Abstract

We consider a natural analogue of the graph linear arrangement problem for posets. Let $P = (X, \prec)$ be a poset that is not an antichain, and let $\lambda : X \rightarrow [n]$ be an order-preserving bijection, that is, a linear extension of $P$. For any relation $a \prec b$ of $P$, the distance between $a$ and $b$ in $\lambda$ is $\lambda(b) - \lambda(a)$. The average relational distance of $\lambda$, denoted $\text{dist}_P(\lambda)$, is the average of these distances over all relations in $P$. We show that we can find a linear extension of $P$ that maximises $\text{dist}_P(\lambda)$ in polynomial time. Furthermore, we show that this maximum is at least $\frac{1}{3}(|X| + 1)$, and this bound is extremal.

1 Introduction

In graph theory, the linear arrangement problem, or optimal arrangement problem, or wire-length problem, is the following. Given a graph $G = (V, E)$
where \(|V| = n\) and \(|E| = m\), find a function amongst all bijective functions \(f : V \rightarrow [n]\) that minimises

\[
\frac{1}{m} \sum_{a, b \in E} |f(a) - f(b)|.
\]

Note that the factor \(1/m\) makes no difference to this problem and is generally omitted. Also, since

\[
\sum_{ab \in V^{(2)}} |f(a) - f(b)| = \sum_{1 \leq i < j \leq n} (j - i) = \frac{1}{6}(n + 1)n(n - 1),
\]

we see that the maximisation problem for a given graph \(G\) is equivalent to the minimisation problem for its complement. (Note: \(A^{(2)}\) denotes the set of all unordered pairs of a set \(A\).)

The linear arrangement problem is known to be NP-hard (see [3]), and furthermore, there are few classes of graphs for which this problem is known to be polynomial-time solvable. The problem, which is fairly well studied, falls inside a more general class of problems called graph layout problems. These ask for an ordering of graph vertices so as to optimise some objective function of edge lengths. For a survey of such problems, see [1].

We formulate a natural analogue of the linear arrangement problem for posets. Given a poset \(P = (X, \prec)\) with \(|X| = n\), a linear extension \(\lambda\) of \(P\) is a bijection, \(\lambda : P \rightarrow [n]\), which satisfies the condition that \(\lambda(a) < \lambda(b)\) whenever \(a \prec b\) for every pair of elements \(a, b \in X\). We write \(\Lambda_P\) for the set of all linear extensions of \(P\).

Given a linear extension \(\lambda\) of \(P = (X, \prec)\) and \(a, b \in X\) with \(a \prec b\), we define the distance from \(a\) to \(b\) in \(\lambda\) to be \(\text{dist}(a, b; \lambda) = \lambda(b) - \lambda(a)\). The average relational distance in \(\lambda\), \(\text{dist}_P(\lambda)\), is given by

\[
\text{dist}_P(\lambda) = \frac{1}{m} \sum_{(a, b) : a \prec b} \text{dist}(a, b; \lambda) = \frac{1}{m} \sum_{(a, b) : a \prec b} (\lambda(b) - \lambda(a)),
\]

where \(m\) is the number of comparable pairs in \(P\). For this to be well defined, we require that \(m > 0\), and so we shall assume throughout that \(P\) is not an antichain.

Clearly \(\text{dist}_P\) is a natural function to consider on \(\Lambda_P\), and in this note, we give some of its properties. In contrast to the linear arrangement problem for graphs, we show in Section 2 that an element of \(\Lambda_P\) maximising \(\text{dist}_P\) can be found in polynomial time. The same result was obtained simultaneously and independently by Howard et al. in the preceding article [4]. Our algorithm is
very simple and makes use of some of the ideas of a simple polynomial-time algorithm for a poset version of the maxcut problem [5].

We make some remarks to give some context to the problem of maximising $\text{dist}_P$.

**Remark** Our problem is not simply a restriction of the graph linear arrangement problem to comparability graphs of posets: we are maximising over linear extensions of $P$ rather than arbitrary bijections. (The *comparability graph* of a poset $P = (X, \prec)$ is the graph on $X$ whose edges are the comparable pairs in $P$.)

