REGULAR SUBGRAPHS OF UNIFORM HYPERGRAPHS

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Abstract. We prove that for every integer $r \geq 2$, an n-vertex k-uniform hypergraph H containing no r-regular subgraphs has at most $(1+o(1))\binom{n-1}{k-1}$ edges if $k \geq r+1$ and n is sufficiently large. Moreover, if $r \in \{3,4\}$, $r \mid k$ and k,n are both sufficiently large, then the maximum number of edges in an n-vertex k-uniform hypergraph containing no r-regular subgraphs is exactly $\binom{n-1}{k-1}$, with equality only if all edges contain a specific vertex v. We also ask some related questions.

1. Introduction

What are the graphs containing no r-regular subgraphs? For $r = 2$, the answer is easy, they are forests. However, the question becomes much harder when r is larger than two. Complete characterizations of graphs with no r-regular subgraphs seem impossible even for the case $r = 3$. So it is natural to ask how many edges can a graph with no r-regular subgraphs have. Pyber [14] showed that there exists a constant c_r such that all n-vertex graphs with at least $c_r n \log n$ edges have an r-regular subgraph. On the other hand, Pyber, Rödl and Szemerédi [15] proved that there exists a graph with $\Omega(n \log\log n)$ edges having no r-regular subgraphs for any $r \geq 3$. The gap between the two bounds still remains open.

It is also natural to consider the same question for hypergraphs, both uniform and non-uniform hypergraphs. Mubayi and Verstraëte [12] proved that for every even integer $k \geq 4$, there exists n_k such that for $n \geq n_k$, each n-vertex k-uniform hypergraph H with no 2-regular subgraphs has at most $\binom{n-1}{k-1}$ edges, and equality holds if and only if H is a *full k-star*, that is, a k-uniform hypergraph consisting of all possible edges of size k containing a given vertex. For non-uniform hypergraphs, it is easy to see that an n-vertex hypergraph H with no r-regular subgraphs has at most $2^{n-1} + r - 2$ edges. One example for the equality is a *full star*, that is, a hypergraph consisting of all possible edges containing a given vertex with additional $r-2$ smallest edges not containing the given vertex. The author and Kostochka [10] proved that if $n \geq 425$ and $n > r$, a hypergraph H with no r-regular subgraphs contains $2^{n-1} + r - 2$ edges only if H is a *full star* with $r-2$ additional edges. One can ask a similar question for linear hypergraphs. Dellamonica et al. [3] showed that the maximum number of edges in a linear 3-uniform hypergraph with no two-regular subgraphs is $\Omega(n \log n)$ and $O(n^{3/2}(\log n)^5)$ and they asked whether every linear 3-uniform hypergraph with no 3-regular subgraphs has at most $o(n^2)$ edges. In Section 6, we confirm that this is true.

In this paper, we consider k-uniform hypergraphs with no r-regular subgraphs. The two main results of this paper are the following two theorems.

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Theorem 1.1. *Let k, r be two integers with $r \geq 2, k \geq r+1$. Then there exists n_k such that for $n > n_k$ any n-vertex k-uniform hypergraph H with no r-regular subgraphs has at most $(1+o(1))\binom{n-1}{k-1}$ edges. Moreover, if $k \geq 2r+1$ and $|H| \geq (1-n^{-\frac{2}{3r^2}})\binom{n-1}{k-1}$, then there exists a vertex v which belongs to at least $(1-n^{-\frac{1}{6r^2}})\binom{n-1}{k-1}$ edges.*

Theorem 1.2. *Let k, r be two integers with $r \in \{3,4\}$, $k \geq 140r$ and $r \mid k$. Then there exists n_k such that for $n > n_k$ any n-vertex k-uniform hypergraph H with no r-regular subgraphs has at most $\binom{n-1}{k-1}$ edges. Moreover, the equality holds if and only if H is a full k-star.*

Our proofs of theorems develop ideas in [12]. In Section 3 and Section 4, we prove Theorem 3.1 and Theorem 4.1 which together imply Theorem 1.1. In Section 5, we prove Theorem 1.2. In Section 6, we show some examples which somewhat explain necessity of each condition in each theorem and we also pose some further questions.

2. Preliminaries

We say H has an *r-regular subgraph* if there exists a collection of edges in $E(H)$ which all together cover each vertex in a nonempty set exactly r-times and no other vertices. We write $V(H)$ and $E(H)$ for the set of vertices and the set of edges in a hypergraph H, respectively. We denote the size of H by $|H| := |E(H)|$. Also log denotes $\log_2$ and *s-set* denotes a set of size s. First, we introduce the following simple observation which we use several times in the paper.

Observation 2.1. *For $t > 1$ and $n \geq 2k$, if an n-vertex k-uniform hypergraph H has at least $t\binom{n-1}{k-1}$ edges, then H contains a matching of size $\max\{2, \lceil \frac{t}{k} \rceil\}$.*

Proof. If $t \leq 2k$, it is obvious by Erdős-Ko-Rado theorem [6]. Assume $t > 2k$. We greedily choose disjoint edges from H. If we choose $\ell < \lceil \frac{t}{k} \rceil$ disjoint edges, the number of edges intersecting at least one of them is at most $\ell k\binom{n-1}{k-1} < t\binom{n-1}{k-1}$. Thus we can choose an edge disjoint from all previous ones to extend the matching. We can do this until we get $\lceil \frac{t}{k} \rceil$ disjoint edges to get a matching of size $\max\{2, \lceil \frac{t}{k} \rceil\}$. $\square$

The following is another simple observation which we will use later.

Observation 2.2. *Let r, k', k be integers with $k = rk'$ and $\mathcal{B} \subset \binom{[k]}{k'}$ satisfying that for any r-equipartition $A_1, \dots, A_r$ of $[k]$, there exists a part A_i with $e_i \in \mathcal{B}$. Then*

$$|\mathcal{B}| \geq \frac{1}{r}\binom{k}{k'}.$$

Proof. We pick an r-equipartition $\mathcal{A} = (A_1, \dots, A_r)$ of $[k]$ uniformly at random. For a set $B \in \binom{[k]}{k'}$, we say $B \in \mathcal{A}$ if $B = A_i$ for some $1 \leq i \leq r$. Since $\mathcal{A}$ is chosen uniformly at random, for any k'-set $B \in \binom{[k]}{k'}$,

$$\mathbb{P}[B \in \mathcal{A}] = r\binom{k}{k'}^{-1}.$$

Since for any $\mathcal{A}$, there exists $B \in \mathcal{B}$ which satisfy $B \in \mathcal{A}$,

$$\mathbb{E}[|\{B : B \in \mathcal{A}, B \in \mathcal{B}\}|] \geq 1.$$

On the other hand,

$$1 \le \mathbb{E}[|\{B : B \in \mathcal{A}, B \in \mathcal{B}\}|] = \sum_{B \in \mathbb{B}} \mathbb{P}[B \in \mathcal{A}] \le r \binom{k}{k'}^{-1} |\mathcal{B}|.$$

Therefore $|B| \ge \frac{1}{r}\binom{k}{k'}$. □

The following is a theorem from [8] concerning about the size of hypergraph without a matching of certain size.

