edges of H arbitrarily). \square

7. CONSTRUCTING HAMILTON CYCLES VIA BALANCED EXTENSIONS

Recall that a cyclic system can be viewed as a blow-up of a Hamilton cycle. Given a cyclic system (G, \mathcal{Q}, C) and a balanced extension \mathcal{BE} of a set \mathcal{M} of ordered directed matchings, our aim is to extend each path sequence in \mathcal{BE} into a Hamilton cycle using edges of G . Moreover, each Hamilton cycle has to be consistent with a distinct $M \in \mathcal{M}$. This is achieved by the following lemma, which is the key step in proving Lemma 3.1.

Lemma 7.1. *Suppose that $0 < 1/m \ll \varepsilon_0, \varepsilon', 1/k \ll 1/\ell, \rho \leq 1$, that $0 \leq \mu, \rho \ll 1$ and that $m, k, \ell, q \in \mathbb{N}$ with $q \leq (1 - \mu - \rho)m$. Let (G, \mathcal{Q}, C) be a $(k, m, \mu, \varepsilon')$ -cyclic system and let $\mathcal{M} = \{M_1, \dots, M_q\}$ be a set of q ordered directed matchings. Suppose that $\mathcal{BE} = \{PS_1, \dots, PS_q\}$ is a balanced extension of \mathcal{M} with respect to (\mathcal{Q}, C) and parameters (ε_0, ℓ) such that for each $s \leq q$, $M_s \subseteq PS_s$. Then there exist q Hamilton cycles C_1, \dots, C_q in $G + \mathcal{BE}$ such that for all $s \leq q$, C_s contains PS_s and is consistent with M_s , and such that $C_1 - PS_1, \dots, C_q - PS_q$ are edge-disjoint subgraphs of G .*

Lemma 7.1 will be used both in the two cliques case (i.e. to prove Lemma 3.1) and in the bipartite case (i.e. to prove Lemma 3.2).

We now give an outline of the proof of Lemma 7.1, where for simplicity we assume that the path sequences in the balanced extension \mathcal{BE} are edge-disjoint from each other. Our first step is to remove a sparse subdigraph H from G (see Lemma 7.2), and set $G' := G - H$. Next we extend each path sequence in \mathcal{BE} into a (directed) 1-factor using edges of G' such that all these 1-factors are edge-disjoint from each other (see Lemma 7.3). Finally, we use edges of H to transform the 1-factors into Hamilton cycles (see Lemma 7.6).

The following lemma enables us to find a suitable sparse subdigraph H of G . Recall that (ε, d, d^*, c) -superregularity was defined in Section 4.4.

Lemma 7.2. *Suppose that $0 < 1/m \ll \varepsilon' \ll \gamma \ll \varepsilon \ll 1$ and $0 \leq \mu \ll \varepsilon$. Let G be a bipartite graph with vertex classes U and W of size m such that $d(v) = (1 - \mu \pm \varepsilon')m$ for all $v \in V(G)$. Then there is a spanning subgraph H of G which satisfies the following properties:*

- (i) H is $(\varepsilon, 2\gamma, \gamma, 3\gamma)$ -superregular.
- (ii) Let $G' := G - H$. Then $d_{G'}(v) = (1 - \mu \pm 4\gamma)m$ for all $v \in V(G)$.

Proof. Note that $\delta(G) \geq (1 - \mu - \varepsilon')m \geq (1 - \varepsilon^3)m$ as $\varepsilon', \mu \ll \varepsilon \ll 1$. Thus, whenever $A \subseteq U$ and $B \subseteq W$ are sets of size at least εm , then

$$(7.1) \quad e_G(A, B) \geq (|B| - \varepsilon^3 m)|A| \geq (1 - \varepsilon^2)|A||B|.$$

Let H be a random subgraph of G which is obtained by including each edge of G with probability 2γ . (7.1) implies that whenever $A \subseteq U$ and $B \subseteq W$ are sets of size at least εm then

$$(7.2) \quad 2\gamma(1 - \varepsilon^2)|A||B| \leq \mathbb{E}(e_H(A, B)) \leq 2\gamma|A||B|.$$

Further, for all $u, u' \in V(H)$,

$$(7.3) \quad \mathbb{E}(|N_H(u) \cap N_H(u')|) \leq 4\gamma^2 m$$

and

$$(7.4) \quad 3\gamma m/2 \leq \mathbb{E}(\delta(H)), \mathbb{E}(\Delta(H)) \leq 2\gamma m.$$

Thus, (7.2)–(7.4) together with Proposition 4.1 imply that, with high probability, H is an $(\varepsilon, 2\gamma, \gamma, 3\gamma)$ -superregular pair. Since $\Delta(H) \leq 3\gamma m$ by (Reg3) and $\varepsilon' \ll \gamma$, G' satisfies (ii). \square

7.1. Transforming a balanced extension into 1-factors. The next lemma will be used to extend each locally balanced path sequence PS belonging to a balanced extension \mathcal{BE} into a (directed) 1-factor using edges from G' . We will select the edges from G' in such a way that (apart from the path sequences) the 1-factors obtained are edge-disjoint.

Lemma 7.3. *Suppose that $0 < 1/m \ll 1/k \leq \varepsilon \ll \rho, 1/\ell \leq 1$, that $\rho \ll 1$, that $0 \leq \mu \leq 1/4$ and that $q, m, k, \ell \in \mathbb{N}$ with $q \leq (1 - \mu - \rho)m$. Let (G, \mathcal{Q}, C) be a (k, m, μ, ε) -cyclic system, where $C = V_1 \dots V_k$. Suppose that there exists a set \mathcal{PS} of q path sequences PS_1, \dots, PS_q satisfying the following conditions:*

- (i) *Each $PS_s \in \mathcal{PS}$ is locally balanced with respect to C .*
- (ii) *$|V(PS_s) \cap V_i| \leq \varepsilon m$ for all $i \leq k$ and $s \leq q$. Moreover, for each $i \leq k$, there are at most $\ell m/k$ PS_s such that $V(PS_s) \cap V_i \neq \emptyset$.*

Then there exist q directed 1-factors F_1, \dots, F_q in $G + \mathcal{PS}$ such that for all $s \leq q$ $PS_s \subseteq F_s$ and $F_1 - PS_1, \dots, F_q - PS_q$ are edge-disjoint subgraphs of G .

Proof. By changing the values of ρ, μ and ε slightly, we may assume that $\rho m, \mu m \in \mathbb{N}$. For each $s \leq q$ and each $i \leq k$, let $V_i^{s,-}$ (or $V_i^{s,+}$) be the set of vertices in V_i with indegree (or outdegree) one in PS_s . Since each PS_s is locally balanced with respect to C , $|V_i^{s,+}| = |V_{i+1}^{s,-}| \leq \varepsilon m$ for all $s \leq q$ and all $i \leq k$ (where the inequality follows from (ii)). To prove the lemma, it suffices to show that for each $i \leq k$, there exist edge-disjoint directed matchings M_i^1, \dots, M_i^q , so that each M_i^s is a perfect matching in $G[V_i \setminus V_i^{s,+}, V_{i+1} \setminus V_{i+1}^{s,-}]$. The lemma then follows by setting $F_s := PS_s + \sum_{i \leq k} M_i^s$ for each $s \leq q$.