In order to see that the two problems are genuinely different, consider the following example. Let $P^*_r = (X, \prec^*)$ be a poset where $X$ consists of the elements $x, x_1, \ldots, x_r, y, y_1, \ldots, y_r$, and where $x \prec^* x_i$ for $i = 1, \ldots, r$ and $y \prec^* y_i$ for $i = 1, \ldots, r$. Thus $P^*_r$ has $2r + 2$ elements and $2r$ relations.

We note that all linear extensions $\lambda$ of $P^*_r$ in which $x$ and $y$ are the first two elements, that is $\{\lambda(x), \lambda(y)\} = \{1, 2\}$, have the same average relational distance, and moreover these linear extensions turn out to maximise the average relational distance. For the ordering $y, x, x_1, \ldots, x_r, y_1, \ldots, y_r$, this distance is

$$\frac{1}{2r} \left( \sum_{i=1}^r i + \sum_{i=1}^r (r + 1 + i) \right) = \frac{1}{2r} \left( r(r + 1) + 2 \sum_{i=1}^r i \right) = \frac{2r(r + 1)}{2r} = r + 1.$$

However, if we permit arbitrary bijections from $X$ to $[2r + 2]$, then the average relational distance is maximised by the bijection that orders the elements $x, y_1, \ldots, y_r, x_1, \ldots, x_r, y$, and its average relational distance is

$$\frac{1}{2r} \left( 2 \sum_{i=1}^r (r + i) \right) = \frac{2r^2 + r(r + 1)}{2r} = \frac{3r + 1}{2} > r + 1 \quad \text{for } r \geq 2.$$

**Remark** For posets, the maximisation and minimisation problems are not equivalent in the sense they are for graphs. Indeed, we believe that the problem of minimising $\text{dist}_P(\lambda)$ over all linear extensions $\lambda$ of $P$ is NP-hard.

**Remark** The maximisation problem we have described for posets is equivalent to the following minimisation problem: given a poset $P$, minimise over all linear extensions $\lambda$ of $P$, the function

$$\sum_{a \parallel b} |\lambda(a) - \lambda(b)|,$$
where $a \parallel b$ denotes that $a$ and $b$ are incomparable in $P$. This problem is related to the linear discrepancy of a poset $P$, denoted by $ld(P)$, and defined as the minimum over all linear extensions $\lambda$ of $P$, of the function

$$\max_{a \parallel b} |\lambda(a) - \lambda(b)|.$$  

This problem has been studied by Fishburn, Tanenbaum, and Trenk [6, 2], and is in turn related to the bandwidth problem for graphs, another graph layout problem. Fishburn, Tanenbaum, and Trenk showed that the linear discrepancy of a poset $P$ is always equal to the bandwidth of $Inc(P)$, where $Inc(P)$, the incomparability graph of $P$, is the graph on $X$ whose edges are the incomparable pairs of $P$.

Moving away from the algorithmic problem, in Section 3 we prove the following extremal bound on $dist_P$: for any poset $P$ on $n$ elements that is not an antichain, we have

$$\max_{\lambda \in \Lambda_P}(dist_P(\lambda)) \geq \frac{1}{3}(n + 1).$$

Note that equality holds in the above bound for $P = C_n$, the chain on $n$ elements. Exactly the same bound holds for the corresponding graph problem, and it is trivial to prove. Given a graph $G = (V, E)$, take a random bijection $f : V \to [n]$. It is easy to see that

$$\mathbb{E}\left(\frac{1}{m} \sum_{ab \in E} |f(a) - f(b)|\right) = \frac{1}{m} \sum_{ij \in [n]^2} |j - i| \mathbb{P}(\exists ab \in E : f(a)f(b) = ij)$$

$$= \frac{1}{m} \left(\frac{1}{6}(n - 1)n(n + 1)\right) \frac{m}{\binom{n}{2}}$$

$$= \frac{1}{3}(n + 1).$$

Now the existence of the desired bijection is ensured.

The bound for posets is proved in a similar way except that the expectation is bounded rather than computed.

