Theorem 4.1 (Cauchy–Schwarz inequality). If $X$ is an inner product space then

$$|⟨x, y⟩| ≤ ∥x∥ · ∥y∥.$$  \hfill (4.2)

Proof. First note that

$$0 ≤ \left\|u∥v − ∥v∥u\right\|^2 = 2∥u∥∥v∥^2 - 2∥u∥ ∥v∥\Re⟨u,v⟩.$$  \hfill (4.3)

Therefore,

$$\Re⟨u,v⟩ ≤ ∥u∥∥v∥$$  \hfill (4.4)

for all $u, v ∈ X$. Choose now $θ ∈ [0, 2π]$ such that $e^{iθ}⟨x, y⟩ = |⟨x, y⟩|$ then (4.2) follows immediately from (4.4) with $u = e^{iθ}x$ and $v = y$.

END OF LECTURE 10

Remark. Equality in $|⟨x, y⟩| = ∥x∥ · ∥y∥$ holds if and only if $x$ and $y$ are linearly dependent. This follows from (4.3) using the nondegeneracy of the norm.

Lemma 4.2. If $X$ is an inner product space and $∥·∥$ is given by (4.1) then $(X, ∥·∥)$ is a normed vector space.

Proof. As we said, only the triangle inequality remains to be verified. If $x, y ∈ X$ then by Cauchy–Schwarz,

$$∥x + y∥^2 = ∥x∥^2 + ∥y∥^2 + 2\Re⟨x,y⟩$$

$$≤ ∥x∥^2 + ∥y∥^2 + 2∥x∥∥y∥ = (∥x∥ + ∥y∥)^2.$$  \hfill □

Another consequence of the Cauchy–Schwarz inequality is the continuity of the inner product.

Lemma 4.3. Let $(x_\\n)$ and $(y_\\n)$ be two sequences in the inner product space $X$ converging to $x$ and $y$ respectively. Then the sequence $(⟨x_\\n, y_\\n⟩)$ converges to $⟨x, y⟩$.

Proof.

$$|⟨x_\\n, y_\\n⟩ - ⟨x, y⟩| = |⟨x_\\n - x, y_\\n⟩ + ⟨x, y_\\n - y⟩|$$

$$≤ |⟨x_\\n - x, y_\\n⟩| + |⟨x, y_\\n - y⟩|$$

$$≤ ∥x_\\n - x∥∥y_\\n∥ + ∥x∥∥y_\\n - y∥ → 0$$

as $n → ∞$ since $∥y_\\n∥ → ∥y∥$ by Proposition 2.3.  \hfill □
Theorem 4.4. Let \((X, \langle \cdot, \cdot \rangle)\) be an inner product space over \(\mathbb{F}\) and let \(\| \cdot \|\) denote the induced norm. Then
\[
\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2
\]
(4.5)
for all \(x, y \in X\). We call this the Parallelogram law.

Conversely, if \((X, \| \cdot \|)\) is a normed vector space such that the Parallelogram law (4.5) holds then there exists an inner product \(\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}\) which induces the norm \(\| \cdot \|\); i.e. \(\langle x, x \rangle^{1/2} = \|x\| \quad \text{for each} \quad x \in X\).

Remark. One can use Theorem 4.4 to show that the Banach space \((\ell^\infty, \| \cdot \|_\infty)\) is not an inner product space:

Take \(x = (1, 0, 0, \ldots), y = (0, 1, 0, 0, \ldots)\). Then \(\|x\|_\infty = 1, \|y\|_\infty = 1, \|x+y\|_\infty = 1, \|x-y\|_\infty = 1\). The parallelogram law is not satisfied as \(1^2 + 1^2 \neq 2 \cdot 1^2 + 1^2\).

Proof of Theorem 4.4. If \((X, \langle \cdot, \cdot \rangle)\) is an inner product space then one can show (4.5) holds by expanding the left-hand side using the linearity of the inner product.