Fix any $i \leq k$. Without loss of generality (by relabelling the PS_s if necessary) we may assume that there exists an integer s_0 such that $V_i^{s,+} \neq \emptyset$ for all $s \leq s_0$ and $V_i^{s,+} = \emptyset$ for all $s_0 < s \leq q$. By (ii), $s_0 \leq \ell m/k$. Suppose that for some s

with $1 \leq s \leq s_0$ we have already found our desired matchings M_i^1, \dots, M_i^{s-1} in $G[V_i, V_{i+1}]$. Let

$$V'_i := V_i \setminus V_i^{s,+}, \quad V'_{i+1} := V_{i+1} \setminus V_{i+1}^{s,-} \quad \text{and} \quad G_s := G[V'_i, V'_{i+1}] - \sum_{s' < s} M_i^{s'}.$$

Note that each $v \in V'_i$ satisfies

$$d_{G_s}^+(v) \geq d_G^+(v, V_{i+1}) - (|V_{i+1}^{s,-}| + s_0) \geq (1 - \mu - (2\varepsilon + \ell/k))m \geq (1 - \mu - \sqrt{\varepsilon})m$$

by (Sys2) and the fact that $1/k \leq \varepsilon \ll 1/\ell$. Similarly, each $v \in V'_{i+1}$ satisfies $d_{G_s}^-(v) \geq (1 - \mu - \sqrt{\varepsilon})m$. Thus G_s contains a perfect matching M_i^s (this follows, for example, from Hall's theorem). So we can find edge-disjoint matchings $M_i^1, \dots, M_i^{s_0}$ in $G[V_i, V_{i+1}]$.

Let G' be the subdigraph of $G[V_i, V_{i+1}]$ obtained by removing all the edges in $M_i^1, \dots, M_i^{s_0}$. Since $V_i^{s,+} = \emptyset$ for all $s_0 < s \leq q$ (and thus also $V_{i+1}^{s,-} = \emptyset$ for all such s), in order to prove the lemma it suffices to find $q - s_0$ edge-disjoint perfect matchings in G' . Each $v \in V_i$ satisfies

$$d_{G'}^+(v) = d_G^+(v, V_{i+1}) \pm s_0 = d_G^+(v, V_{i+1}) \pm \ell m/k = (1 - \mu \pm \sqrt{\varepsilon})m$$

by (Sys2) and the fact that $1/k \leq \varepsilon \ll 1/\ell$. Similarly, each $v \in V_{i+1}$ satisfies $d_{G'}^-(v) = (1 - \mu \pm \sqrt{\varepsilon})m$. Set $\rho' := \rho + s_0/m$. Note that $\rho'm \in \mathbb{N}$ and $\rho \leq \rho' \leq \rho + \ell/k \leq 2\rho$ as $1/k \ll 1/\ell, \rho$. Hence, $\varepsilon \ll \rho' \ll 1$. Thus we can apply Lemma 4.2 with $G', \rho', \sqrt{\varepsilon}$ playing the roles of $\Gamma, \rho, \varepsilon$ to obtain $(1 - \mu - \rho')m$ edge-disjoint perfect matchings in G' . Since $(1 - \mu - \rho')m = (1 - \mu - \rho)m - s_0 \geq q - s_0$, there exists $q - s_0$ edge-disjoint perfect matchings $M_i^{s_0+1}, \dots, M_i^q$ in G' . This completes the proof of the lemma. \square

7.2. Merging cycles to obtain Hamilton cycles. Recall that we have removed a sparse subdigraph H from G and that $G' = G - H$. Our final step in the proof of Lemma 7.1 is to merge the cycles from each of the 1-factors F_s returned by Lemma 7.3 to obtain edge-disjoint (directed) Hamilton cycles. We will apply Lemma 7.4 to merge the cycles of each F_s , using the edges in H . However, the Hamilton cycles obtained in this way might not be consistent with the matching $M_s \in \mathcal{M}$ that lies in PS_s . Lemma 7.5 is designed to deal with this issue.

Lemma 7.4 was proved in [10] and was first used to construct approximate Hamilton decompositions in [15]. Roughly speaking, it asserts the following: suppose that we have a 1-factor F where most of the edges wind around a cycle $C = V_1 \dots V_k$. Suppose also that we have a digraph H which winds around C . (More precisely, H is the union of superregular pairs $H[V_i, V_{i+1}]$.) Then we can transform F into a Hamilton cycle C' by using a few edges of H . The crucial point is that when applying this lemma, the edges in $C' - F$ can be taken from a small number of the superregular pairs $H[V_i, V_{i+1}]$ (i.e. the set J in Lemma 7.4 will be very small compared to k). In this way, we can transform many 1-factors F into edge-disjoint Hamilton cycles without using any of the pairs $H[V_i, V_{i+1}]$ too often. This in turn means that we will be able to transform all of our 1-factors into edge-disjoint Hamilton cycles by using the edges of a single sparse graph H .

Lemma 7.4. *Suppose that $0 < 1/m \ll d' \ll \varepsilon \ll d \ll \zeta, 1/t \leq 1/2$. Let V_1, \dots, V_k be pairwise disjoint clusters, each of size m , and let $C = V_1 \dots V_k$ be a directed cycle on these clusters. Let H be a digraph on $V_1 \cup \dots \cup V_k$ and let $J \subseteq E(C)$. For each edge $V_i V_{i+1} \in J$, let $V_i^1 \subseteq V_i$ and $V_{i+1}^2 \subseteq V_{i+1}$ be such that $|V_i^1| = |V_{i+1}^2| \geq m/100$ and such that $H[V_i^1, V_{i+1}^2]$ is $(\varepsilon, d', \zeta d', td'/d)$ -superregular. Suppose that F is a 1-regular digraph with $V_1 \cup \dots \cup V_k \subseteq V(F)$ such that the following properties hold:*

- (i) *For each edge $V_i V_{i+1} \in J$ the digraph $F[V_i^1, V_{i+1}^2]$ is a perfect matching.*
- (ii) *For each cycle D in F there is some edge $V_i V_{i+1} \in J$ such that D contains a vertex in V_i^1 .*
- (iii) *Whenever $V_i V_{i+1}, V_j V_{j+1} \in J$ are such that J avoids all edges in the segment $V_{i+1} C V_j$ of C from V_{i+1} to V_j , then F contains a path P_{ij} joining some vertex $u_{i+1} \in V_{i+1}^2$ to some vertex $u'_j \in V_j^1$ such that P_{ij} winds around C .*

Then we can obtain a directed cycle on $V(F)$ from F by replacing $F[V_i^1, V_{i+1}^2]$ with a suitable perfect matching in $H[V_i^1, V_{i+1}^2]$ for each edge $V_i V_{i+1} \in J$.

Lemma 7.5. *Suppose that $0 < 1/m \ll \gamma \ll d' \ll \varepsilon \ll d \ll \zeta, 1/t \leq 1/2$. Let V_1, \dots, V_k be pairwise disjoint clusters, each of size m , and let $C = V_1 \dots V_k$ be a directed cycle on these clusters. Let $1 \leq i \leq k$ be fixed and let $V_i^1 \subseteq V_i$ and $V_{i+1}^2 \subseteq V_{i+1}$ be such that $|V_i^1| = |V_{i+1}^2| \geq m/100$. Suppose that $H = H[V_i^1, V_{i+1}^2]$ is an $(\varepsilon, d', \zeta d', td'/d)$ -superregular bipartite digraph. Let $X = \{x_1, \dots, x_p\} \subseteq V_i^1$ with $|X| \leq \gamma m$. Suppose that C' is a directed cycle with $V_1 \cup \dots \cup V_k \subseteq V(C')$ such that $C'[V_i^1, V_{i+1}^2]$ is a perfect matching. Then we can obtain a directed cycle on $V(C')$ from C' that visits the vertices x_1, \dots, x_p in order by replacing $C'[V_i^1, V_{i+1}^2]$ with a suitable perfect matching in $H[V_i^1, V_{i+1}^2]$.*