### 2 Maximisation of $dist_P$

In this section, we give a simple characterisation of linear extensions of $P$ that maximise $dist_P$. A polynomial-time algorithm immediately follows from this characterisation. The results in this section have been proved (with
essentially the same proofs) simultaneously and independently by Howard et al. [4].

We begin with some notation. Given a poset $P = (X, \prec)$ and $x \in X$, we define

$$u(x) = |\{y \in X : y \succ x\}|$$

and

$$d(x) = |\{y \in X : y \prec x\}|.$$  

For $A, B \subseteq X$, we define

$$e_P(A) = |\{(a, b) : a \prec b \text{ and } a, b \in A\}|$$

and

$$e_P(A, B) = |\{(a, b) : a \prec b \text{ and } a \in A, b \in B\}|.$$  

In practice, $A$ will generally be a down-set and $B$ an up-set of $P$, with $A$ and $B$ disjoint.

Define $h : X \to \mathbb{Z}$, where $h(x) = d(x) - u(x)$ for each $x \in X$. Observe that $h$ is a strictly increasing function on $P$, that is, whenever $a, b \in X$ with $a \prec b$, we have $h(a) < h(b)$.

Now $h$ imposes a partial order $P_h = (X, \prec_h)$ on $X$ defined as follows. For $a, b \in X$, we have that $a \prec_h b$ if and only if $h(a) < h(b)$. (Note that $P_h$ is a linear ordering if and only if $h$ is injective.) Since $h$ is an increasing function with respect to $P$, we see that any linear extension of $P_h$ is also a linear extension of $P$. We assert that the linear extensions of $P$ that maximise $\text{dist}_P$ are precisely the linear extensions of $P_h$.

**Theorem 2.1** Given a poset $P = (X, \prec)$, an element of $\Lambda_P$ maximises $\text{dist}_P$ if and only if it is a linear extension of $P_h$.

The following corollary follows easily from the previous theorem.

**Corollary 2.2** Given a poset $P = (X, \prec)$, an element of $\Lambda_P$ that maximises $\text{dist}_P$ can be found in time polynomial in $|X|$.

**Proof** Clearly the values of $h$ can be found in time polynomial in $|X|$, and we can sort the elements of $X$ according to their $h$-values in time polynomial in $|X|$ to give a linear extension of $P_h$, which, by Theorem 2.1, maximises $\text{dist}_P$.  

Here is the proof of Theorem 2.1.
\textbf{Proof} (of Theorem 2.1) Fix a linear extension $\lambda$ of $P$. Let $A_i$ be the set of the first $i-1$ elements of $P$ in $\lambda$, and let $B_i$ be the remaining elements of $P$, that is

\begin{align*}
A_i &= \{ \lambda^{-1}(j) : j < i \} \quad \text{and} \\
B_i &= \{ \lambda^{-1}(j) : j \geq i \} .
\end{align*}

We let $e_i = e_P(A_i, B_i)$, the number of comparable pairs of $P$ from $A_i$ to $B_i$. Given a comparable pair $(a, b)$ of $P$, where $a \prec b$, we note that $(a, b)$ is counted in $e_P(A_i, B_i)$ for values of $i$ satisfying $\lambda(a) < i \leq \lambda(b)$. Therefore the comparable pair $(a, b)$ is counted precisely $\lambda(b) - \lambda(a) = \text{dist}(a, b; \lambda)$ times in $\sum_{i=1}^{n} e_i$. Hence

\begin{align*}
\frac{1}{m} \sum_{i=1}^{n} e_i &= \frac{1}{m} \sum_{(a, b) : a \prec b} \text{dist}(a, b; \lambda) = \text{dist}_P(\lambda).
\end{align*}

We now evaluate $e_i$ in terms of $h$. Since for each $i$, $A_i$ is a down-set of $P$ disjoint from $B_i$, which is an up-set of $P$, we have

\begin{align*}
e_i &= e_P(X) - e_P(A_i) - e_P(B_i) \\
&= \sum_{x \in X} d(x) - \sum_{x \in A_i} d(x) - \sum_{x \in B_i} u(x) \\
&= \sum_{x \in B_i} d(x) - \sum_{x \in B_i} u(x) \\
&= \sum_{x \in B_i} h(x) = \sum_{j=i}^{n} h(\lambda^{-1}(j)).
\end{align*}

(This calculation is derived from [5], Theorem 4.2.)

