Finite-dimensional normed vector spaces

**Proposition 3.5.** Let \((X, \| \cdot \|)\) be an \(n\)-dimensional normed vector space for some \(n \in \mathbb{N}\) and let \(\{e_1, \ldots, e_n\}\) be a basis for \(X\). Then there exist constants \(a, b > 0\) such that

\[
a \sum_{i=1}^{n} |a_i| \leq \left\| \sum_{i=1}^{n} a_i e_i \right\| \leq b \sum_{i=1}^{n} |a_i|
\]

(3.4)

for all \((a_1, \ldots, a_n) \in \mathbb{R}^n\).

**Proof.** Note that \(\|(a_1, \ldots, a_n)\|_1 = \sum_{i=1}^{n} |a_i|\) and if \(\|(a_1, \ldots, a_n)\|_1 = 0\) then \(a_i = 0\) for all \(1 \leq i \leq n\) and (3.4) is satisfied with any \(a\) and \(b\). Assume therefore \(\|(a_1, \ldots, a_n)\|_1 \neq 0\). By homogeneity it suffices to show that

\[
a \leq \left\| \sum_{i=1}^{n} a_i e_i \right\| \leq b
\]

(3.5)

for all \((a_1, \ldots, a_n) \in \mathbb{R}^n\) with \(\|(a_1, \ldots, a_n)\|_1 = 1\) since for \((\alpha_1, \ldots, \alpha_n) \neq 0\) inequality (3.4) follows from (3.5) applied to the renormalised vector \((\alpha'_1, \ldots, \alpha'_n)\) given by \(\alpha'_i = \alpha_i / \|(\alpha_1, \ldots, \alpha_n)\|_1\) for each \(i = 1, \ldots, n\).

To prove (3.5), define a mapping

\[F : K \to \mathbb{R}; \quad F(a_1, \ldots, a_n) = \left\| \sum_{i=1}^{n} a_i e_i \right\|\]

where

\[K := \{(a_1, \ldots, a_n) \in \mathbb{R}^n : \|(a_1, \ldots, a_n)\|_1 = 1\}.
\]

It is easy to see that \(K \subseteq \mathbb{R}^n\) is a closed and bounded set. It is therefore compact. We now show that \(F(a_1, \ldots, a_n) > 0\) for every \((a_1, \ldots, a_n) \in K\). Indeed, since \(\{e_1, \ldots, e_n\}\) is a linearly independent system it follows that

\[(a_1, \ldots, a_n) \in K \Rightarrow \sum_{i=1}^{n} a_i e_i \neq 0\]

and consequently \(F(a_1, \ldots, a_n) > 0\) for all \((a_1, \ldots, a_n) \in K\).

By Proposition 2.3, \(F\) is continuous and therefore is bounded above and below and attains these bounds; i.e. there exist points \(A, B \in K\) such that

\[F(A) \leq F(a_1, \ldots, a_n) \leq F(B)\]

for all \((a_1, \ldots, a_n) \in K\). Denote \(a = F(A)\) and \(b = F(B)\), then \(a, b > 0\) and (3.5) is satisfied. \(\square\)
Corollary 3.6. Let $X$ be a finite dimensional vector space, let $\| \cdot \|$ and $||| \cdot |||$ be two norms on $X$. Then $\| \cdot \| \simeq ||| \cdot |||$.

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis for $X$. By Proposition 3.5, there are real positive numbers $a, b$ and $a', b'$ such that

$$a \sum_{i=1}^{n} |a_i| \leq \| \sum_{i=1}^{n} a_i e_i \| \leq b \sum_{i=1}^{n} |a_i|$$

and

$$a' \sum_{i=1}^{n} |a_i| \leq ||| \sum_{i=1}^{n} a_i e_i ||| \leq b' \sum_{i=1}^{n} |a_i|.$$ 

Let $x \in X$, find $(a_1, \ldots, a_n)$ such that $x = \sum_{i=1}^{n} a_i e_i$. Therefore,

$$\|x\| = \| \sum_{i=1}^{n} a_i e_i \| \leq b \sum_{i=1}^{n} |a_i| \leq \frac{b}{a'} ||| \sum_{i=1}^{n} a_i e_i ||| = \frac{b}{a'} \|x\|$$

and

$$\|x\| = \| \sum_{i=1}^{n} a_i e_i \| \geq a \sum_{i=1}^{n} |a_i| \geq \frac{a}{b'} ||| \sum_{i=1}^{n} a_i e_i ||| = \frac{a}{b'} \|x\|,$$

hence $\frac{a}{b'} \|x\| \leq \|x\| \leq \frac{b}{a'} \|x\|$ for all $x \in X$. \hfill $\square$

Remark. Exercise 4b in PS2 claims that there are two non-equivalent norms on $\ell^1$: namely, $\| \cdot \|_1$ and $||| \cdot |||_\infty$. Then Corollary 3.6 implies that $\ell^1$ is infinite-dimensional. Finally, as every $\ell^p \supseteq \ell^1$ for $1 \leq p \leq \infty$ (exercise 5 PS1), we get that all $\ell^p$ are infinite-dimensional.