Proof. Pick ν and τ such that $\gamma \ll \nu \ll \tau \ll d'$. For every $u \in V_i^1$, starting at u we move along the cycle C' (but in the opposite direction to the orientation of the edges) and let $f(u)$ be the first vertex on C' in V_{i+1}^2 . (Note that $f(u)$ exists since $C'[V_i^1, V_{i+1}^2]$ is a perfect matching. Moreover, $f(u) \neq f(v)$ if $u \neq v$.) Define an auxiliary digraph A on V_{i+1}^2 such that $N_A^+(f(u)) := N_H^+(u)$. So A is obtained by identifying each pair $(u, f(u))$ into one vertex with an edge from $(u, f(u))$ to $(v, f(v))$ if H has an edge from u to $f(v)$. So Lemma 4.6 applied with $d', d/t$ playing the roles of d, μ implies that A is a robust (ν, τ) -outexpander. Moreover, $\delta^+(A), \delta^-(A) \geq \zeta d' |V_{i+1}^2| = \zeta d' |A|$ by (Reg4). Thus Theorem 4.3 implies that A has a Hamilton cycle visiting $f(x_1), \dots, f(x_p)$ in order, which clearly corresponds to a perfect matching M in H with the desired property. \square

The above proof idea is actually quite similar to that for Lemma 7.4 itself. We now apply Lemmas 7.4 and 7.5 to each 1-factor F_s given by Lemma 7.3 and obtain edge-disjoint Hamilton cycles that are consistent with the M_s .

Lemma 7.6. *Suppose that $0 < 1/m \ll \varepsilon_0, 1/k \ll \gamma \ll \varepsilon \ll 1$, that $\gamma \ll 1/\ell \leq 1$ and that $q, m, k, \ell \in \mathbb{N}$. Let $\mathcal{Q} = \{V_1, \dots, V_k\}$ be a (k, m) -equipartition of a vertex set V and let $C = V_1 \dots V_k$ be a directed cycle. Let $\mathcal{M} = \{M_1, \dots, M_q\}$ be a set of ordered directed matchings. Suppose that $\mathcal{BE} = \{PS_1, \dots, PS_q\}$ is a balanced extension of \mathcal{M} with respect to (\mathcal{Q}, C) and parameters (ε_0, ℓ) . Furthermore, suppose that there*

exist 1-regular digraphs F_1, \dots, F_q on V such that for each $s \leq q$, $PS_s \subseteq F_s$ and such that $F_s - PS_s$ winds around C . Let H be a digraph on V which is edge-disjoint from each of $F_1 - PS_1, \dots, F_q - PS_q$ and such that $H[V_i, V_{i+1}]$ is $(\varepsilon, 2\gamma, \gamma, 3\gamma)$ -superregular for all $i \leq k$. Then there exist q Hamilton cycles C_1, \dots, C_q in $F_1 + \dots + F_q + H$ such that C_s contains PS_s and is consistent with M_s for all $s \leq q$ and such that $C_1 - F_1, \dots, C_q - F_q$ are edge-disjoint subgraphs of H .

Proof. Recall from (BE2) that for each $s \leq q$ there is some $i_s \leq k$ such that PS_s is a V_{i_s} -extension of M_s . In particular, $M_s \subseteq PS_s$. Let I_s be the set consisting of all $i \leq k$ such that $V_i \cap V(PS_s) \neq \emptyset$. Since \mathcal{BE} is a balanced extension with parameters (ε_0, ℓ) , (BE3) implies that for every $i \leq k$ we have

$$(7.5) \quad |\{s : i \in I_s\}| \leq \ell m/k.$$

For each $s \leq q$ in turn, we are going to show that there exist Hamilton cycles C_1, \dots, C_s in $F_1 + \dots + F_s + H$ such that

- (a_s) $PS_{s'} \subseteq C_{s'}$ and $C_{s'}$ is consistent with $M_{s'}$ for all $s' \leq s$,
- (b_s) $E(C_{s'} - F_{s'}) \subseteq \bigcup_{i \in I_{s'}} E(H[V_i, V_{i+1}])$ for all $s' \leq s$,
- (c_s) $C_1 - F_1, \dots, C_s - F_s$ are pairwise edge-disjoint.

So suppose that for some s with $1 \leq s \leq q$ we have already constructed C_1, \dots, C_{s-1} . We now construct C_s as follows. Let $H_s := H - \sum_{s' < s} (C_{s'} - F_{s'})$. Define a new constant d such that $\varepsilon \ll d \ll 1$.

Our first task is to apply Lemma 7.4 to F_s to merge all the cycles in F_s into a Hamilton cycle using only edges of H_s . For each $i \in I_s$, let V_i^- be the set of vertices in V_i with indegree one in PS_s and let V_i^+ be the set of vertices in V_i with outdegree one in PS_s . Set $V_i^1 := V_i \setminus V_i^+$ and $V_{i+1}^2 := V_{i+1} \setminus V_{i+1}^-$. Since PS_s is locally balanced, $|V_i^+| = |V_{i+1}^-| \leq \varepsilon_0 m$ for all $i \in I_s$ (where the inequality holds by (BE3)). By (b_{s-1}) and (7.5), $H_s[V_i, V_{i+1}]$ is obtained from $H[V_i, V_{i+1}]$ by removing at most $|\{s' < s : i \in I_{s'}\}| \leq \ell m/k \leq \varepsilon^2 \gamma m$ edges from each vertex (as $1/k \ll \varepsilon, \gamma, 1/\ell$). So by Proposition 4.5, $H_s[V_i, V_{i+1}]$ is still $(2\varepsilon, 2\gamma, \gamma/2, 3\gamma)$ -superregular for each $i \in I_s$. Recall that $|V_i \setminus V_i^1| = |V_{i+1} \setminus V_{i+1}^2| \leq \varepsilon_0 m$. Hence $H_s[V_i^1, V_{i+1}^2]$ is $(4\varepsilon, 2\gamma, \gamma/4, 6\gamma)$ -superregular by Proposition 4.4 and thus also $(4\varepsilon, 2\gamma, \gamma/4, 4\gamma/d)$ -superregular.

Let $E_s := \{V_i V_{i+1} : i \in I_s\}$. Our aim is to apply Lemma 7.4 with $F_s, E_s, H_s, 4\varepsilon, 2\gamma, 2, 1/8$ playing the roles of $F, J, H, \varepsilon, d', t, \zeta$. Our assumption that $F_s - PS_s$ winds around C implies that for each $i \in I_s$, $F_s[V_i^1, V_{i+1}^2]$ is a perfect matching. So Lemma 7.4(i) holds. Note that every final vertex of a nontrivial path in PS_s must lie in $\bigcup_{i \in I_s} V_i^1$, implying Lemma 7.4(ii). Finally, recall that $|V_i^1|, |V_{i+1}^2| \geq (1 - \varepsilon_0)m$ for all $i \in I_s$. Together with our assumption that $F_s - PS_s$ winds around C , this easily implies Lemma 7.4(iii). So we can apply Lemma 7.4 to obtain a Hamilton cycle C'_s which is constructed from F_s by replacing $F_s[V_i^1, V_{i+1}^2]$ with a suitable perfect matching in $H_s[V_i^1, V_{i+1}^2]$ for each $i \in I_s$. In particular, $PS_s \subseteq C'_s$.