Now we have that

\begin{equation}
\text{dist}_P(\lambda) = \frac{1}{m} \sum_{i=1}^{n} e_i = \frac{1}{m} \sum_{i=1}^{n} \sum_{j=i}^{n} h(\lambda^{-1}(j)) = \frac{1}{m} \sum_{i=1}^{n} i h(\lambda^{-1}(i)).
\end{equation}

We now see from the formula above that a linear extension $\lambda$ of $P$ maximises $\text{dist}_P$ if and only if $h(\lambda^{-1}(i))$ is an increasing function of $i$, that is, if and only if $\lambda$ is a linear extension of $P_h$. This proves our claim and completes the proof.

Alternatively, one can prove that maximising $\text{dist}_P$ is polynomial-time solvable by repeatedly performing local optimisations: given a linear extension $\lambda$ of $P$, if we can switch a consecutive pair of elements in $\lambda$ to obtain
a linear extension for which \( \text{dist}_P \) is larger, then we make the switch. We iterate this process until no more switches can be made. It is easy to prove that what remains is an optimal linear extension. The proof above gives the explicit formula (1), which might prove to be useful elsewhere.

3 An Extremal Bound for \( \text{dist}_P \)

In this section, we prove the following theorem.

**Theorem 3.1** For every poset \( P \) that is not an antichain, there exists a linear extension \( \lambda^* \) such that

\[
\text{dist}_P(\lambda^*) \geq \frac{1}{3}(n + 1).
\]

**Proof** Pick a linear extension \( \mu \) of \( P \) uniformly at random. We shall prove in Lemma 3.3 that

\[
\mathbb{E}(\text{dist}_P(\mu)) \geq \frac{1}{3}(n + 1).
\]

This then ensures the existence of the desired linear extension. \( \square \)

We give some notation. Fix a poset \( P = (X, \prec) \), where \( |X| = n \). For \( i, j \in [n] \) with \( i < j \), write

\[
N_P(i, j) = \{ \mu : \mu \text{ is a linear extension of } P, \mu^{-1}(i) \prec \mu^{-1}(j) \},
\]

and let \( n_P(i, j) = |N_P(i, j)| \), the number of linear extensions of \( P \) in which the element in the \( i^{th} \) position is below (in \( P \)) that in the \( j^{th} \) position. We have the following lemma.

**Lemma 3.2** Let \( P = (X, \prec) \) be a poset and let \( i, j, i', j' \) be elements of \( [n] \), with \( i \leq i' < j' \leq j \). Then

\[
n_P(i, j) \geq n_P(i', j').
\]

**Proof** It is sufficient to prove that

\[
n_P(a, b + 1) \geq n_P(a, b)
\]

for all applicable \( a, b \in [n] \) with \( a < b \). Indeed we can then conclude by induction that \( n_P(a, b + k_1) \geq n_P(a, b) \), for all \( k_1 \in [n - b] \), and furthermore by symmetry, we have \( n_P(a, b) \geq n_P(a + k_2, b) \), for all \( k_2 \in [b - a] \). Thus, we have

\[
n_P(a, b + k_1) \geq n_P(a + k_2, b),
\]
and setting \( a = i, \ b = j', \ k_1 = j - j', \) and \( k_2 = i' - i \) gives the desired inequality.

Fix \( a, b \in [n] \) with \( a < b \). In order to show that \( n_P(a, b + 1) \geq n_P(a, b) \), we give an injection from \( N_P(a, b) \) to \( N_P(a, b + 1) \) as follows.