Theorem 2.3. [8] *For $s \ge 1$ and $n \ge 4s$, if H is an n-vertex 3-uniform hypergraph with no matching of size s, then*

$$|H| \le \binom{n}{3} - \binom{n-s+1}{3}.$$

Now we introduce the notion of *sunflower.* Erdős and Rado [7] introduced the following notion of sunflower in connection with some problems in Number Theory. It is also called a Δ-*system.*

Definition 2.4. *A family of p sets is a p-sunflower if the intersections of any two sets in the family are all the same. Let $q(k,p)$ be the least integer q such that every k-uniform family of q sets contains a p-sunflower.*

They also showed that $q(k,p)$ exists. It means that if a k-uniform hypergraph has no p-sunflower, then the number of edges in the hypergraph is bounded by $q(k,p)$. In particular, they proved the following.

Theorem 2.5. [7]

$$(p-1)^k \le q(k,p) \le (p-1)^k k!$$

They also conjectured that $q(k,p) \le c_p^k$ for some constant c_p. Abbott, Hanson, and Sauer [1] and later Füredi and Kahn (see [5]) improved the upper bound of Theorem 2.5. The most recent result on the topic is the following result by Kostochka.

Theorem 2.6 (Kostochka [11])**.** *For $p \ge 3$ and $\alpha > 1$, there exists $D(p,\alpha)$ such that $q(k,p) \le D(p,\alpha)k!(\frac{(\log\log\log k)^2}{\alpha \log\log k})^k$.*

Essentially, Theorem 2.6 implies that there exists a constant $c(p)$ such that $q(k,p) \le \frac{k^k}{(\log\log k)^{k/2}}$ for k at least $c(p)$. By using Theorem 2.6, we prove the following lemma which is a variation of Lemma 1 in [12]. Note that the proof is identical to the proof of Lemma 1 in [12] except the part using Theorem 2.6.

Lemma 2.7. *There exists a constant $c(r)$ such that the following holds. Let k, r be integers and H be a k-uniform hypergraph on n vertices containing no r-regular subgraphs with maximum degree $\Delta = \Delta(H)$. If $|H| \ge c(r)\Delta k$, then*

$$|H| \le \frac{6n^{k/(k-1)}\Delta^{(k-2)/(k-1)}}{(\log\log \frac{|H|}{k\Delta})^{\frac{1}{2(k-1)}}}.$$

Proof. Let $m = \lfloor \frac{|H|}{k\Delta}$. Suppose $|H| \geq c(r)k\Delta$ and

$$|H| > \frac{6n^{k/(k-1)}\Delta^{(k-2)/(k-1)}}{(\log\log m)^{\frac{1}{2(k-1)}}}$$

for a contradiction. Then $m \geq c(r)$. These assumptions imply that

$$m^{k-1} = (\frac{|H|}{k\Delta})^{k-1} \geq \frac{6^{k-1}n^k\Delta^{k-2}}{k^{k-1}\Delta^{k-1}(\log\log m)^{1/2}} \geq \frac{1}{k\Delta(\log\log m)^{1/2}}\frac{6^{k-1}n^k}{k^{k-2}}.$$

So, we get

$$(2.1)\quad (k\Delta)^m \geq (\frac{6^{k-1}n^k}{m^{k-1}k^{k-2}(\log\log m)^{1/2}})^m > \frac{k^{2m}m^m}{(\log\log m)^{m/2}}(\frac{3n}{mk})^{mk} > \frac{k^{2m}m^m}{(\log\log m)^{m/2}}\binom{n}{mk}.$$

Now we count the matchings of size m in H. We may greedily pick edges $e_1, e_2, \cdots, e_m$ so that edges are disjoint. At first, we have $|H|$ choices for e_1. In each step, we exclude all edges intersecting previously chosen edges from the list of choices. Then we exclude at most $k\Delta$ edges in each step. Thus we conclude that the number of matchings of size m in H is at least

$$(2.2)\qquad \frac{1}{m!}\prod_{i=0}^{m-1}(|H| - k\Delta i) = \frac{1}{m!}|H|^m\prod_{i=0}^{m-1}(1-\frac{k\Delta i}{|H|}) \geq \frac{1}{m!}|H|^m\prod_{i=0}^{m-1}(1-\frac{i}{m}) \geq (k\Delta)^m.$$

Because the number of mk-sets in $V(H)$ is $\binom{n}{mk}$, (2.1) and (2.2) together assert that there are at least $\frac{k^{2m}m^m}{(\log\log m)^{1/2}} \geq q(m,r)$ distinct matchings $M_1, M_2, \cdots, M_{q(m,r)}$ covering exactly the same set M of size mk. Consider the following auxiliary hypergraph $\mathcal{H}$ with

$$V(\mathcal{H}) = \{e \in H\}, E(\mathcal{H}) = \{M_i : i = 1, \cdots, q(m,r)\}.$$

Note that a vertex in $\mathcal{H}$ is an edge in H, and an edge in $\mathcal{H}$ is a matching of size m in H. By Theorem 2.6 there are at least r distinct matchings $M_{i_1}, \cdots, M_{i_r}$ which together form an r-sunflower in $\mathcal{H}$. By the definition of r-sunflower, there exists a set M' such that $M' = M_{i_j} \cap M_{i_{j'}}$ for any $j, j' \in [r]$ with $j \neq j'$. Then $M_{i_j} - M'$ for $j = 1, 2, \cdots, r$ are r disjoint matchings covering the same set $M - \bigcup_{e\in M'} e$. Thus $\bigcup_{j=1}^{r}(M_{i_j} - M)$ gives us an r-regular subgraph of H, it is a contradiction. □

We also use the following theorem of Pikhurko and Verstraëte in several places.

Theorem 2.8. [13] *For $k \geq 3$, if H is an n-vertex k-uniform hypergraph with at least $\frac{7}{4}\binom{n-1}{k-1}$ edges, then H contains two pairs of sets $\{A, B\}, \{C, D\}$ so that*

$$A \cap B = C \cap D = \emptyset, A \cup B = C \cup D.$$

Now we introduce new hypergraphs $H(k,\ell)$, and $H'(k,\ell)$ which will be useful for proving several claims later.

Definition 2.9. *Let A, B be two disjoint $(k-\ell)$-sets and $Y = \{u_1, \ldots, u_\ell, v_1, \ldots, v_\ell\}$. For two integer k, ℓ with $k > \ell$, we define $H(k,\ell)$ to be the $2k$-vertex k-uniform hypergraph on the ground set $A \cup B \cup Y$ satisfying the following,*

$$E(H'(k,\ell)) = \{e \cup Z : |e \cap \{u_i, v_i\}| = 1 \text{ for all } i = 1, 2, \cdots, \ell, |e| = l, Z \in \{A, B\}\}.$$

We call each of A and B a stationary part, and vertices in them stationary vertices. Also we call vertices in Y dynamic vertices and let $V_d(H(k,\ell))$ denote Y.

Note that if e is an edge of $H(k,\ell)$, then there exist indices $i_1,\ldots,i_s,j_1,\ldots,j_{\ell-s}$ with $\{i_1,\ldots,i_s\}\cup\{j_1,\ldots,j_{\ell-s}\}=\{1,\ldots,\ell\}$ such that

$$e=\{u_{i_1},\ldots,u_{i_s},v_{j_1},\ldots,v_{j_{\ell-s}}\}\cup Z \text{ for } Z\in\{A,B\}.$$

Then $e'=\{v_{i_1},\ldots,v_{i_s},u_{j_1},\ldots,u_{j_{\ell-s}}\}\cup Z'$ for $Z'=(A\cup B)\setminus Z$ is also an edge in $H(k,\ell)$. Thus the following holds.