Let $H'_s := H_s - (C'_s - F_s)$. Recall that M_s is an ordered directed matching, say $M_s = \{e_1, \dots, e_r\}$, and that PS_s is a V_{i_s} -extension of M_s . For each $j \leq r$, let P_j be the path in PS_s containing e_j and let x_j denote the final vertex of P_j . Hence x_1, \dots, x_r are distinct and lie in $V_{i_s}^1$. Together with (BE3) this implies that $r \leq \varepsilon_0 m$. Note that $H'_s[V_{i_s}^1, V_{i_s+1}^2]$ is obtained from $H_s[V_{i_s}^1, V_{i_s+1}^2]$ by removing a

perfect matching, namely $C'_s[V_{i_s}^1, V_{i_s+1}^2]$. So by Proposition 4.5, $H'_s[V_{i_s}^1, V_{i_s+1}^2]$ is still $(8\varepsilon, 2\gamma, \gamma/8, 4\gamma/d)$ -superregular. Apply Lemma 7.5 with $C'_s, i_s, H'_s[V_{i_s}^1, V_{i_s+1}^2], \varepsilon_0, 8\varepsilon, 2\gamma, 2, 1/16$ playing the roles of $C', i, H, \gamma, \varepsilon, d', t, \zeta$ to obtain a Hamilton cycle C_s which visits x_1, \dots, x_r in this order and is constructed from C'_s by replacing the perfect matching $C'_s[V_{i_s}^1, V_{i_s+1}^2]$ with a suitable perfect matching in $H'_s[V_{i_s}^1, V_{i_s+1}^2]$. In particular, $PS_s \subseteq C_s$.

Note that $E(C_s - F_s) \subseteq \bigcup_{i \in I_s} E(H_s[V_i, V_{i+1}])$, so (b_s) and (c_s) hold. Since $PS_s \subseteq C_s$ and x_j is the final vertex of P_j and since $e_j \in E(P_j)$, it follows that C_s visits the edges e_1, \dots, e_r in order. So C_s is consistent with M_s , implying (a_s). \square

Proof of Lemma 7.1. Let $\mathcal{Q} = \{V_1, \dots, V_k\}$. By relabeling the V_i if necessary, we may assume that $C = V_1 \dots V_k$. Define new constants γ and ε such that $\varepsilon_0, \varepsilon', 1/k \ll \gamma \ll \varepsilon, \rho, 1/\ell$ and $\mu \ll \varepsilon \ll 1$. For each $i \leq k$ we apply Lemma 7.2 to (the underlying undirected graph of) $G[V_i, V_{i+1}]$ in order to obtain a spanning subdigraph H of G which satisfies the following properties:

- (i') For each $i \leq k$, $H[V_i, V_{i+1}]$ is $(\varepsilon, 2\gamma, \gamma, 3\gamma)$ -superregular.
- (ii') Let $G' := G - H$. Then (G', \mathcal{Q}, C) is a $(k, m, \mu, 4\gamma)$ -cyclic system.

Indeed, (ii') follows easily from Lemma 7.2(ii) and the definition of a $(k, m, \mu, 4\gamma)$ -cyclic system. Recall that $\mathcal{BE} = \{PS_1, \dots, PS_q\}$ with $M_s \subseteq PS_s$ for all $s \leq q$. Our next aim is to apply Lemma 7.3 with $G', \mathcal{BE}, 4\gamma$ playing the roles of $G, \mathcal{PS}, \varepsilon$ to obtain 1-factors F_s extending the PS_s . Note that (BE1) and (BE3) imply that conditions (i) and (ii) of Lemma 7.3 hold. So we can apply Lemma 7.3 to obtain q (directed) 1-factors F_1, \dots, F_q in $G' + \mathcal{BE}$ such that $PS_s \subseteq F_s$ for all $s \leq q$ and $F_1 - PS_1, \dots, F_q - PS_q$ are edge-disjoint subgraphs of G' . Recall from (ii') and (Sys2) that G' (and thus also $F_s - PS_s$) winds around C . So we can apply Lemma 7.6 to obtain q Hamilton cycles C_1, \dots, C_q in $F_1 + \dots + F_q + H$ such that C_s contains PS_s and is consistent with M_s for all $s \leq q$, and such that $C_1 - F_1, \dots, C_q - F_q$ are edge-disjoint subgraphs of H . Since H and G' are edge-disjoint, $C_1 - PS_1, \dots, C_q - PS_q$ are edge-disjoint subgraphs of G . \square

We can now put everything together to prove the approximate decomposition lemma in the two cliques case. First we apply Lemma 6.3 to obtain cyclic systems and sparse subgraphs $H_{A,j}$ and $H_{B,j}$. Then we apply Lemma 6.4 to balance out the exceptional systems into balanced extensions. Next, we apply Lemma 7.1 to A and B separately to extend the balanced extensions into Hamilton cycles.

Proof of Lemma 3.1. Apply Lemma 6.3 to G, \mathcal{P} and \mathcal{J} to obtain (for each $1 \leq j \leq (K-1)/2$) pairs of tuples $(G_{A,j}, \mathcal{Q}_A, C_{A,j}, H_{A,j}, \mathcal{J}_{A,j}^*)$ and $(G_{B,j}, \mathcal{Q}_B, C_{B,j}, H_{B,j}, \mathcal{J}_{B,j}^*)$ which satisfy (a₁)–(a₇). Fix $j \leq (K-1)/2$. Write $\mathcal{J}_{A,j}^* = \{J_{A,\text{dir},1}^*, \dots, J_{A,\text{dir},q}^*\}$, where

$$(7.6) \quad q := |\mathcal{J}_{A,j}^*| \leq (1 - 4\mu - 3\rho)m$$

by (a₃). We now apply Lemma 6.4 with $\mathcal{J}_{A,j}^*, \mathcal{Q}_A, C_{A,j}, H_{A,j}, K, 5K\sqrt{\varepsilon_0}$ playing the roles of $\mathcal{M}, \mathcal{Q}, C, H, k, \varepsilon$ to obtain an orientation $H_{A,j,\text{dir}}$ of $H_{A,j}$ and a balanced extension \mathcal{BE}_j of $\mathcal{J}_{A,j}^*$ with respect to $(\mathcal{Q}_A, C_{A,j})$ and parameters $(10K\sqrt{\varepsilon_0}, 3)$.

(Note that (a₃) and (a₅) imply conditions (i) and (ii) of Lemma 6.4.) Write $\mathcal{BE}_j := \{PS_1, \dots, PS_q\}$ such that $J_{A,\text{dir},s}^* \subseteq PS_s$ for all $s \leq q$. So (BE1) implies that $PS_1 - J_{A,\text{dir},1}^*, \dots, PS_q - J_{A,\text{dir},q}^*$ are edge-disjoint subgraphs of $H_{A,j,\text{dir}}$. Since $(G_{A,j,\text{dir}}, \mathcal{Q}_A, C_{A,j})$ is a $(K, m, 4\mu, 5/K)$ -cyclic system by (a₆), (7.6) implies that we can apply Lemma 7.1 as follows:

	$G_{A,j,\text{dir}}$	\mathcal{Q}_A	$C_{A,j}$	K	$\mathcal{J}_{A,j}^*$	q	4μ	3ρ	$10K\sqrt{\varepsilon_0}$	$5/K$	3
plays role of	G	\mathcal{Q}	C	k	\mathcal{M}	q	μ	ρ	ε_0	ε'	ℓ

In this way we obtain q directed Hamilton cycles $C'_{A,j,1}, \dots, C'_{A,j,q}$ in $G_{A,j,\text{dir}} + \mathcal{BE}_j$ such that $C'_{A,j,s}$ contains PS_s and is consistent with $J_{A,\text{dir},s}^*$ for all $s \leq q$. Moreover, $C'_1 - J_{A,\text{dir},1}^*, \dots, C'_q - J_{A,\text{dir},q}^*$ are edge-disjoint subgraphs of $G_{A,j,\text{dir}} + H_{A,j,\text{dir}}$. Repeat this process for all $j \leq (K-1)/2$.

