Exact Minimum Codegree Threshold for K_4^--factors

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Joint work with
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Forcing a copy of F in a (hyper)graph H

- Turán's theorem: G n-vertex graph,

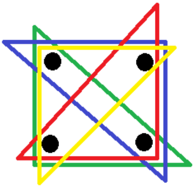
$$e(G) > \left(1 - \frac{1}{r-1}\right)\binom{n}{2}$$

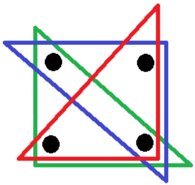
$$\implies K_r \subseteq G$$

- The Erdős–Stone theorem determines the asymptotic threshold for all graphs F
(replace r with $\chi(F)$ in Turán's theorem)

Forcing a copy of F in a (hyper)graph H

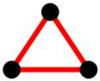
- Far less is known about the corresponding problem for k-graphs (i.e. k vertices in each edge)
- For example, for 3-graphs still open for K_4^3 (4 vertices, 4 edges) and K_4^- (4 vertices, 3 edges).

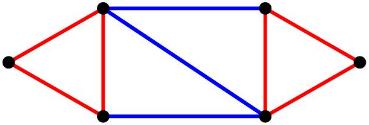




- An *F*-tiling in G is a collection of vertex-disjoint copies of F in G.
- An F-tiling is perfect if it covers all vertices in G.

F





perfect F-tiling

Remarks

- Perfect F-tilings also known as F-factors, perfect F-packings and perfect F-matchings.
- If $F = K_2$ then perfect F-tiling $\iff$ perfect matching.
- Edge density problem not as interesting, so instead look at minimum degree problem.

Perfect K_r-tilings

Theorem (Hajnal, Szemerédi '70)

G graph, $|G| = n$ where $r|n$ and

$$\delta(G) \geq (r-1)\,n/r$$

$\Rightarrow$ G contains a perfect K_r-tiling.

- Corrádi and Hajnal ('64) proved triangle case
- Easy to see that that Hajnal–Szemerédi theorem best possible.
- Kühn and Osthus '09 characterised, up to an additive constant, $\delta(G)$ that forces perfect F-tiling for any F.

Perfect tilings in hypergraphs

- Akin to the Turán problem, it seems perfect tiling problems are much harder in the hypergraph case.
- H k-graph, $1 \leq \ell < k$
- $d_H(v_1, \ldots, v_\ell) = \#$ edges containing $v_1, \ldots, v_\ell$
- minimum ℓ-degree $\delta_\ell(H)$ = minimum over all $d_H(v_1, \ldots, v_\ell)$
- $\delta_1(H)$ = minimum vertex degree
- $\delta_{k-1}(H)$ = minimum codegree

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Previous results for k-graph tiling

Define $\delta(n, F) = \min\{m :$ every k-graph H on n vertices with $\delta_{k-1}(H) \geq m$ contains a perfect F-tiling$\}$.

- Kühn, Osthus (asymptotic, 2006), Czygrinow, DeBiasio, Nagle (2013): $n \geq n_0$,

$$\delta(n, K_4^3 - 2e) = \begin{cases} n/4 + 1 & \text{if } n \in 8\mathbb{N} \\ n/4 & \text{otherwise.} \end{cases}$$

- Lo, Markström (asymptotic, 2015); Keevash, Mycroft (2015): $n \geq n_0$,

$$\delta(n, K_4^3) = \begin{cases} 3n/4 - 2 & \text{if } n \in 8\mathbb{N} \\ 3n/4 - 1 & \text{otherwise.} \end{cases}$$

- Mycroft 2015: $\delta(n, F)$ asymptotically for complete k-partite k-graphs.
  Gao, Han 2015+: $\delta(n, C_6^3)$ exactly; Czygrinow 2015: loose cycles in 3-graphs exactly.

Main result

Let H be a 3-graph on n vertices, where $4 \mid n$.

Theorem (Lo, Markström 2014)

If $\delta_2(H) \geq n/2 + o(n)$, then H contains a perfect K_4^--tiling.

Theorem (Han, Lo, T., Zhao 2015+)

If $n \geq n_0$ and $\delta_2(H) \geq n/2 - 1$, then H contains a perfect K_4^--tiling.

Lower bound Construction:

$V = A \cup B$, $|B| \in \{\frac{n}{2}, \frac{n}{2} - 1\}$ and $3 \nmid |B|$.

Every edge intersects A in 1 or 3 vertices.

Every copy of K_4^- intersects B in 0 or 3 vertices.

Proof overview

A K_4^--tiling $\mathcal{K}$ absorbs a set U outside $V(\mathcal{K})$ if there is a K_4^--tiling on $V(\mathcal{K}) \cup U$.

Lemma (Absorbing Lemma)

If $\delta(H) \geq (\frac{1}{2} - \gamma)n$, then H contains a small absorbing K_4^--tiling unless H is in the extremal case.

Lemma (Almost Tiling Lemma)

Assume $1/n \ll \varepsilon \ll \gamma$. If $\delta(H) \geq (\frac{1}{2} - \gamma)n$, then H contains a K_4^--tiling on $n - \varepsilon n$ vertices.

Lemma (Extremal Case)

H is in the extremal case and $\delta(H) \geq \frac{n}{2} - 1 \Longrightarrow \exists$ a perfect K_4^--tiling in H.

Sketch of the proof of the Absorbing Lemma

Lemma (Absorbing Lemma)

If $\delta(H) \geq (\frac{1}{2} - \gamma)n$, then H contains a small absorbing K_4^--tiling unless H is in the extremal case.

- To obtain an absorbing set it suffices to show there are 'many' (x, y)-*connectors* for each $x, y \in V(H)$.
- Case 1: For every $x \in V(H)$, x forms a copy of K_4^- with many edges e.
- $\implies$ $\exists$ many (x, y)-connectors for $\geq (1/4 - o(1))n$ vertices $y \in V(H)$
- $\implies$ H can be partitioned into at most 4 *closed* components $V_1, \dots, V_m$.
- Now show we can merge $V_1, \dots, V_m$ into a single closed component.

Sketch of the proof of the Absorbing Lemma

Lemma (Absorbing Lemma)

If $\delta(H) \geq (\frac{1}{2} - \gamma)n$, then H contains a small absorbing K_4^--tiling unless H is in the extremal case.

- Case 2: There exists a $v \in V(H)$, s.t. v forms a copy of K_4^- with *few* edges e.
- $\implies$ $\exists$ a partition X, Y of $V(H)$ s.t. almost all XXY- and XYY-edges lie in H
- $\implies$ H has an absorbing set.

Open problems

- Even less is known about minimum vertex degree conditions that force perfect tilings in k-graphs.

- What is $\delta_1(n, K_4^-)$?
  What is $\delta_1(n, K_4^3)$?
  Han, Zhao (2015) determined $\delta_1(n, K_4^3 - 2e)$.
- Prove a Hajnal-Szemerédi theorem for 3-graphs, e.g., determining $\delta_2(n, K_t^3)$ for all $t > 4$.