(2.3) For an edge e in $H(k,\ell)$, there exists $e'\in H(k,\ell)$ with $e\cap e'=\emptyset, e\cup e'=V(H(k,\ell))$.

Definition 2.10. *Let A,B,C,D be four distinct $(k-\ell)$-sets satisfying $A\cup B=C\cup D$, $A\cap B=C\cap D=\emptyset$, and $Y=\{u_1,\ldots,u_\ell,v_1,\ldots,v_\ell\}$. For two integer k,ℓ with $k>\ell$, we define $H'(k,\ell)$ to be the $2k$-vertex k-uniform hypergraph on the ground set $A\cup B\cup Y$ satisfying the following,*

$$E(H'(k,\ell))=\{e\cup Z: |e\cap\{u_i,v_i\}|=1 \text{ for all } i=1,2,\cdots,\ell, |e|=l, Z\in\{A,B,C,D\}\}.$$

We call each of A,B,C and D a stationary part, and vertices in them stationary vertices. Also we call vertices in Y dynamic vertices.

Indeed, $H(k,\ell)$ is a k-uniform hypergraph which resembles the complete $(\ell+1)$-partite $(\ell+1)$-uniform hypergraph with all parts size two. Because of the resemblance, its Turan number is related to the Turan number of $(\ell+1)$-partite $(\ell+1)$-graph. The lemma below is proved by Erdős, and we use it to find the Turan number of $H(k,\ell)$.

Lemma 2.11. [4] *Let S be a set of N elements $y_1,y_2,\cdots,y_N$ and let A_i for $1\le i\le n$ be subsets of S with size k. If $\sum_{i=1}^n|A_i|\ge\frac{nN}{w}$ for some w and $n\ge 8w^2$, then there are 2 distinct A_{i_1},A_{i_2} so that*

$$|A_{i_1}\cap A_{i_2}|\ge\frac{N}{2w^2}.$$

Proposition 2.12. *Let k,ℓ be integers with $k>\ell$. Then for $n>2k$, any k-uniform hypergraph H with $2n^{k-2^{-\ell}}$ edges contains a copy of $H(k,\ell)$ as a subgraph. Moreover, if $k\ge\ell+3$, then it also contains a copy of $H'(k,\ell)$.*

Proof. We use induction on ℓ. For $\ell=0$, assume we have an n-vertex k-uniform hypergraph H with $2n^{k-1}$ edges. By Observation 2.1 and the fact $n^{k-1}>\binom{n-1}{k-1}$, we get $H(k,0)$ which is a matching of size two for any k. If $k\ge 3$ and $\ell=0$, then Theorem 2.8 implies that H contains $H'(k,\ell)$, which consists of two pairs of disjoint edges with the same union.

For $k=2,\ell=1$, Turan number for the cycle of length 4 gives us the conclusion about $H(k,\ell)$. Assume now that every n-vertex k-uniform hypergraph with $2n^{k-2^{-\ell+1}}$ edges contains a copy of $H(k,\ell-1)$ for $n>2(k-1)$ and $k\ge 3,\ell\ge 1$. If an n-vertex k-uniform hypergraph H with $n>2k$ contains at least $2n^{k-2^{-\ell}}$ edges, we let $x_1,x_2,\cdots,x_n$ be the vertices of H and $y_1,\cdots,y_N$ with $N=\binom{n}{k-1}$ be the all $(k-1)$-sets in $V(H)$. Let $A_i=\{y_j: y_j\cup x_i\in H, 1\le j\le N\}$. Because each edges of H contains k sets y_i,

$$\sum_{i=1}^n|A_i|=k|H|\ge 2kn^{k-2^{-\ell}}>\frac{nN}{\frac{n^{2^{-\ell}}}{2(k-1)!}},$$

we may apply Lemma 2.11 with

$$N = \binom{n}{k-1}, w = \frac{n^{2^{-\ell}}}{2(k-1)!}.$$

For $n > 2k, n \geq 8w^2 = \frac{2n^{2^{1-\ell}}}{(k-1)!^2}$ is satisfied, thus there are two sets A_{i_1}, A_{i_2} for which

$$|A_{i_1} \cap A_{i_2}| \geq \frac{1}{2}\binom{n}{k-1}(2(k-1)!n^{-2^{-\ell}})^2 > 2n^{k-1-2^{-\ell+1}}.$$

By induction hypothesis, $A_{i_1} \cap A_{i_2}$ contains H', a copy of $H(k-1, \ell-1)$. Then

$$\{\{z\} \cup e : e \in E(H'), z \in \{x_{i_1}, x_{i_2}\}\}$$

is a copy of $H(k, \ell)$. Thus H must contain a copy of $H(k, l)$. We get the conclusion for $H'(k, \ell)$ in the same logic. □

Since $H(k, \ell)$ contains 2^ℓ distinct matchings of size 2 covering the same ground set, if a hypergraph contains a copy of $H(k, \ell)$, then it contains an r-regular subgraphs for all $1 \leq r \leq 2^\ell$. Also, $H'(k, \ell)$ contains an r-regular subgraphs for all $1 \leq r \leq 2^{\ell+1}$.

3. Approximate size of H

In this section, we prove the following Theorem 3.1 by showing that most of the edges in H contain only one vertex of high degree. Note that we only consider the case when $r \geq 3$ because the case of $r = 2$ is already done in [12]. We let $\ell := \lceil \log r \rceil$, $0 < \alpha \leq \frac{1}{2}$ be a number which we decide later, and we let

$$D := n^{k-1-\alpha}(\log\log n)^{\frac{1}{4(k-1)}}. \tag{3.1}$$

We let T denote the set of vertices of H of degree at least D and set $t := |T|$. Since $tD \leq k|H|$,

$$t \leq D^{-1}k|H|. \tag{3.2}$$

We also define $H_i := \{e \in H : |e \cap T| = i\}$ for $i \leq k$, and $G := \{e \in H_1 : \nexists f \in H_1 : e \setminus T = f \setminus T\}$. Then, it is obvious that $|G| \leq \binom{n-1}{k-1}$. Note that $r + 1$ is always at least $2^{\ell-1} + 2$.