Write $\mathcal{J} = \{J_1, \dots, J_{|\mathcal{J}|}\}$. Recall from (a₂) that the $\mathcal{J}_{A,1}^*, \dots, \mathcal{J}_{A,(K-1)/2}^*$ partition $\{J_{A,\text{dir}}^* : J \in \mathcal{J}\}$. Therefore, we have obtained $|\mathcal{J}|$ directed Hamilton cycles $C'_{A,1}, \dots, C'_{A,|\mathcal{J}|}$ on vertex set A . Moreover, by relabelling the J_s if necessary, we may assume that $C'_{A,s}$ is consistent with $(J_s)_{A,\text{dir}}^*$ for all $s \leq |\mathcal{J}|$. Furthermore, (a₄) implies that the undirected versions of $C'_{A,1} - (J_1)_{A,\text{dir}}^*, \dots, C'_{A,|\mathcal{J}|} - (J_{|\mathcal{J}|})_{A,\text{dir}}^*$ are edge-disjoint spanning subgraphs of $G[A]$.

Similarly we obtain directed Hamilton cycles $C'_{B,1}, \dots, C'_{B,|\mathcal{J}|}$ on vertex set B so that $(J_s)_{B,\text{dir}}^* \subseteq C'_{B,s}$ for all $s \leq |\mathcal{J}|$. Let H_s be the undirected graph obtained from $C'_{A,s} + C'_{B,s} - J_s^* + J_s$ by ignoring all the orientations of the edges. Recall that $J_1, \dots, J_{|\mathcal{J}|}$ are edge-disjoint exceptional systems and that they are edge-disjoint from the $C'_{A,s} + C'_{B,s} - J_s^*$ by (EC3). So $H_1, \dots, H_{|\mathcal{J}|}$ are edge-disjoint spanning subgraphs of G . Finally, Proposition 6.2 implies that $H_1, \dots, H_{|\mathcal{J}|}$ are indeed as desired in Lemma 3.1. \square

8. THE BIPARTITE CASE

Roughly speaking, the idea in this case is to reduce the problem of finding the desired edge-disjoint Hamilton cycles in G to that of finding suitable Hamilton cycles in an almost complete balanced bipartite graph. This is achieved by considering the graphs J_{dir}^* , which we define in the next subsection. The main steps are similar to those in the proof of Lemma 3.1 (in fact, we re-use several of the lemmas, in particular Lemma 7.1).

We will construct the graphs J_{dir}^* , which are based on balanced exceptional systems J , in Section 8.1. In Section 8.2 we describe a decomposition of G into blown-up Hamilton cycles. We will construct balanced extensions in Section 8.3 (this is more difficult than in the two cliques case). Finally, we obtain the desired Hamilton cycles using Lemma 7.1 (in the same way as in the two cliques case).

8.1. Defining the graphs J_{dir}^* for the bipartite case. Let \mathcal{P} be a (K, m, ε) -partition of a vertex set V and let J be a balanced exceptional system with respect to \mathcal{P} (as defined in Section 3.2). We construct J^* in two steps. First we will construct a matching J_{AB}^* on $A \cup B$ and then J^* . Since each maximal path in J has

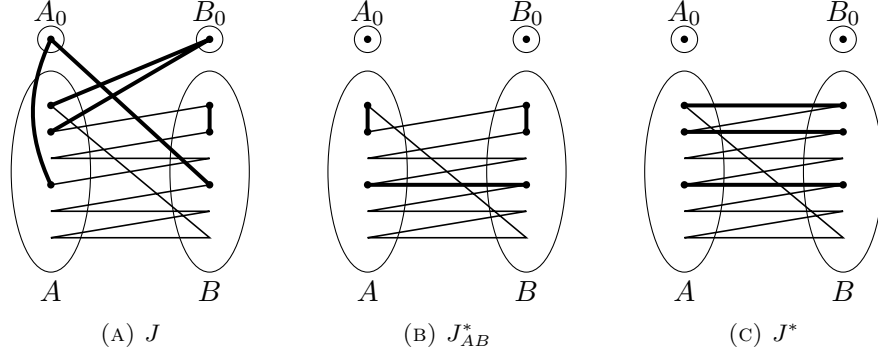


FIGURE 2. The thick lines illustrate the edges of J , J_{AB}^* and J^* , respectively.

endpoints in $A \cup B$ and internal vertices in V_0 by (BES1), a balanced exceptional system J naturally induces a matching J_{AB}^* on $A \cup B$. More precisely, if $P_1, \dots, P_{\ell'}$ are the non-trivial paths in J and x_i, y_i are the endpoints of P_i , then we define $J_{AB}^* := \{x_i y_i : i \leq \ell'\}$. Thus J_{AB}^* is a matching by (BES1) and $e(J_{AB}^*) \leq e(J)$. Moreover, J_{AB}^* and $E(J)$ cover exactly the same vertices in A . Similarly, they cover exactly the same vertices in B . So (BES3) implies that $e(J_{AB}^*[A]) = e(J_{AB}^*[B])$. We can write $E(J_{AB}^*[A]) = \{x_1 x_2, \dots, x_{2s-1} x_{2s}\}$, $E(J_{AB}^*[B]) = \{y_1 y_2, \dots, y_{2s-1} y_{2s}\}$ and $E(J_{AB}^*[A, B]) = \{x_{2s+1} y_{2s+1}, \dots, x_{s'} y_{s'}\}$, where $x_i \in A$ and $y_i \in B$. Define $J^* := \{x_i y_i : 1 \leq i \leq s'\}$. Note that

$$(8.1) \quad e(J^*) = e(J_{AB}^*) \leq e(J).$$

All edges of J^* are called *fictive edges*. We say that an (undirected) cycle D is *consistent with J^** if D contains J^* and (there is an orientation of D which) visits the vertices $x_1, y_1, x_2, \dots, y_{s'-1}, x_{s'}, y_{s'}$ in this order. The following proposition is proved in [4]. It is illustrated in Figure 2.

Proposition 8.1. *Let \mathcal{P} be a (K, m, ε) -partition of a vertex set V . Let G be a graph on V and let J be a balanced exceptional system with respect to \mathcal{P} . If $J \subseteq G$ and D is a Hamilton cycle of $G[A \cup B] + J^*$ which is consistent with J^* , then $D - J^* + J$ is a Hamilton cycle of G .*

Again, we will actually need a directed version, which follows immediately from the above. For this, define J_{dir}^* to be the ordered directed matching $\{f_1, \dots, f_{s'}\}$ such that f_i is a directed edge from x_i to y_i for all $i \leq s'$. So J_{dir}^* consists only of AB -edges. Similarly to the undirected case, we say that a directed cycle D_{dir} is *consistent with J_{dir}^** if D_{dir} contains J_{dir}^* and visits the edges $f_1, \dots, f_{s'}$ in this order.

Proposition 8.2. *Let \mathcal{P} be a (K, m, ε) -partition of a vertex set V . Let G be a graph on V and let J be a balanced exceptional system with respect to \mathcal{P} such that $J \subseteq G$. Suppose that D_{dir} is a directed Hamilton cycle on $A \cup B$ such that D_{dir} is consistent with J_{dir}^* . Furthermore, suppose that $D - J^* \subseteq G$, where D is the cycle obtained*

from D_{dir} after ignoring the directions of all edges. Then $D - J^* + J$ is a Hamilton cycle of G .