For the converse statement, we include the ideas for the case \(\mathbb{F} = \mathbb{R}\). Assume the parallelogram law (4.5) is satisfied for all \(x, y \in X\); define \(\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}\) by
\[
\langle x, y \rangle := \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2).
\]
(4.6)
It is clear that \(\langle x, x \rangle = \|x\|^2\) for each \(x \in X\) by the 2nd property of \(\| \cdot \|\). Hence the 1st property of \(\langle \cdot, \cdot \rangle\) is satisfied. It is also clear that \(\langle \cdot, \cdot \rangle\) is symmetric using the 2nd property of \(\| \cdot \|\): \(\langle y, x \rangle = \langle x, y \rangle\). It remains to verify the linearity condition. To this end, the proof follows the steps:

1. \(2 \langle \frac{x+y}{2}, z \rangle = \langle x, z \rangle + \langle y, z \rangle\),
2. \(\langle (x+y), z \rangle = \langle x, z \rangle + \langle y, z \rangle\),
3. \(\langle \alpha x, z \rangle = \alpha \langle x, z \rangle\) for all \(\alpha \in \mathbb{Q}\),
4. \(\langle \alpha x, z \rangle = \alpha \langle x, z \rangle\) for all \(\alpha \in \mathbb{R}\).

Details of the proof (not covered in lectures) are below.

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1The proof for \(\mathbb{F} = \mathbb{C}\) proceeds in a similar way under the definition, for all \(x, y \in X\)
\[
\langle x, y \rangle := \frac{1}{4} \left[ (\|x + y\|^2 - \|x - y\|^2) + i(\|x + iy\|^2 - \|x - iy\|^2) \right].
Note first that
\[
2 \left\langle \frac{x + y}{2}, z \right\rangle = \frac{1}{2} \left( \left\| \frac{x + y}{2} + z \right\|^2 - \left\| \frac{x + y}{2} - z \right\|^2 \right) \\
= \frac{1}{2} \left( \left\| \frac{x + y}{2} + z \right\|^2 + \left\| \frac{x - y}{2} \right\|^2 \right) - \frac{1}{2} \left( \left\| \frac{x + y}{2} - z \right\|^2 + \left\| \frac{x - y}{2} \right\|^2 \right) \\
= \frac{1}{4} \left( \left\| \frac{x + y}{2} + z \right\|^2 + \left\| \frac{x - y}{2} \right\|^2 \right) - \frac{1}{4} \left( \left\| \frac{x + y}{2} - z \right\|^2 + \left\| \frac{x - y}{2} \right\|^2 \right) \\
= \frac{1}{4} \left( \|x + z\|^2 + \|y + z\|^2 - \|x - z\|^2 - \|y - z\|^2 \right),
\]
where equality marked * follows from the parallelogram law (4.5) which we assumed holds for all pairs of vectors from $X$. Hence
\[
2 \left\langle \frac{x + y}{2}, z \right\rangle = \left\langle x, z \right\rangle + \left\langle y, z \right\rangle \tag{4.7}
\]
for all $x, y, z \in X$. Taking $y = 0$ in (4.7) gives
\[
2(x/2, z) = \left\langle x, z \right\rangle
\]
for all $x \in X$ and hence, using (4.7) again,
\[
\left\langle x, z \right\rangle + \left\langle y, z \right\rangle = 2(\langle x + y \rangle/2, z) = \langle x + y, z \rangle \tag{4.8}
\]
for all $x, y, z \in X$. Note that (4.8) immediately implies
\[
\left\langle x, z \right\rangle - \left\langle y, z \right\rangle = \left\langle x - y, z \right\rangle \tag{4.9}
\]
for all $x, y, z \in X$.

It remains to verify that $\left\langle \alpha x, y \right\rangle = \alpha \left\langle x, y \right\rangle$ for all $\alpha \in \mathbb{R}$ and all $x, y \in X$. To see this, first note that (4.8) with $x = y$ implies that $2(\langle x, z \rangle) = (2x, z)$ for all $x, z \in X$. Hence, by induction,
\[
m(x, z) = \langle mx, z \rangle \tag{4.10}
\]
for all $m \in \mathbb{N}$ and all $x, z \in X$. However, (4.10) implies $n(\langle x/n, z \rangle) = \langle x, z \rangle$ for all $n \in \mathbb{N}$ and all $x, z \in X$. Hence,
\[
\frac{m}{n} \langle x, z \rangle = \left\langle \frac{m}{n} x, z \right\rangle \tag{4.11}
\]
for all $m, n \in \mathbb{N}$ and all $x, z \in X$.

We now claim that, for fixed $x, z \in X$ and $\alpha_n \to \alpha$ we have
\[
\left\langle \alpha_n x, z \right\rangle \to \left\langle \alpha x, z \right\rangle.
\]
Granted, it follows from (4.11) and the density of $Q$ in $\mathbb{R}$ that
\[ \alpha \langle x, z \rangle = \langle \alpha x, z \rangle \quad (4.12) \]
for all $\alpha \geq 0$ and all $x, z \in X$.