Theorem 3.1. *For integer k, r, ℓ with $k > r \geq 3$, $\ell = \lceil \log r \rceil$, there exists an integer n_k such that for $n \geq n_k$ any n-vertex k-uniform hypergraph H with no r-regular subgraphs satisfies the following.*

$$\text{If } k > 2^{\ell-1} + 2, \text{ then } |H| \leq \binom{n-1}{k-1} + 3n^{k-1-\frac{1}{2r^2-2}}.$$
$$\text{If } k = 2^{\ell-1} + 2, \text{ then } |H| \leq \binom{n-1}{k-1} + 3n^{k-1}(\log\log n)^{-\frac{1}{4(k-1)}}.$$

Proof. First we suppose the conclusion does not hold. We may assume we have a counterexample H such that $|H|$ is equal to the stated upper bound by deleting some edges if necessary and assume n is large enough. Since n is large enough, $|H| \leq n^{k-1}/k$ and (3.2) implies

$$t \leq D^{-1}k|H| \leq n^{\alpha}(\log\log n)^{-\frac{1}{4(k-1)}}. \tag{3.3}$$

Claim 3.2.

$$|H_0| \le \max\left\{ n^{k-1-\alpha+\frac{1}{2k^2}}, n^{k-1+\frac{1-(k-2)\alpha}{k-1}}(\log\log n)^{-\frac{1}{4(k-1)}} \right\}.$$

$$|H \setminus (H_0 \cup H_1)| \le n^{k-2+2\alpha}(\log\log n)^{-\frac{1}{2(k-1)}}.$$

Proof. First, we estimate $|H_0|$. Since edges in H_0 do not intersect T, the maximum degree of H_0 is less than D. We apply Lemma 2.7 to H_0, then we get

$$|H_0| \le \max\left\{ c(r)k\Delta(H_0), \frac{6n^{k/(k-1)}D^{(k-2)/(k-1)}}{(\log\log\frac{|H|}{kD})^{\frac{1}{2(k-1)}}} \right\}$$

where $c(r)$ is the constant from Lemma 2.7. Since n is large enough,

$$c(r)k\Delta(H_0) \le c(r)kD \le c(r)kn^{k-1-\alpha}(\log\log n)^{\frac{1}{4(k-1)}} \le n^{k-1-\alpha+\frac{1}{2k^2}}.$$

Also since n is large, $\frac{|H|}{kD} \ge \binom{n}{k-1}/(kn^{k-1-\alpha}(\log\log n)^{\frac{1}{4(k-1)}}) \ge n^{\alpha/2}$ holds. Thus for large enough n,

$$\frac{6n^{k/(k-1)}D^{(k-2)/(k-1)}}{(\log\log\frac{|H|}{kD})^{\frac{1}{2(k-1)}}} \le \frac{6n^{k/(k-1)}D^{(k-2)/(k-1)}}{(\log\log n^{\alpha/2})^{\frac{1}{2(k-1)}}} \overset{(3.1)}{\le} n^{k-1+\frac{1-(k-2)\alpha}{k-1}}(\log\log n)^{-\frac{1}{4(k-1)}}.$$

Last, every edge in $H \setminus (H_0 \cup H_1)$ contains two vertices of T and $k-2$ vertices of $V(H)$. By (3.3),

$$|H \setminus (H_0 \cup H_2)| \le \binom{|T|}{2} n^{k-2} \le n^{k-2+2\alpha}(\log\log n)^{-\frac{1}{2(k-1)}}.$$

□

Claim 3.3.

$$|H_1| \le |G| + n^{k-1-2^{-\ell+2}+2\alpha}(\log\log n)^{-\frac{1}{2(k-1)}}.$$

Proof. Note that $k \ge 2^{\ell-1}+2 \ge l+2$. We consider $H_1 \setminus G$. Then every edge e in $H_1 \setminus G$ satisfies $|e \cap (V(H)-T)| = k-1$. Also a $(k-1)$-set $e \setminus T$ lies in at least two edges of H. For each pair $\{u,u'\} \subset T$, we consider the $(k-1)$-uniform hypergraph

$$H_{\{u,u'\}} = \{e' \in \binom{A}{k-1} : e' \cup \{u\} \in E(H_1 \setminus G), e' \cup \{u'\} \in E(H_1 \setminus G)\}.$$

By the definition of G, every edge in $H_1 \setminus G$ belongs to $H_{\{u,u'\}}$ for at least one pair $\{u,u'\}$. However, if $H_{\{u,u'\}}$ contains a copy of $H'(k-1,\ell-2)$, then the copy together with u,u' form a copy of $H'(k,\ell-1)$ in H, which gives us an r-regular subgraph of H. Thus Proposition 2.12 and the fact that $k-1 = 2^{\ell-1}+1 \ge \ell-2+3$ for $\ell \ge 2$ imply $|H_{\{u,u'\}}| \le 2n^{k-1-2^{-\ell+2}}$. Thus we get

$$|H_1 \setminus G| \le \sum_{\{u,u'\}\in\binom{T}{2}} |H_{\{u,u'\}}| \le \binom{|T|}{2} 2n^{k-1-2^{-\ell+2}} \overset{(3.3)}{\le} n^{k-1-2^{-\ell+2}+2\alpha}(\log\log n)^{-\frac{1}{2(k-1)}}.$$

□

If $k > 2^{\ell-1}+2$, we choose $\alpha = \frac{1}{2(k-2)} + 2^{-\ell}$, then we get

$$\max\{k-1-\alpha+\frac{1}{2k^2}, k-1+\frac{1}{k-1}-\frac{(k-2)\alpha}{k-1}, k-1-2^{-\ell+2}+2\alpha, k-2+2\alpha\} \le k-1-\frac{1}{2r^2-2}.$$

Hence, $|H\setminus G| = |H_0| + |H_1\setminus G| + |H\setminus(H_0\cup H_1)| \le 3n^{k-1-\frac{1}{2r^2-2}}$. We conclude that for large enough n,

$$|H| \le \binom{n-1}{k-1} + 3n^{k-1-\frac{1}{2r^2-2}}.$$

If $k = 2^{\ell-1}+2$, then we choose $\alpha = 2^{-\ell+1} = \frac{1}{k-2}$, then we get $|H\setminus G| = |H_0| + |H_1\setminus G| + |H\setminus (H_0\cup H_1)| \le 3n^{k-1}(\log\log n)^{-\frac{1}{4(k-1)}}$ and we conclude that for large enough n,

$$|H| \le \binom{n-1}{k-1} + 3n^{k-1}(\log\log n)^{-\frac{1}{4(k-1)}}$$

Therefore, we can take n_k large enough so that the Theorem holds. $\square$

Remark 3.4. *If $k \ge 2^\ell+3$, then we may choose $\alpha := \frac{3\cdot 2^\ell+4}{2^\ell(3\cdot 2^\ell+5)}$, $D := n^{k-1-\alpha}$ and go through the argument above. Then we can conclude $|H\setminus G| \le 3n^{k-1-\frac{1}{2^{\ell-1}(3\cdot 2^\ell+5)}}$ for any n-vertex k-uniform hypergraph H with no r-regular subgraphs when n is large enough. In order to get Theorem 4.1, we assume*

$$\alpha := \frac{3\cdot 2^\ell+4}{2^\ell(3\cdot 2^\ell+5)}, D := n^{k-1-\alpha}, \ell := \lceil \log r\rceil$$

throughout the paper.

4. Asymptotic structure of H

In this section, we want to show that the asymptotic structure of H is close to a full k-star. We let G be as we define in the previous section, and $\alpha = \frac{3\cdot 2^\ell+4}{2^\ell(3\cdot 2^\ell+5)}$, $D = n^{k-1-\alpha}$, $\ell = \lceil\log r\rceil$ as in Remark 3.4 and T denote the set of vertices of H of degree at least D. Then we still have (3.3). We also define

$$G' := \{e\setminus T : e\in G\},\ G_x := \{e\setminus\{x\} : x\in e, e\in G\}.$$

In order to prove Theorem 4.1, we count the copies of $H(k-1,\ell+1)$ in G' and show that there exists a vertex v such that almost all copies of $H(k-1,\ell+1)$ consist of $(k-1)$-sets in G_v. We define β as follows and use throughout the paper, $\beta := k^{4k}n^{-\frac{1}{2^{\ell-1}(3\cdot 2^\ell+5)}}$.