8.2. Finding systems. The following lemma gives a decomposition of an almost complete bipartite graph G into blown-up Hamilton cycles (together with an associated decomposition of exceptional systems). Its proof is almost the same as that of Lemma 6.3, so we omit it here. The only difference is that instead of Walecki's theorem we use a result of Auerbach and Laskar [1] to decompose the complete bipartite graph $K_{K,K}$ into Hamilton cycles, where K is even.

Lemma 8.3. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll 1/K \ll \rho \ll 1$ and $0 \leq \mu \ll 1$, where $n, K \in \mathbb{N}$ and K is even. Suppose that G is a graph on n vertices and $\mathcal{P} = \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$ is a (K, m, ε_0) -partition of $V(G)$. Furthermore, suppose that the following conditions hold:*

- (a) $d(v, B_i) = (1 - 4\mu \pm 4/K)m$ and $d(w, A_i) = (1 - 4\mu \pm 4/K)m$ for all $v \in A$, $w \in B$ and $1 \leq i \leq K$.
- (b) There is a set \mathcal{J} which consists of at most $(1/4 - \mu - \rho)n$ edge-disjoint exceptional systems with parameter ε_0 in G .
- (c) \mathcal{J} has a partition into K^4 sets $\mathcal{J}_{i_1, i_2, i_3, i_4}$ (one for all $1 \leq i_1, i_2, i_3, i_4 \leq K$) such that each $\mathcal{J}_{i_1, i_2, i_3, i_4}$ consists of precisely $|\mathcal{J}|/K^4$ (i_1, i_2, i_3, i_4) -BES with respect to \mathcal{P} .

Then for each $1 \leq j \leq K/2$, there is a tuple $(G_j, \mathcal{Q}, C_j, H_j, \mathcal{J}_j)$ such that the following assertions hold, where $\mathcal{Q} := \{A_1, \dots, A_K, B_1, \dots, B_K\}$:

- (a₁) Each of $C_1, \dots, C_{K/2}$ is a directed Hamilton cycle on \mathcal{Q} such that the undirected versions of these cycles form a Hamilton decomposition of the complete bipartite graph whose vertex classes are $\{A_1, \dots, A_K\}$ and $\{B_1, \dots, B_K\}$.
- (a₂) $\mathcal{J}_1, \dots, \mathcal{J}_{K/2}$ is a partition of \mathcal{J} .
- (a₃) Each \mathcal{J}_j has a partition into K^4 sets $\mathcal{J}_{j, i_1, i_2, i_3, i_4}$ (one for all $1 \leq i_1, i_2, i_3, i_4 \leq K$) such that $\mathcal{J}_{j, i_1, i_2, i_3, i_4}$ consists of (i_1, i_2, i_3, i_4) -BES with respect to \mathcal{P} and $|\mathcal{J}_{j, i_1, i_2, i_3, i_4}| \leq (1 - 4\mu - 3\rho)m/K^4$.
- (a₄) $G_1, \dots, G_{K/2}, H_1, \dots, H_{K/2}$ are edge-disjoint subgraphs of $G[A, B]$.
- (a₅) $H_j[A_i, B_{i'}]$ is a $(11K + 248/K)\varepsilon_0 m$ -regular graph for all $j \leq K/2$ and all $i, i' \leq K$.
- (a₆) For each $j \leq K/2$, there exists an orientation $G_{j, \text{dir}}$ of G_j such that $(G_{j, \text{dir}}, \mathcal{Q}, C_j)$ is a $(2K, m, 4\mu, 5/K)$ -cyclic system.

8.3. Constructing balanced extensions. Let $\mathcal{P} = \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$ be a (K, m, ε) -partition of a vertex set V , let $\mathcal{Q} := \{A_1, \dots, A_K, B_1, \dots, B_K\}$ and let $C = A_1 B_1 A_2 B_2 \dots A_K B_K$ be a directed cycle. Given a set \mathcal{J} of balanced exceptional systems with respect to \mathcal{P} , we write $\mathcal{J}_{\text{dir}}^* := \{J_{\text{dir}}^* : J \in \mathcal{J}\}$. So $\mathcal{J}_{\text{dir}}^*$ is a set of ordered directed matchings and thus it makes sense to construct a balanced extension of $\mathcal{J}_{\text{dir}}^*$ with respect to (\mathcal{Q}, C) . (Recall that balanced extensions were defined in Section 5.2.)

Now consider any of the tuples $(G_j, \mathcal{Q}, C_j, H_j, \mathcal{J}_j)$ guaranteed by Lemma 8.3. We will apply the following lemma to find a balanced extension of $(\mathcal{J}_j)_{\text{dir}}^*$ with respect to (\mathcal{Q}, C_j) , using edges of H_j (after a suitable orientation of these edges). So the

lemma is a bipartite analogue of Lemma 6.4. However, the proof is more involved than in the two cliques case.

Lemma 8.4. *Suppose that $0 < 1/n \ll \varepsilon \ll 1/K \ll 1$, where $n, K \in \mathbb{N}$. Let $\mathcal{P} = \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$ be a (K, m, ε) -partition of a set V of n vertices. Let $\mathcal{Q} := \{A_1, \dots, A_K, B_1, \dots, B_K\}$ and let $C := A_1 B_1 A_2 B_2 \dots A_K B_K$ be a directed cycle. Suppose that there exist a set \mathcal{J} of edge-disjoint balanced exceptional systems with respect to \mathcal{P} and parameter ε and a graph H such that the following conditions hold:*

- (i) \mathcal{J} can be partitioned into K^4 sets $\mathcal{J}_{i_1, i_2, i_3, i_4}$ (one for all $1 \leq i_1, i_2, i_3, i_4 \leq K$) such that $\mathcal{J}_{i_1, i_2, i_3, i_4}$ consists of (i_1, i_2, i_3, i_4) -BES with respect to \mathcal{P} and $|\mathcal{J}_{i_1, i_2, i_3, i_4}| \leq m/K^4$.
- (ii) For each $v \in A \cup B$ the number of all those $J \in \mathcal{J}$ for which v is incident to an edge in J is at most $2\varepsilon n$.
- (iii) $H[A_i, B_{i'}]$ is a $(11K + 248/K)\varepsilon m$ -regular graph for all $i, i' \leq K$.

Then there exist an orientation H_{dir} of H and a balanced extension \mathcal{BE} of $\mathcal{J}_{\text{dir}}^*$ with respect to (\mathcal{Q}, C) and parameters $(12\varepsilon K, 12)$ such that each path sequence in \mathcal{BE} is obtained from some $J_{\text{dir}}^* \in \mathcal{J}_{\text{dir}}^*$ by adding edges of H_{dir} .

The proof proceeds roughly as follows. Consider any $J \in \mathcal{J}_{i_1, i_2, i_3, i_4}$. We extend J_{dir}^* into a locally balanced path sequence in two steps. For this, recall that J_{dir}^* consists only of edges from $A_{i_1} \cup A_{i_2}$ to $B_{i_3} \cup B_{i_4}$. In the first step, we construct a path sequence PS that is an A_{i_1} -extension of J_{dir}^* by adding suitable $B_{i_3} A_{i_1}$ - and $B_{i_4} A_{i_1}$ -edges from H to J_{dir}^* . In the second step, we locally balance PS in such a way that (BE1)–(BE3) are satisfied.

Proof. First we decompose H into H' and H'' such that $H'[A_i, B_{i'}]$ is a $11\varepsilon K m$ -regular graph for all $i, i' \leq K$ and $H'' := H - H'$. Hence $H''[A_i, B_{i'}]$ is a $248\varepsilon m/K$ -regular graph for all $i, i' \leq K$.