To see the claim, note that by (4.6) and continuity of the norm and linear combinations (Proposition 2.3)
\[ \langle \alpha_n x, z \rangle = \frac{1}{4} \left( \left\| \alpha_n x + z \right\|^2 - \left\| \alpha_n x - z \right\|^2 \right) \to \frac{1}{4} \left( \left\| \alpha x + z \right\|^2 - \left\| \alpha x - z \right\|^2 \right) = \langle \alpha x, z \rangle. \]

Finally, for $\alpha < 0$, note first that (4.9) implies
\[ \langle -x, z \rangle = \langle 0 - x, z \rangle = \langle 0, z \rangle - \langle x, z \rangle = 0 - \langle x, z \rangle = -\langle x, z \rangle \]
and therefore, by (4.12)
\[ \alpha \langle x, z \rangle = -(\alpha) \langle x, z \rangle = -\langle -\alpha x, z \rangle = \langle \alpha x, z \rangle \]
as required. This completes the proof of Theorem 4.4.

\[ \square \]

**Definition (Orthogonal vectors, orthogonal sets)**

Elements $y$ and $z$ in the inner product space $X$ are said to be **orthogonal** if $\langle y, z \rangle = 0$.

**Example (Collections of orthogonal vectors)**

Let $X = \ell^2$ and $\{e^{(n)}\}_{n \in \mathbb{N}}$ be the standard basis of $\ell^2$ (see Remark before Theorem 3.9). Then $e^{(n)}$ and $e^{(m)}$ are orthogonal if and only if $n \neq m$.

**Theorem 4.5 (Pythagoras’ theorem).** Let $X$ be an inner product space and suppose $x, y \in X$ are orthogonal. Then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

\section{Hilbert spaces}

**Definition (Hilbert space)**

An inner product space that is a Banach space with respect to the norm associated to the inner product is called a **Hilbert space**.

**Example (Hilbert spaces)**

1. If $X = \mathbb{F}^d$ then we have seen that $X$ is an inner product space. Since $X$ is a finite-dimensional vector space it is a Banach space by Corollary 3.7. Thus $\mathbb{F}^d$ is a Hilbert space.
2. The space $\ell^2$ is a Hilbert space (the space $\ell^2$ is a Banach space).

3. The inner product space $C[a, b]$ with the product

$$\langle f, g \rangle = \int_a^b f(t)g(t)dt$$

as in the Example in the beginning of §4, is not a Hilbert space. This follows from the fact that $(C[a, b], \| \cdot \|_2)$:

$$\|f\|_2 = \left(\int_a^b |f(t)|^2 dt\right)^{1/2}, \quad f \in C[a, b]$$

is not a Banach space. The proof of this fact is similar to the proof of the fact that $(C[a, b], \| \cdot \|_1)$ is not a Banach space (see Example before definition of equivalent norms).

END OF LECTURE 11

Distances and minimising vectors.

We address the question of whether the distance from an element $x$ of a Hilbert space $H$ to a given subset $M$,

$$\text{dist}(x, M) := \inf\{\|x - y\| : y \in M\},$$

is attained or not.

**Theorem 5.1.** Let $M$ be a closed convex subset of the Hilbert space $H$. Then for every $x \in H$ there exists a unique $y_0 \in M$ such that $\text{dist}(x, M) = \|x - y_0\|$.

**Remark.** Of course, if $x \in M$ then $y_0 = x$.

**Proof.** If $\delta := \text{dist}(x, M)$ then there exists a sequence $(y_n) \in M$ such that

$$\delta_n := \|x - y_n\| \to \delta. \quad (5.1)$$

We claim that $(y_n)$ is a Cauchy sequence. Indeed, due to convexity of $M$,

$$\|\frac{y_n + y_m}{2} - x\| \geq \delta.$$

Thus using the Parallelogram Law (4.5) we get

$$0 \leq \|y_n - y_m\|^2 = -\|y_n + y_m - 2x\|^2 + 2(\|y_n - x\|^2 + \|y_m - x\|^2)
\leq -(2\delta^2) + 2(\delta_n^2 + \delta_m^2) \to 0$$

as $m, n \to \infty$ because of (5.1). We therefore conclude that $(y_n)$ is a Cauchy sequence. Since $H$ is a Banach space, it converges to some point $y_0 \in H$. As $M$ is closed and $y_n \in M$ for all $n$ we get $y_0 \in M$. Moreover, $\|x - y_0\| = \lim \|x - y_n\| = \lim \delta_n = \delta$. 

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**§5. Hilbert spaces**