§2. Normed vector spaces

For $p \in [1, \infty]$ the $\ell^p$ spaces enjoy the following nesting property.

**Remark.** If $1 \leq p \leq q \leq \infty$ then $\ell^p \subseteq \ell^q$.

**Proof.** See Problem Sheet 1.

**Convergence**

We conclude this section by introducing the notion of convergence in normed vector spaces.

**Definition (Converging & diverging sequence)**

Let $(X, \| \cdot \|)$ be a normed vector space and let $(x_n)_{n \in \mathbb{N}} \subseteq X$ be a sequence in $X$. If there exists $y \in X$ such that

$$\|x_n - y\| \to n \to \infty 0$$

then $(x_n)$ is said to **converge** to $y$, which we write as $x_n \to y = \lim_{n \to \infty} x_n$.

**Example.** Let $x_n = (1,1,\ldots,1,0,0,\ldots) \in \ell^\infty$. We show that $x_n \not\to y = (1,1,1,\ldots)$ in $\ell^\infty$.

Indeed, $x_n - y = (0,0,\ldots,0, -1, -1, \ldots)$, hence $\|x_n - y\|_\infty = 1 \not\to 0$.

END OF LECTURE 4

The following results are reassuring.

**Proposition 2.3.** Let $(X, \| \cdot \|)$ be a normed vector space.

1. If $(\alpha_n)$ is a convergent sequence in $\mathbb{R}$ and $(x_n)$ is a convergent sequence in $X$ with limits $\alpha$ and $x$ respectively, then $(\alpha_n x_n)$ is a convergent sequence in $X$ with limit $\alpha x$.

2. If $(x_n)$ and $(y_n)$ are convergent sequences in $X$ with limits $x$ and $y$ respectively then $(x_n + y_n)$ is a convergent sequence in $X$ with limit $x + y$.

3. If $(x_n)$ is a convergent sequence in $X$ with limit $x$ then $(\|x_n\|)$ is a convergent sequence in $[0, \infty)$ with limit $\|x\|$.

**Proof.**

1. By the triangle inequality,

$$0 \leq \|\alpha_n x_n - \alpha x\| = \|\alpha_n(x_n - x) + (\alpha_n - \alpha)x\| \leq |\alpha_n|\|x_n - x\| + |\alpha_n - \alpha|\|x\| \to n \to \infty |\alpha| \cdot 0 + 0 \cdot \|x\| = 0.$$ 

Hence $\|\alpha_n x_n - \alpha x\| \to 0$. By definition this means $\alpha_n x_n \to \alpha x$.

2. We use $0 \leq \|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0$ to show $(x_n + y_n) \to (x + y)$.

3. Using $0 \leq \|x_n\| - \|x\| \leq \|x_n - x\| \to 0$ we get the required. $$\square$$
Definition (Cauchy sequence)

Let \((X, \| \cdot \|)\) be a normed vector space and let \((x_n)\) be a sequence of vectors in \(X\). Then \((x_n)\) is said to be a \textit{Cauchy sequence} if

\[
\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n, m \geq N \quad (\|x_n - x_m\| < \varepsilon).
\]

This is sometimes written as \(\|x_n - x_m\| \to n, m \to \infty\) 0.

**Lemma 2.4.** Let \((X, \| \cdot \|)\) be a normed vector space and let \((x_n)\) be a convergent sequence in \(X\). Then the following statements hold.

1. The limit of \((x_n)\) is unique.
2. \((x_n)\) is Cauchy.

**Proof.** Suppose that \(\|x_n - x\|\) and \(\|x_n - y\|\) tend to zero as \(n\) tends to infinity. By the triangle inequality,

\[
0 \leq \|x - y\| \leq \|x - x_n\| + \|x_n - y\| \to n \to \infty 0.
\]

This means \(\|x - y\| = 0\) and consequently \(x = y\), which proves (1). For (2), apply the triangle inequality to get

\[
\|x_n - x_m\| \leq \|x_n - x\| + \|x - x_m\| \to n, m \to \infty 0.
\]

\[\square\]

§3 Banach spaces

**Definition (Complete space, Banach space)**

The normed vector space \((X, \| \cdot \|)\) is said to be \textit{complete} if every Cauchy sequence is convergent in \(X\). A \textit{Banach space} is a complete normed vector space.