Write $\mathcal{J} := \{J_1, \dots, J_{|\mathcal{J}|}\}$. For each $s \leq |\mathcal{J}|$, we will extend $J_{s, \text{dir}}^* := (J_s)_{\text{dir}}^*$ into a path sequence PS_s satisfying the following conditions:

- (α_s) Suppose that $J_s \in \mathcal{J}_{i_1, i_2, i_3, i_4}$. Then PS_s is an A_{i_1} -extension of $J_{s, \text{dir}}^*$ consisting of precisely $e(J_s^*)$ vertex-disjoint directed paths of length two.
- (β_s) $V(PS_s) = V(J_{s, \text{dir}}^*) \cup A'_s$, where $A'_s \subseteq A_{i_1} \setminus V(J_{s, \text{dir}}^*)$ and $|A'_s| = e(J_s^*)$.
- (γ_s) $PS_s - J_{s, \text{dir}}^*$ is a matching of size $e(J_s^*)$ from B'_s to A'_s , where $B'_s := V(J_{s, \text{dir}}^*) \cap (B_{i_3} \cup B_{i_4})$.
- (δ_s) Let M_s be the set of undirected edges obtained from $PS_s - J_{s, \text{dir}}^*$ after ignoring all the orientations. Then M_1, \dots, M_s are edge-disjoint matchings in H' .
- (ε_s) PS_s consists only of edges from $A_{i_1} \cup A_{i_2}$ to $B_{i_3} \cup B_{i_4}$, and from $B_{i_3} \cup B_{i_4}$ to A_{i_1} .

Note that (β_s) and (γ_s) together imply (ε_s). Suppose that for some s with $1 \leq s \leq |\mathcal{J}|$ we have already constructed PS_1, \dots, PS_{s-1} . We will now construct PS_s as follows. Let i_1, i_2, i_3, i_4 be such that $J_s \in \mathcal{J}_{i_1, i_2, i_3, i_4}$ and let $H'_s := H' - (M_1 + \dots +$

M_{s-1}). (BES4) implies that

$$(8.2) \quad e(J_{s,\text{dir}}^*) = e(J_s^*) \stackrel{(8.1)}{\leq} e(J_s) \leq \varepsilon n \leq 3\varepsilon Km \quad \text{and} \quad |V(J_{s,\text{dir}}^*) \cap A_{i_1}| \leq 3\varepsilon Km.$$

Consider any $s' < s$. Recall from the definition of $J_{s',\text{dir}}^*$ that $V(J_{s',\text{dir}}^*)$ is the set of all those vertices in $A \cup B$ which are covered by edges of $J_{s'}$. Together with $(\beta_{s'})$ and $(\gamma_{s'})$ this implies that a vertex $v \in B$ is covered by $M_{s'}$ if and only if v is incident to an edge of $J_{s'}$. Together with (ii) this in turn implies that for all $v \in B$ we have

$$\begin{aligned} d_{H_s^*}(v, A_{i_1}) &\geq d_{H'}(v, A_{i_1}) - \sum_{s' < s} d_{M_{s'}}(v) \geq 11K\varepsilon m - 2\varepsilon n \geq 11K\varepsilon m - 5K\varepsilon m \\ &\stackrel{(8.2)}{\geq} |V(J_{s,\text{dir}}^*) \cap A_{i_1}| + e(J_s^*). \end{aligned}$$

Note that $e(J_s^*) = |V(J_{s,\text{dir}}^*) \cap (B_{i_3} \cup B_{i_4})| = |B'_s|$. So we can greedily find a matching M_s of size $e(J_s^*)$ in $H'_s[A_{i_1} \setminus V(J_{s,\text{dir}}^*), B'_s]$ (which therefore covers all vertices in B'_s). Orient all edges of M_s from B'_s to A_{i_1} and call the resulting directed matching $M_{s,\text{dir}}$. Set

$$PS_s := J_{s,\text{dir}}^* + M_{s,\text{dir}}.$$

Note that PS_s consists of precisely $e(J_s^*)$ directed paths of length two whose final vertices lie in A_{i_1} , so (α_s) – (ε_s) hold by our construction. This shows that we can obtain path sequences $PS_1, \dots, PS_{|\mathcal{J}|}$ satisfying (α_s) – (ε_s) for all $s \leq |\mathcal{J}|$.

The following claim provides us with a ‘reservoir’ of edges which we will use to balance out the edges of each PS_s and thus extend each PS_s into a path sequence PS'_s which is locally balanced with respect to C .

Claim. H'' contains $|\mathcal{J}|$ subgraphs $H''_1, \dots, H''_{|\mathcal{J}|}$ satisfying the following properties for all $s \leq |\mathcal{J}|$ and all $i, i' \leq K$:

- (a₁) If PS_s contains an $A_i B_{i'}$ -edge, then H''_s contains a matching between $A_{i'}$ and B_i of size $30\varepsilon Km$.
- (a₂) If PS_s contains a $B_i A_{i'}$ -edge, then H''_s contains a matching between A_{i+1} and $B_{i'-1}$ of size $30\varepsilon Km$.
- (a₃) $H''_1, \dots, H''_{|\mathcal{J}|}$ are edge-disjoint and for all $s \leq |\mathcal{J}|$ the matchings guaranteed by (a₁) and (a₂) are edge-disjoint.

So if PS_s contains both an $A_i B_{i'}$ -edge and a $B_{i'-1} A_{i+1}$ -edge, then H''_s contains a matching between $A_{i'}$ and B_i of size $60\varepsilon Km$.

To prove the claim, first recall that $H''[A_i, B_{i'}]$ is a $248\varepsilon m/K$ -regular graph for all $i, i' \leq K$. So $H''[A_i, B_{i'}]$ can be decomposed into $248\varepsilon m/K$ perfect matchings. Each perfect matching can be split into $1/(31\varepsilon K)$ matchings, each of size at least $30\varepsilon Km$. Therefore $H''[A_i, B_{i'}]$ contains $8m/K^2$ edge-disjoint matchings, each of size at least $30\varepsilon Km$. (i) and (ε_s) together imply that for any $i, i' \leq K$, the number of PS_s containing an $A_i B_{i'}$ -edge is at most

$$\sum_{(i_1, i_2, i_3, i_4) : i \in \{i_1, i_2\}, i' \in \{i_3, i_4\}} |\mathcal{J}_{i_1, i_2, i_3, i_4}| \leq 4m/K^2.$$

Recall that $H''[A_{i'}, B_i]$ contains $8m/K^2$ edge-disjoint matchings, each of size at least $30\epsilon m$. Thus we can assign a distinct matching in $H''[A_{i'}, B_i]$ of size $30\epsilon m$ to each PS_s that contains an $A_i B_{i'}$ -edge. Additionally, we can also assign a distinct matching in $H''[A_{i+1}, B_{i'-1}]$ of size $30\epsilon m$ to each PS_s that contains a $B_i A_{i'}$ -edge. For all $s \leq |\mathcal{J}|$, let H''_s be the union of all those matchings assigned to PS_s . Then $H''_1, \dots, H''_{|\mathcal{J}|}$ are as desired in the claim.

For each $s \leq |\mathcal{J}|$, we will now add suitable edges from H''_s to PS_s in order to obtain a path sequence PS'_s which is locally balanced with respect to C . So fix $s \leq |\mathcal{J}|$ and let e_1, \dots, e_ℓ denote the edges of PS_s . Note that $\ell = 2e(J_s^*) \leq 6K\epsilon m$ by (γ_s) and (8.2). For each $r \leq \ell$, we will find a directed edge f_r satisfying the following conditions:

- (b₁) If e_r is an $A_i B_{i'}$ -edge, then f_r is an $A_{i'} B_i$ -edge.
- (b₂) If e_r is a $B_i A_{i'}$ -edge, then f_r is a $B_{i'-1} A_{i+1}$ -edge.
- (b₃) The undirected version of $\{f_1, \dots, f_\ell\}$ is a matching in H''_s and vertex-disjoint from $V(PS_s)$.