Theorem 4.1. *For integer k, r, ℓ with $\ell = \lceil\log r\rceil$, $k\ge 2^\ell+3$, there exists an integer n_k such that the following holds. If H is an n-vertex k-uniform hypergraph H with no r-regular subgraphs and $|H| \ge \binom{n-1}{k-1} - n^{k-1-\frac{1}{2^{\ell-1}(3\cdot 2^\ell+5)}}$, then there exists a vertex v in H such that*

$$|G_v| \ge (1-\beta)|G'|$$

with $\beta = k^{4k}n^{-\frac{1}{2^{\ell-1}(3\cdot 2^\ell+5)}}$.

Proof.
We take a k-uniform hypergraph H with no r-regular subgraphs satisfying $|H| \geq \binom{n-1}{k-1} - n^{k-1-\frac{1}{2^{\ell-1}(3\cdot 2^{\ell}+5)}}$. Then by Remark 3.4, we know

$$|G| = |G'| \geq \binom{n-1}{k-1} - 4n^{k-1-\frac{1}{2^{\ell-1}(3\cdot 2^{\ell}+5)}}. \tag{4.1}$$

We pick v such that

$$|G_v| = \max_{x\in V(H)} |G_x|.$$

For a contradiction, we assume $|G_v| < (1-\beta)|G'|$. For each $(k-1)$-set e in G', we define $g(e) := x$ if $e \in G_x$. Let $R_i(G')$ be the set of all pairs $\{f_1, f_2\}$ of $(k-1)$-sets in G' so that $|f_1 \cap f_2| = i$, and let $R'_i(G')$ be the subset of $R_i(G')$ of all pairs $\{f_1, f_2\}$ such that $g(f_1) \neq g(f_2)$, and let

$$R'(G') := R'_{\ell}(G') \cup R'_{\ell+1}(G').$$

For a hypergraph F, we define $P(F)$ to be the set of copies of $H(k-1, \ell+1)$ in F.

$$P(F) := \{H' : H' \subset F, H' \simeq H(k-1, \ell+1)\}.$$

Let $P_0(G')$ be the set of copies of $H(k-1, \ell+1)$ so that the copy does not contain a pair $\{f_1, f_2\}$ in $R'(G')$ in the way that all vertices in $f_1 \cap f_2$ are dynamic vertices in the copy of $H(k-1, \ell+1)$. Let $P_1(G')$ be the set of copies of $H(k-1, \ell+1)$ so that the copy contains at least one pair $\{f_1, f_2\}$ in $R'(G')$ in the way that all vertices in $f_1 \cap f_2$ are dynamic vertices of the copy of $H(k-1, l+1)$.

Let K be the complete $(k-1)$-graph on $V(G')$. To count the number of copies of $H(k-1, \ell+1)$ in K, we choose two disjoint $(k-1)$-sets, and choose $\ell+1$ vertices from one part and match them with other $\ell+1$ vertices on the other part. In this manner, one copy of $H(k-1, \ell+1)$ is counted exactly $2^{\ell+1}$ times which is the number of pairs in $H(k-1, \ell+1)$. Thus we get

$$|P(K)| = \frac{1}{2^{\ell+2}} \binom{k-1}{\ell+1} \frac{(k-1)!}{(k-\ell-2)!} \binom{n-1}{k-1} \binom{n-k}{k-1}. \tag{4.2}$$

Since n is large enough and $(\frac{n}{k})^k \leq \binom{n}{k}$ holds,

$$|P(K)| \geq \frac{1}{2^{\ell+2}} \binom{n-1}{k-1} \binom{n-k-1}{k-1} \geq k^{-2k} n^{2k-2} \tag{4.3}$$

Thus (4.1) implies

$$|K \setminus G'| \leq 4n^{k-1-\frac{1}{2^{\ell-1}(3\cdot 2^{\ell}+5)}}. \tag{4.4}$$

Firs we show a lower bound on $|P(G')|$.

Claim 4.2. $|P(G')| > (1-\beta)|P(K)|$

Proof. It is enough to show $|P(K) \setminus P(G')| < \beta |P(K)|$. Note that any copy $H(k-1, \ell-1)$ in $P(K) \setminus P(G')$ must contain an edge $e \in K \setminus G'$. Also by (2.3), we can find another edge e' satisfying

$$e' \in H(k-1, \ell+1) \text{ with } e \cap e' = \emptyset \text{ and } e \cup e' = V(H(k-1, \ell+1)).$$

Thus, in order to count $P(K) \setminus P(G')$, we take a $(k-1)$-set e in $K \setminus G'$ and a $(k-1)$-set e' in K disjoint from e. There are $|K \setminus G'|$ ways to choose e, and for fixed e, there are $\binom{n-k}{k-1}$ ways to choose e'. Then there are at most $\binom{k-1}{\ell+1}\frac{(k-1)!}{(k-\ell-2)!}$ copies of $H(k-1,\ell+1)$ containing both e, e'. By (4.4), the definition of β, and the fact that $(\frac{n}{s})^s \leq \binom{n}{s}$, we have

$$|K \setminus G'| \leq 4n^{k-1-\frac{1}{2^{\ell-1}(3\cdot 2^{\ell}+5)}} = 4k^{-4k}\beta n^{k-1} < \frac{\beta}{2^{\ell+2}}\binom{n-1}{k-1},$$

Thus,

$$\begin{aligned}|P(K) \setminus P(G')| &\leq \binom{k-1}{\ell+1}\frac{(k-1)!}{(k-\ell-2)!}|K \setminus G'|\binom{n-k}{k-1}\\ &< \frac{\beta}{2^{\ell+2}}\binom{k-1}{\ell+1}\frac{(k-1)!}{(k-\ell-2)!}\binom{n-1}{k-1}\binom{n-k}{k-1} \overset{(4.2)}{\leq} \beta|P(K)|.\end{aligned}$$

□

Now we estimate $|P(G')|$ to show a contradiction.

Claim 4.3. $|P_1(G')| < \frac{1}{2}\beta|P(K)|$

Proof. First, we count the number of pairs $\{f_1, f_2\}$ in $R_i'(G')$ with $i \in \{\ell, \ell+1\}$. We take two disjoint $(k-1-i)$-sets e_1, e_2, and two distinct vertices $x, y \in T$. Let $p(e_1, e_2, x, y)$ be the collection of i-sets h so that $f_1 = e_1 \cup h, f_2 = e_2 \cup h$ and $g(f_1) = x, g(f_2) = y$. If $p(e_1, e_2, x, y)$ contains a copy of $H(i, \ell-1)$, then H contains $2^{\ell} \geq r$ distinct matchings of size two covering the same ground set giving an r-regular subgraph, a contradiction. Thus $p(e_1, e_2, x, y)$ does not contain $H(i, \ell-1)$, so it has at most $2n^{i-\frac{1}{2^{\ell-1}}}$ edges by Proposition 2.12. There are at most $\binom{n-1}{k-i-1}^2$ choices for $\{e_1, e_2\}$ and $\binom{|T|}{2}$ choices for $\{x, y\}$. So,

$$|R_i'(G')| \leq \sum_{\{e_1,e_2,x,y\}} 2n^{i-\frac{1}{2^{\ell-1}}} \leq 2n^{i-\frac{1}{2^{\ell-1}}}\binom{n-1}{k-i-1}^2\binom{|T|}{2}.$$