**Remark.** When \(X = \mathbb{R}\) (with the corresponding absolute value function as the norm), Cauchy’s convergence principle states that the converse of Lemma 2.4 (2) is true. This relies on the completeness axiom for the real numbers. However, for general normed vector spaces the converse of Lemma 2.4 (2) is not true.

**Example (Banach spaces)**

1. For \(p \in [1, \infty]\), the space \((\ell^p, \| \cdot \|_p)\) is a Banach space. We shall prove this now for \(p = \infty\). \textit{See Problem Sheet 2 for} \(p \in [1, \infty)\).
§3. Banach spaces

Proof that \((\ell^\infty, \| \cdot \|_\infty)\) is a Banach space. Let \((x_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in \(\ell^\infty\), and write

\[ x_n = (a_{1,n}, a_{2,n}, a_{3,n}, \ldots) =: (a_{k,n})_{k \in \mathbb{N}}. \]

For each \(\varepsilon > 0\), let \(N_\varepsilon \in \mathbb{N}\) be such that \(\|x_n - x_m\| < \varepsilon\) whenever \(n, m \geq N_\varepsilon\). Then, for \(n, m \geq N_\varepsilon\) and each \(k \in \mathbb{N}\), we have

\[ |a_{k,n} - a_{k,m}| \leq \|x_n - x_m\|_\infty < \varepsilon \tag{3.1} \]

and therefore \((a_{k,n})_{n \in \mathbb{N}}\) is a Cauchy sequence in \(\mathbb{R}\). By the completeness axiom, for every fixed \(k \in \mathbb{N}\) there exists \(y_k \in \mathbb{R}\) such that \(a_{k,n} \to y_k\). We let \(y := (y_k)_{k \in \mathbb{N}}\) and claim that \(y \in \ell^\infty\) and moreover \(\|x_n - y\|_\infty \to 0\). Firstly, fixing \(n \geq N_\varepsilon\) and \(k \geq 1\) and letting \(m \to \infty\) in (3.1), it follows that for all \(k \in \mathbb{N}\) and all \(n \geq N_\varepsilon\)

\[ |a_{k,n} - y_k| \leq \varepsilon. \tag{3.2} \]

Thus, for every \(k \geq 1\) we get \(|y_k| \leq \varepsilon + |x_k| \leq \varepsilon + \|x_n\|_\infty\) and so \(\|y\|_\infty = \sup_{k \in \mathbb{N}} |y_k| \leq \varepsilon + \|x_n\|_\infty < \infty\) which means \(y \in \ell^\infty\).

For the remaining claim, taking supremum over \(k \in \mathbb{N}\) in (3.2), we conclude that \(\|x_n - y\|_\infty \leq \varepsilon\) for all \(m \geq N_\varepsilon\). This implies \(\|x_n - y\| \to 0\).

Remark. Note that simply \(a_{k,n} \to y_k\) for every \(k\) does not imply \(x_n \to y\) in \(\ell^\infty\).

Consider the vectors \(e_n = (0, \ldots, 0, 1, 0, \ldots)\) with all coordinates zeros except for \(n\)th coordinate which is equal to 1. Then \(a_{k,n} \to 0\) for each \(k\) so that all coordinates of \(y\) are zeros. However \(\|e_n - (0, 0, 0, \ldots)\|_\infty = \|e_n\|_\infty = 1\) does not tend to zero!

Moreover, the sequence \((e_n)\) has no limit at all;

\[ \|e_n - e_m\|_\infty = \|(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots)\|_\infty = 1, \]

so the quantity \(\|e_n - e_m\|_\infty\) never becomes less than \(\varepsilon = 0.5\). This implies \((e_n)\) is not a Cauchy sequence and hence does not converge by Lemma 2.4 (2).

END OF LECTURE 5

2. For \(p \in [1, \infty]\) and \(d \in \mathbb{N}\), the space \((\mathbb{R}^d, \| \cdot \|_p)\) is a Banach space. We prove this later.

3. For \(a, b \in \mathbb{R}\), the space \((C[a, b], \| \cdot \|_\infty)\) is a Banach space. However, \((C[a, b], \| \cdot \|_1)\) is not a Banach space. These two statements are proved below.

Open and closed sets

Notation. Let \((X, \| \cdot \|)\) be a normed vector space. If \(x \in X\) and \(r \in (0, \infty)\) then

\[ B(x, r) := \{ y \in X : \|y - x\| < r \} \]