Suppose \( \lambda \in N_P(a, b) \), so that \( \lambda^{-1}(a) \prec \lambda^{-1}(b) \). If \( \lambda^{-1}(b) \prec \lambda^{-1}(b + 1) \), then \( \lambda^{-1}(a) \prec \lambda^{-1}(b + 1) \) by transitivity, so that \( \lambda \in N_P(a, b + 1) \), and in this case, we set \( \theta(\lambda) = \lambda \).

If \( \lambda^{-1}(b) \) is incomparable to \( \lambda^{-1}(b + 1) \) in \( P \), then let \( \mu \) be the same linear extension as \( \lambda \) with the order of \( \lambda^{-1}(b) \) and \( \lambda^{-1}(b + 1) \) reversed. More precisely,

\[
\mu(x) = \begin{cases} 
  x & \text{if } x \in X \setminus \{\lambda^{-1}(b), \lambda^{-1}(b + 1)\}; \\
  b & \text{if } x = \lambda^{-1}(b + 1); \\
  b + 1 & \text{if } x = \lambda^{-1}(b). 
\end{cases}
\]

It is easy to see that \( \mu \) is a linear extension of \( P \) and furthermore, we have that \( \mu^{-1}(a) \prec \mu^{-1}(b + 1) \), so that \( \mu \in N_P(a, b + 1) \). In this case, we set \( \theta(\lambda) = \mu \).

It is easy to see that \( \theta \) is injective, and this completes the proof. \( \square \)

We are now ready to prove Lemma 3.3, which then completes the proof of Theorem 3.1.

**Lemma 3.3** Let \( P = (X, \prec) \) be a poset that is not an antichain, and let \( \lambda \) be a linear extension of \( P \) chosen uniformly at random from \( \Lambda_P \). Then

\[
\mathbb{E}(\text{dist}_P(\lambda)) \geq \frac{1}{3}(n + 1).
\]

**Proof** Observe first that

\[
\text{dist}_P(\lambda) = \frac{1}{m} \sum_{i,j \in [n]: i < j} (j - i)I_{ij},
\]

where \( I_{ij} \) is the indicator function of the event that \( \lambda^{-1}(i) \prec \lambda^{-1}(j) \). Taking expectations of both sides, we have that

\[
\mathbb{E}(\text{dist}_P(\lambda)) = \frac{1}{m} \sum_{i,j \in [n]: i < j} (j - i)\mathbb{P}(\lambda^{-1}(i) \prec \lambda^{-1}(j)).
\]
Since, for any fixed linear extension \( \lambda \) of \( P \), we have \( \lambda^{-1}(i) \prec \lambda^{-1}(j) \) for exactly \( m \) pairs \((i, j)\), then

\[
\frac{1}{m} \sum_{i,j \in [n]: i < j} \mathbb{P}(\lambda^{-1}(i) \prec \lambda^{-1}(j)) = 1.
\]

(These probabilities are not necessarily equal as they were in the graph version of the problem.) Let \( \mathcal{I} \) denote the set of intervals of the form \([i, j]\), where \( i, j \in [n] \) and \( i < j \). Let \( p_{[i,j]} := \frac{1}{m} \mathbb{P}(\lambda^{-1}(i) \prec \lambda^{-1}(j)) \) be the components of a vector \( p \in [0,1]^\mathcal{I} \). Now we have that

\[
\mathbb{E}(\text{dist}_P(\lambda)) = \sum_{[i,j] \in \mathcal{I}} (j - i) p_{[i,j]} =: \phi(p).
\]

Then \( p \) satisfies the following:

1. \( p_{[i,j]} \geq 0 \) for all \([i, j] \in \mathcal{I}\) (2)
2. \( \sum_{[i,j] \in \mathcal{I}} p_{[i,j]} = 1 \) (3)
3. and \( p_{[i,j]} \geq p_{[i',j']} \) whenever \([i', j'] \subseteq [i, j]\). (4)

The set of inequalities (4) is a consequence of Lemma 3.2. Let \( S \) be the set of vectors in \([0,1]^\mathcal{I}\) that satisfy (2) (3) and (4). Then we have that

\[
\mathbb{E}(\text{dist}_P(\lambda)) \geq \min_{p \in S} \phi(p).
\]

Note that \( \phi \) has a minimum in \( S \) since \( \phi \) is continuous and \( S \) is closed and bounded. Let \( p^* \in S \) be the vector with all its components equal (to \( \binom{n}{2}^{-1} \)).