For each pair in $R_i'(G')$, we can complete a copy of $H(k-1, \ell+1)$ by adding i more vertices from outside and choosing $\ell+1-i$ vertices each from e_1, e_2 to play a role of dynamic vertices and match those dynamic vertices. Thus each pair in $R_i'(G')$ is contained in at most $(\ell+1)!\binom{k-i-1}{\ell+1-i}^2\binom{n-1}{i}$ copies of $H(k-1, \ell+1)$. Thus by the fact that $-\frac{1}{2^{\ell-1}} + 2\alpha = -\frac{1}{2^{\ell-1}(3\cdot 2^{\ell}+5)}$ and the definition of β,

$$\begin{aligned}|P_1(G')| &= \sum_{i=\ell}^{\ell+1} |R_i'(G')|(\ell+1)!\binom{k-i-1}{\ell+1-i}^2\binom{n-1}{i}\\ &\leq \sum_{i=\ell}^{\ell+1} 2n^{i-\frac{1}{2^{\ell-1}}}\binom{n-1}{k-i-1}^2\binom{|T|}{2}(\ell+1)!\binom{k-i-1}{\ell+1-i}^2\binom{n-1}{i}\\ &\overset{(3.3)}{\leq} \sum_{i=\ell}^{\ell+1} k^2(\ell+1)!n^{i-\frac{1}{2^{\ell-1}}+2\alpha}\binom{n-1}{k-i-1}^2\binom{n-1}{i}\\ &\leq \sum_{i=\ell}^{\ell+1} k^2(\ell+1)!n^{i-\frac{1}{2^{\ell-1}}+2\alpha}n^{2k-2i-2}n^{i}\\ &\leq 2k^2k!n^{2k-2-\frac{1}{2^{\ell-1}}+2\alpha} < \frac{1}{2}k^{4k}n^{-\frac{1}{2^{\ell-1}(3\cdot 2^{\ell}+5)}}k^{-2k}n^{2k-2} \overset{(4.3)}{\leq} \frac{1}{2}\beta|P(K)|\end{aligned}$$

□

Before we estimate $|P_0(G')|$, we prove the following claim.

Claim 4.4. *Let H' be a copy of $H(k-1,\ell+1)$ in $P_0(G')$. Then there exists a vertex $x \in T$ so that every $(k-1)$-set e in H' is contained in G_x.*

Proof. Reminde that $V_d(H')$ denote the set of dynamic vertices of H'. We consider a graph $G_{H'}$ such that

$$V(G_{H'}) := \{f \in H'\},$$
$$E(G_{H'}) := \{f_1 f_2 : f_1, f_2 \in H', g(f_1) = g(f_2), |f_1 \cap f_2| \in \{\ell, \ell+1\}, f_1 \cap f_2 \subset V_d(H')\}.$$

Let A, B be two stationary parts in H'. Consider two $(k-1)$-sets f_1, f_2 in H' such that $|f_1 \setminus f_2| = |f_1 \setminus f_2| = 1$ and both f_1, f_2 contain A. We consider an edge $e \in H'$ such that $e = (f_1 \setminus A) \cup B$. Then, e does not share any stationary vertices with f_1 or f_2, $|f_1 \cap e| = \ell+1$, and $|f_2 \cap e| = \ell$. Thus e is adjacent to both f_1 and f_2 in $G_{H'}$. Thus any two $(k-1)$-sets f, f' in H' containing A with $|f \setminus f'| = 1$ are in the same component of $G_{H'}$. Since being in the same component is transitive, all $(k-1)$-sets in H' containing A are in the same component in $G_{H'}$. All fs containing B are in the same component in $G_{H'}$ by the same logic. Also there are edges between f_1 and e, so $G_{H'}$ is connected. On the other hand, if two $(k-1)$-sets f_1, f_2 are adjacent in $G_{H'}$, then $g(f_1) = g(f_2)$ because of the definition of $P_0(G')$. This fact and connectedness of $G_{H'}$ together imply that there exists a vertex $x \in T$ such that every edge in H' belongs to G_x. □

Claim 4.5. $|P_0(G')| < (1 - \frac{3}{2}\beta)|P(K)|$

Proof. A copy of $H(k-1,\ell+1)$ in G' consists of $2^{\ell+1}$ pairs of two disjoint $(k-1)$-sets all in G_x for some $x \in T$. To count the number of copies of $H(k-1,\ell+1)$ in $P_0(G')$, we choose two disjoint $(k-1)$-sets e, e' with $g(e) = g(e')$, and choose $\ell+1$ elements from one and match them with $\ell+1$ vertices in the other side. This can be done in $\binom{k-1}{\ell+1}\frac{(k-1)!}{(k-\ell-2)!}$ ways. Also, each $H(k-1,\ell+1)$ is counted $2^{\ell+1}$ times from each pair in this counting. So,

$$|P_0(G')| \le \frac{1}{2^{\ell+1}} \binom{k-1}{\ell+1} \frac{(k-1)!}{(k-\ell-2)!} \sum_{x \in T'} \binom{|G_x|}{2}. \tag{4.5}$$

By convexity, the right side of (4.5) is maximized when $|G_v| = (1-\beta)|G'|$ and $|G_u| = \beta|G'|$ for another vertex $u \in T$ and $|G_x| = 0$ for other x. And $\binom{|G'|}{2} \le \frac{1}{2}\binom{n-1}{k-1}^2 \le \frac{1}{2}(1+\frac{k}{n})\binom{n-1}{k-1}\binom{n-k-1}{k-1}$. Becuase of (4.2) and the fact $\frac{k}{n} + 2\beta^2 < \frac{1}{2}\beta$,

$$|P_0(G')| \le ((1-\beta)^2 + \beta^2)(1+\frac{k}{n})|P(K)| \le (1 - \frac{3}{2}\beta)|P(K)|.$$

□

In total,

$$|P(G')| = |P_0(G')| + |P_1(G')| < (1 - \frac{3}{2}\beta)|P(K)| + \frac{1}{2}\beta|P(K)| \leq (1-\beta)|P(K)|.$$

However, it contradicts to Claim 4.2. Therefore, we conclude $|G_v| \geq (1-\beta)|G'|$. Note that $2^\ell + 3 \geq 2r+1$ and $n^{-\frac{2}{3r^2}} \leq k^{4k} n^{-\frac{1}{2^{\ell-1}(3\cdot 2^\ell+5)}} \leq n^{-\frac{1}{6r^2}}$ for large enough n. Thus Theorem 3.1, Remark 3.4 and Theorem 4.1 together imply Theorem 1.1. $\square$

If $r \in \{3,4\}$, then $\ell = 2$, so we get $-\frac{1}{2^{\ell-1}(3\cdot 2^\ell+5)} = -\frac{1}{34}$, thus $\beta = k^{4k}n^{-\frac{1}{34}}$. So the above proof actually gives the following.