Suppose that for some $r \leq \ell$ we have already constructed f_1, \dots, f_{r-1} . Suppose that e_r is an $A_i B_{i'}$ -edge. (The argument for the other case is similar.) By (a₁), $H''_s[A_{i'}, B_i]$ contains a matching of size $30K\epsilon m$. Note by (α_s) and (b₃) that

$$|V(PS_s \cup \{f_1, \dots, f_{r-1}\})| \leq 3e(J_s^*) + 2(r-1) < 5\ell \leq 30K\epsilon m.$$

Hence there exists an edge in $H''_s[A_{i'}, B_i]$ that is vertex-disjoint from $PS_s \cup \{f_1, \dots, f_{r-1}\}$. Orient one such edge from $A_{i'}$ to B_i and call it f_r . In this way, we can construct f_1, \dots, f_ℓ satisfying (b₁)–(b₃).

Let PS'_s be digraph obtained from PS_s by adding all the edges f_1, \dots, f_ℓ . Note that PS'_s is a locally balanced path sequence with respect to C . (Indeed, PS'_s is locally balanced since $\{e_r, f_r\}$ is locally balanced for each $r \leq \ell$.) Let i_1, i_2, i_3, i_4 be such that $J \in \mathcal{J}_{i_1, i_2, i_3, i_4}$. Then the following properties hold:

- (c₁) PS'_s is an A_{i_1} -extension of $J_{s, \text{dir}}^*$.
- (c₂) $|V(PS'_s) \cap A_i|, |V(PS'_s) \cap B_i| \leq 12\epsilon Km$ for all $i \leq K$.
- (c₃) If $V(PS'_s) \cap A_i \neq \emptyset$, then $i \in \{i_1, i_2, i_3, i_4, i_3 + 1, i_4 + 1\}$.
- (c₄) If $V(PS'_s) \cap B_i \neq \emptyset$, then $i \in \{i_1 - 1, i_1, i_2, i_3, i_4\}$.

Indeed, (c₁) is implied by (α_s) and the definition of PS'_s . Since $e(PS'_s) = 2e(PS_s) = 4e(J_s^*)$, (c₂) holds by (8.2). Finally, (c₃) and (c₄) are implied by (ϵ_s) , (b₁) and (b₂) as $J_s \in \mathcal{J}_{i_1, i_2, i_3, i_4}$.

Note that $PS'_1 - J_{1, \text{dir}}^*, \dots, PS'_{|\mathcal{J}|} - J_{|\mathcal{J}|, \text{dir}}^*$ are pairwise edge-disjoint and let $\mathcal{BE} := \{PS'_1, \dots, PS'_{|\mathcal{J}|}\}$. We claim that \mathcal{BE} is a balanced extension of $\mathcal{J}_{\text{dir}}^*$ with respect to (\mathcal{Q}, C) and parameters $(12\epsilon K, 12)$. To see this, recall that $\mathcal{Q} = \{A_1, \dots, A_K, B_1, \dots, B_K\}$ is a $(2K, m)$ -equipartition of $V' := V \setminus (A_0 \cup B_0)$. Clearly, (BE1) holds with V' playing the role of V . (c₃) and (i) imply that for every $i \leq K$ there are at most $6m/K$ $PS'_s \in \mathcal{BE}$ such that $V(PS'_s) \cap A_i \neq \emptyset$. A similar statement also holds for each B_i . So together with (c₂), this implies (BE3), where $2K$ plays the role of k in (BE3). As remarked after the definition of a balanced extension, this implies the ‘moreover

part' of (BE2). So (BE2) holds too. Therefore \mathcal{BE} is a balanced extension, so the lemma follows (by orienting the remaining edges of H arbitrarily). \square

Proof of Lemma 3.2. Apply Lemma 8.3 to obtain tuples $(G_j, \mathcal{Q}, C_j, H_j, \mathcal{J}_j)$ for all $j \leq K/2$ satisfying (a₁)–(a₆). Fix $j \leq K/2$ and write $\mathcal{J}_j := \{J_{j,1}, \dots, J_{j,|\mathcal{J}_j|}\}$. Next, apply Lemma 8.4 with $\mathcal{J}_j, C_j, H_j, \varepsilon_0$ playing the roles of $\mathcal{J}, C, H, \varepsilon$ to obtain an orientation $H_{j,\text{dir}}$ of H_j and a balanced extension \mathcal{BE}_j of \mathcal{J}_j with respect to (\mathcal{Q}, C_j) and parameters $(12\varepsilon_0 K, 12)$. (Note that (a₃) and (a₅) imply conditions (i) and (iii) of Lemma 8.4. Condition (ii) follows from Lemma 3.2(d).) So we can write $\mathcal{BE}_j := \{PS_{j,1}, \dots, PS_{j,|\mathcal{J}_j|}\}$ such that $(J_{j,s})_{\text{dir}}^* \subseteq PS_{j,s}$ for all $s \leq |\mathcal{J}_j|$. Each path sequence in \mathcal{BE}_j is obtained from some $(J_{j,s})_{\text{dir}}^*$ by adding edges of $H_{j,\text{dir}}$. Since $(G_{j,\text{dir}}, \mathcal{Q}, C_j)$ is a $(2K, m, 4\mu, 5/K)$ -cyclic system by Lemma 8.3(a₆), we can apply Lemma 7.1 as follows:

	$G_{j,\text{dir}}$	\mathcal{Q}	C_j	$2K$	$\mathcal{J}_{j,\text{dir}}^*$	$ \mathcal{J}_j $	4μ	3ρ	$12\varepsilon_0 K$	$5/K$	12
plays role of	G	\mathcal{Q}	C	k	\mathcal{M}	q	μ	ρ	ε_0	ε'	ℓ

This gives us $|\mathcal{J}_j|$ directed Hamilton cycles $C'_{j,1}, \dots, C'_{j,|\mathcal{J}_j|}$ in $G_{j,\text{dir}} + \mathcal{BE}_j$ such that each $C'_{j,s}$ contains $PS_{j,s}$ and is consistent with $(J_{j,s})_{\text{dir}}^*$. Moreover, (a₄) implies that $C'_{j,1} - (J_{j,1})_{\text{dir}}^*, \dots, C'_{j,|\mathcal{J}_j|} - (J_{j,|\mathcal{J}_j|})_{\text{dir}}^*$ are edge-disjoint subgraphs of $G_{j,\text{dir}} + H_{j,\text{dir}}$. Repeat this process for all $j \leq K/2$.

Recall from Lemma 8.3(a₂) that $\mathcal{J}_1, \dots, \mathcal{J}_{K/2}$ is a partition of \mathcal{J} . Thus we have obtained $|\mathcal{J}|$ directed Hamilton cycles $C'_1, \dots, C'_{|\mathcal{J}|}$ on $A \cup B$ such that each C'_s is consistent with $(J_s)_{\text{dir}}^*$ for some $J_s \in \mathcal{J}$ (and $J_s \neq J_{s'}$ whenever $s \neq s'$). Let H_s be the undirected graph obtained from $C'_s - J_s^* + J_s$ by ignoring all the orientations of the edges. Since $J_1, \dots, J_{|\mathcal{J}|}$ are edge-disjoint exceptional systems, $H_1, \dots, H_{|\mathcal{J}|}$ are edge-disjoint spanning subgraphs of G . Finally, Proposition 8.2 implies that $H_1, \dots, H_{|\mathcal{J}|}$ are indeed as desired in Lemma 3.2. \square

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