We make the following claim.

**Claim 1** We have

\[
\min_{p \in S} \phi(p) = \phi(p^*).
\]

Proving this claim proves the lemma since \( \phi(p^*) = \frac{1}{3}(n + 1) \).

**Proof** Suppose \( p \in S \), and the components of \( p \) are not all equal. We prove the claim by showing that either \( \phi(p) = \phi(p^*) \) or \( p \) does not minimise \( \phi \).

Consider the inclusion order \( Q = (\mathcal{I}, \subseteq) \) on \( \mathcal{I} \). Thinking of \( p \) as a function from \( \mathcal{I} \) to \([0,1]\), we see that \( p \in S \) implies that \( p \) is an increasing function on \( Q \). Consider the vector \( p' \), which is obtained from \( p \) as follows. For \( i < j \), with \( i, j \in [n] \) let

\[
p'_{[i,j]} = \frac{1}{|\mathcal{I}_{j-i}|} \sum_{l \in \mathcal{I}_{j-i}} p_l,
\]

9
where $\mathcal{I}_k \subset \mathcal{I}$ is the set of intervals in $\mathcal{I}$ of length $k$. Thus $p'_I$ is the average of all the components of $p$ corresponding to intervals of the same length as $I$. From this it is easy to see that $\phi(p') = \phi(p)$.

Next, we show that $p' \in S$. Indeed, it is clear that $p'$ satisfies the inequalities (2) and (3). In order to show that $p'$ satisfies (4), it is sufficient to show that $p'$ is an increasing function on $\mathcal{I}$, that is, for each $k \in [n-1]$, we must show that

$$
\frac{1}{|\mathcal{I}_k|} \sum_{I \in \mathcal{I}_k} p_I \leq \frac{1}{|\mathcal{I}_{k+1}|} \sum_{I \in \mathcal{I}_{k+1}} p_I.
$$

Let $[a, a + k]$ be the interval that minimises $p_{[i,j]}$ amongst all intervals $[i, j] \in \mathcal{I}_k$. Consider the following bijection $g : \mathcal{I}_{k+1} \rightarrow \mathcal{I}_k \setminus [a, a + k]$. Let

$$
g([i, i + (k + 1)]) = \begin{cases} 
[i, i + k] & \text{if } i < a; \\
[i + 1, i + k + 1] & \text{if } i \geq a.
\end{cases}
$$

Now, for each $I \in \mathcal{I}_{k+1}$, we have $g(I) \subset I$ and hence $p_{g(I)} \leq p_I$. Therefore

$$
\frac{1}{|\mathcal{I}_{k+1}|} \sum_{I \in \mathcal{I}_{k+1}} p_I \geq \frac{1}{|\mathcal{I}_{k+1}|} \sum_{I \in \mathcal{I}_{k+1}} p_{g(I)} = \frac{1}{|\mathcal{I}_k| \setminus [a, a + k]} \sum_{I \in \mathcal{I}_k \setminus [a, a + k]} p_I
$$

where the last inequality follows by our choice of $[a, a + k]$. Thus we have shown that $p' \in S$.

Now, if all the components of $p'$ are equal, then $\phi(p) = \phi(p') = \phi(p^*)$. If not, then there is some covering pair of $Q$, $I_1 \subset I_2$, for which $p'_{I_1} < p'_{I_2}$. Suppose $p'_{I_1} = p'_{I_2} + \epsilon$, where $\epsilon > 0$. Then, changing $p'$ by increasing $p'_{I_1}$ by $\epsilon/2$ and decreasing $p'_{I_2}$ by $\epsilon/2$ gives a vector $p''$, where it is easy to check that $p'' \in S$ (using the fact that $p' \in S$ and (5)). Furthermore, it is easy to see that $\phi(p'') < \phi(p') = \phi(p)$ and thus $p$ does not minimise $\phi$. This completes the proof of Claim 1, and the lemma.

References