Remark 4.6. *If $r \in \{3,4\}$ and $k \geq 2r+1$, then there exists an integer n_k such that for $n > n_k$ any n-vertex k-uniform hypergraph H with no r-regular subgraphs H with $|H| \geq (1 - k^{4k}n^{-\frac{1}{34}})\binom{n-1}{k-1}$, then there exists a vertex v which belongs to at least $(1-k^{4k}n^{-\frac{1}{34}})\binom{n-1}{k-1}$ edges.*

5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We assume $r \in \{3,4\}$, $k = k'r$, $k' \geq 140$ and that n is sufficiently large. If an n-vertex k-uniform hypergraph H with no r-regular subgraphs contains at least $\binom{n-1}{k-1}$ edges, we may suppose $|H| = \binom{n-1}{k-1}$ by deleting some edges if necessary. If we show that H has to be a full k-star, then it completes the theorem because a full k-star with one more edge e always contains an r-regular subgraph when $r \mid k$ by the following Observation 5.1. Since $r \in \{3,4\}$ implies $\ell = 2$, we have $\beta = k^{4k}n^{-\frac{1}{34}}$. By Remark 4.6, there exists a vertex v with $|H^*| \leq k^{4k}n^{-\frac{1}{34}}\binom{n-1}{k-1}$. We define

$$H^* := \{e \in H : v \notin e\}, \tilde{H} := \{f \in \binom{V(H)}{k} : v \in f, f \notin H\}.$$

Then

$$|H^*| = |\tilde{H}| \leq k^{4k}n^{-\frac{1}{34}}\binom{n-1}{k-1} \tag{5.1}$$

by our assumption and Remark 4.6. To show that H is a full k-star, it is enough to show $|H^*| = 0$. Suppose $|H^*| > 0$ for a contradiction.

Observation 5.1. *If $\{e'_1, e'_2, \cdots, e'_r\}$ is a partition of $e \in H^*$ into r sets of size k' and $g \subset V(G) - \{v\} - e$ is a $(k'-1)$-set, then there exists j such that $(e \setminus e'_j) \cup g \cup \{v\}$ is not an edge of H.*

Proof. Suppose not. Then $e, (e\setminus e'_1) \cup g \cup \{v\}, (e \setminus e'_2) \cup g \cup \{v\}, \cdots, (e \setminus e_r) \cup g \cup \{v\}$ together form an r-regular subgraph, a contradiction. Thus there is a choice j such that $(e \setminus e'_j) \cup g \cup \{v\}$ is not an edge of H. $\square$

Here we define *wedge* as follows and count them to derive a contradiction.

Definition 5.2. *A pair of k-sets (e,f) is a wedge if if it satisfies the following:*
(1) $e \in H^*, f \in \tilde{H}$;
(2) $|e \cap f| = k - k'$.

Let $\Lambda(H)$ be the number of wedges in H. Then the following claims gives us a lower bound for $\Lambda(H)$.

Claim 5.3.

$$\Lambda(H) \geq \frac{1}{r}\binom{k}{k'}\binom{n-k-1}{k'-1}|H^*|.$$

Proof. To count the wedges (e,f) in H^*, instead we count $(e, f \setminus (e \cup \{v\}), e \setminus f)$. First we choose e. There are $|H^*|$ ways to choose e. For each chosen e, there are $\binom{n-k-1}{k'-1}$ ways to choose $(k'-1)$-set S outside $e \cup \{v\}$ playing the role of $f \setminus (e \cup \{v\})$. For fixed e and S, we call a k'-subset D of e (S,e)-*good* if $(e \setminus D) \cup S \cup \{v\} \in H$, and (S,e)-*bad* otherwise. By Observation 5.1, for an r-equipartition $\{e'_1, \ldots, e'_r\}$ of e, at least one of e'_i is S-bad. By Observation 2.2, this implies that for fixed e and S, there are at least $\frac{1}{r}\binom{k}{k'}$ (S,e)-bad subsets of e. For each (S,e)-bad subset D of e, we get a wedge

$$(e, (e \setminus D) \cup S \cup \{v\}).$$

Since those wedges are distinct for distinct (e,S,D), we get

$$\Lambda(H) \geq \frac{1}{r}\binom{k}{k'}\binom{n-k-1}{k'-1}|H^*|.$$

$\square$

Definition 5.4. *A 3-set $T \in \binom{V(H)\setminus\{v\}}{3}$ is good if $d_{\tilde{H}}(T \cup \{v\}) < \frac{1}{8}\binom{n-k-4}{k-4}$ and bad otherwise. A collection W is the collection of all bad 3-sets in H.*

Claim 5.5.

$$|W| \leq |W| \leq k^{5k} n^{3-\frac{1}{34}}.$$

Proof. We count all k-sets in $\tilde{H}$ which contain a bad 3-set. Each bad 3-set T belongs to at most $\frac{1}{8}\binom{n-k-4}{k-4}$ distinct k-sets in $\tilde{H}$. Also, each k-set can contain at most $\binom{k}{3}$ distinct bad 3-sets. Thus the number of k-sets in $\tilde{H}$ containing a bad 3-set is at least

$$\frac{1}{8}\binom{k}{3}^{-1}|W|\binom{n-k-4}{k-4}.$$

From (5.1),

$$\frac{1}{8}\binom{k}{3}^{-1}|W|\binom{n-k-4}{k-4} \leq |\tilde{H}| \leq k^{4k} n^{-\frac{1}{34}}\binom{n-1}{k-1}.$$

Since $\binom{n-1}{k-1} \leq 2k^3 n^3 \binom{n-k-4}{k-4}$ for sufficiently large n, we get

$$|W| \leq 8\binom{k}{3}k^{4k}n^{-\frac{1}{34}}\binom{n-1}{k-1}\binom{n-k-4}{k-4}^{-1} \leq \frac{16}{6}k^{4k+6}n^{3-\frac{1}{34}} \leq k^{5k}n^{3-\frac{1}{34}}$$

since n is sufficiently large and $k \geq 140r$. $\square$

Claim 5.6.

$$\Lambda(H) \leq |\tilde{H}|\binom{k-1}{k'-1}\frac{1.01}{\binom{k'-34}{3}}\binom{n}{k'-3}n^2.$$

Proof. To count the number of wedges (e,f) in H, instead we count $(f, e\cap f, e\setminus f)$. The number of ways to pick f is $|\tilde{H}|$. For fixed f, the number of ways to choose a $(k-k')$-subset D of $f\setminus\{v\}$ which will play a role of $= e\cap f$ is

$$\binom{k-1}{k-k'} = \binom{k-1}{k'-1}.$$

For a $(k-k')$-set D, let H_D be a k'-uniform hypergraph defined as follows

$$H_D := \{e \setminus D : D \subset e \in H^*\}.$$

We partition H_D into the following two hypergraphs,

$$\begin{aligned} H_D^1 &:= \{B \in H_D : B \text{ contains at least 35 disjoint bad 3-sets}\} \\ H_D^2 &:= H_D \setminus H_D^1. \end{aligned}$$

First, since any k'-set in H_D^1 contains 35 disjoint bad 3-sets, $k' \geq 140$ and n is sufficiently large,

$$|H_D^1| \leq |W|^{35}\binom{n}{k'-105} = (k^{5k}n^{3-\frac{1}{34}})^{35}n^{k'-105} \leq k^{165k}n^{k'-1-\frac{1}{34}} < \frac{1}{100k^k}n^{k'-1}. \tag{5.2}$$