Corollary 3.7. Let $(X, \| \cdot \|)$ be a finite dimensional normed vector space. Then $(X, \| \cdot \|)$ is a Banach space.

Proof. Let $e_1, \ldots, e_n$ be a basis for $X$ and $\| \cdot \|_1$ be the $\ell^1$-norm on $X$: $\| \sum_{i=1}^{n} a_i e_i \|_1 := \sum_{i=1}^{n} |a_i|$. Since $(X, \| \cdot \|_1)$ is a Banach space and any two norms on $X$ are equivalent, we conclude (using Remark 4 after the definition of equivalent norms) that $(X, \| \cdot \|)$ is a Banach space too. \hfill $\square$

Corollary 3.8. Let $(X, \| \cdot \|)$ be a finite dimensional normed vector space and let $Y$ be a linear subspace of $X$. Then $Y$ is closed.

Proof. By Corollary 3.7, $(Y, \| \cdot \|)$ is a Banach space. Thus $Y$ is closed by Proposition 3.4. \hfill $\square$
We conclude this section with a brief discussion on compactness.

**Definition (Compact set)**

Let \((X, \| \cdot \|)\) be a normed vector space and \(C \subseteq X\). Then a set \(C\) is said to be **compact** if for every family of open sets \((U_\alpha)_{\alpha \in A}\) such that \(\bigcup_{\alpha \in A} U_\alpha \supseteq C\) there exists a finite subfamily \((U_\alpha_i)_{i=1,\ldots,n}\) such that \(\bigcup_{1 \leq i \leq n} U_\alpha_i \supseteq C\).

[Every open cover has a finite subcover.]

**Theorem (Heine–Borel).** Let \((X, \| \cdot \|)\) be a finite dimensional normed vector space and let \(C \subseteq X\). Then \(C\) is compact if and only if \(C\) is closed and bounded.

**Remark.** In general, compact sets in normed vector spaces are necessarily closed and bounded. However, away from the finite-dimensional case, the converse is not true in general. For example, let \(X = \ell^2\), \(C = \{e^{(n)}, n \geq 1\}\), where \(e^{(n)} = (e_1^{(n)}, e_2^{(n)}, \ldots)\) with

\[
e_j^{(n)} = \delta_{n,j} = \begin{cases} 1 & \text{if } j = n; \\ 0 & \text{otherwise.} \end{cases}
\]

It is clear that \(C\) is bounded. Moreover, for all \(n \neq m\) we have \(\|e_n - e_m\|_2 = \sqrt{2}\) and so \(C\) is closed (the set of accumulation points of \(C\) is empty). However, if we let \(U_n = B(e^{(n)}, \sqrt{2}/3)\) be disjoint open balls around vectors \(e^{(n)}\), then any finite union \(U = \bigcup_{i=1}^{\infty} U_{n_k}\) will not cover the whole \(C\) (letting \(m = \max\{n_1, \ldots, n_N\}\) we get that \(e^{(m+1)} \not\in U\)).

**END OF LECTURE 9**

**Theorem 3.9.** Let \((X, \| \cdot \|)\) be a normed vector space. If \(C = \overline{B}(0, 1)\) is compact then \(X\) is finite dimensional.

**Proof.** The following proof has not been covered in the lectures

Suppose that \(C := \overline{B}(0, 1)\) is compact. Clearly, \(C \subseteq \bigcup_{x \in C} B(x, 1/2)\). By the definition of compactness, there exists a finite subset \(\{x_1, \ldots, x_n\}\) of \(C\) such that

\[
C \subseteq \bigcup_{i=1}^{n} B(x_i, 1/2).
\]

Let \(M\) be the linear span \(\langle x_1, \ldots, x_n \rangle\). Then we have

\[
C \subseteq M + \overline{B}(0, 1/2) = M + \frac{1}{2}B(0, 1) \subseteq M + \frac{1}{2}C,
\]

where for \(A, B \subseteq X\) we define \(A + B = \{a + b \mid a \in A, b \in B\}\). Therefore, using the fact that \(M\) is a linear subspace of \(X\), we get

\[
C \subseteq M + \frac{1}{2} \left( M + \frac{1}{2}C \right) = M + \frac{1}{4}C.
\]