To find an upper bound of $|H_D^2|$, note that any k'-set in H_D^2 contains at least one good 3-set because it contains at most 34 disjoint bad 3-sets and $k'-3\cdot 34 \geq 3$. Thus we first bound the number of pairs (A,T) where T is a good 3-set and $A = B\setminus T$ for some $B \in H_D^2$. There are at most $\binom{n}{k'-3}$ ways to choose A. We claim that for fixed A, there are at most $2\binom{n-k+k'-1}{2}$ distinct good 3-sets T such that $A\cup T \in H_D^2$. Otherwise, by Theorem 2.8, there exists four good 3-sets T_1, T_2, T_3, T_4 with

$$A\cup T_i \in H_D^2, T_1\cup T_2 = T_3 \cup T_4 = T', T'\cap (D\cup\{v\}) = \emptyset \text{ and } T_1\cap T_2 = T_3\cap T_4 = \emptyset.$$

Since each $T_i \cup \{v\}$ belongs to at most $\frac{1}{8}\binom{n-k-4}{k'-4}$ sets in $\tilde{H}$, there are at most $\frac{1}{2}\binom{n-k-4}{k-4}$ many $(k-4)$-sets S outside $D\cup A\cup T'\cup\{v\}$ such that $T_i\cup S\cup\{v\} \notin H$. So there exists a $(k-4)$-set S such that $T_i \cup S\cup \{v\} \in H$ for $i\in[4]$ and $S\cap(D\cup A\cup T'\cup\{v\}) = \emptyset$. Then

$$T_1\cup S\cup\{v\}, \dots, T_4\cup S\cup\{v\}, D\cup A\cup T_1, \dots, D\cup A\cup T_4$$

together contain both 3-regular subgraph and 4-regular subgraph of H, a contradiction. Thus for each A, there are at most $2\binom{n-k+k'-1}{2} \leq n^2$ distinct T's with $A\cup T\in H_D^2$. Thus the number of such pairs (A,T) is at most $\binom{n}{k'-3}n^2$.

Let $B \in H_D^2$, then B does not contain a matching of bad sets of size 35, and $k' \geq 4\cdot 35 = 140$. So we apply Theorem 2.3, then we get that the number of bad sets in B is at most $\binom{k'}{3} - \binom{k'-34}{3}$. Thus there are at least $\binom{k'-34}{3}$ good sets in B. Hence each B yields at least $\binom{k'-34}{3}$ distinct pairs (A,T). So

$$|H_D^2| \leq \frac{1}{\binom{k'-34}{3}}\binom{n}{k'-3}n^2. \tag{5.3}$$

Since $(\frac{n}{k'-1})^{k'-1} \le \binom{n}{k'-1}$ and $k \ge 3k'$, we know $\frac{1}{k^k} n^{k'-1} \le \frac{1}{\binom{k'-34}{3}} \binom{n}{k'-3} n^2$ for large enough n. Thus from (5.2) and (5.3),

$$|H_D| = |H_D^1| + |H_D^2| \le \frac{1}{100k^k} n^{k'-1} + \frac{1}{\binom{k'-34}{3}} \binom{n}{k'-3} n^2 \le \frac{1.01}{\binom{k'-34}{3}} \binom{n}{k'-3} n^2$$

for sufficiently large n. Therefore we get

$$\Lambda(H) \le |\tilde{H}| \binom{k-1}{k'-1} \frac{1.01}{\binom{k'-34}{3}} \binom{n}{k'-3} n^2.$$

□

Therefore, from Claim 5.3 and Claim 5.6, we get

$$\frac{1}{r}\binom{k}{k'}\binom{n-k-1}{k'-1}|H^*| \le \Lambda(H) \le |\tilde{H}| \binom{k-1}{k'-1} \frac{1.01}{\binom{k'-34}{3}} \binom{n}{k'-3} n^2$$

Since n is sufficiently large, $\binom{n}{k'-3} n^2 \le 1.01(k'-1)(k'-2)\binom{n-k-1}{k'-1}$ holds. Also we assumed $|H^*| = |\tilde{H}| > 0$, this yields

$$\frac{1}{r}\binom{k}{k'}\binom{n-k-1}{k'-1} \le \binom{k-1}{k'-1} \frac{1.01}{\binom{k'-34}{3}} \binom{n}{k'-3} n^2 \le \binom{k-1}{k'-1} \frac{1.01^2 (k'-1)(k'-2)}{\binom{k'-34}{3}} \binom{n-k-1}{k'-1}.$$

and by dividing $\binom{n-k-1}{k'-1}$ on both sides, we get

$$\frac{1}{r}\binom{k}{k'} \le \binom{k-1}{k'-1} \frac{1.01^2 (k'-1)(k'-2)}{\binom{k'-34}{3}} = \frac{k'}{k}\binom{k}{k'} \frac{1.01^2 \cdot 6(k'-1)(k'-2)}{(k'-34)(k'-35)(k'-36)}$$

From this and the fact that $\frac{1}{r} = \frac{k'}{k}$, we get

$$\frac{(k'-34)(k'-35)(k'-36)}{(k'-1)(k'-2)} \le 1.01^2 \cdot 6.$$

which is a contradiction since $k' \ge 140$.

6. What happens if r is big or $r \nmid k$?

In the same spirit as Theorem 1.2, we propose the following conjecture.

Conjecture 6.1. *For r, there exist k_r, n_k such that for all $k > k_r$, and $n > n_k$, and $r \mid k$, if H is a k-uniform hypergraph with no r-regular subgraphs, then*

$$|H| \le \binom{n-1}{k-1}$$

and equality holds if and only if H is a full k-star.

The proof of Theorem 1.2 does not extend for the case $r > 4$ because the author does not know how to generalize Theorem 2.8 for more pairs of disjoint edges. However, if the following conjecture is true, then we can prove Conjecture 6.1.

Conjecture 6.2. *For every positive integer r, there exist k_r, n_k and $g(r)$ which satisfy the following. For $k \geq k_r, n \geq n_k$, any n-vertex k-uniform hypergraph H contains more than*

$$g(r)\binom{n-1}{k-1}$$

edges contains distinct edges $A_1, B_1, \cdots, A_r, B_r$ so that $A_i \cap B_i = \emptyset$ for all $i = 1, 2 \cdots, r$ and $A_1 \cup B_1 = A_2 \cup B_2 = \cdots = A_r \cup B_r$.

Note that this conjecture is known to be true for $r = 1, 2$. For $r = 1$, it's Erdős-Ko-Rado Theorem. For $r = 2$ Füredi [9] proved $k_2 = 3, g(2) \leq \frac{7}{2}$ and later Pikhurko and Verstraëte [13] improved it to $g(2) \leq \frac{7}{4}$.

In Theorem 1.2, we assume k is much bigger than r and $r \mid k$. What happens if the conditions do not hold? First, let's see what happens if k is not big enough in terms of r. The author believes that full k-star might be the only extremal example even when $k \geq 2r$ and $r \mid k$. However if $r = k$ then the extremal example is no longer only full k-star. Also, if $r > k$, then $|H|$ can be bigger than $\binom{n-1}{k-1}$. It is straightforward to check the following example.

Example 6.3. *Take an n-vertex full k-star H. We take a non-edge e of H, and an edge e' of H such that $|e \cap e'| = k-1$. Then $H \setminus \{e'\} \cup \{e\}$ does not have r-regular subgraphs when $r = k$, and $H \cup \{e\}$ does not have r-regular subgraphs when $r = k+1$.*

As an example, if r is bigger than k, even $r = k+1$ does not imply $|H| \leq \binom{n-1}{k-1}$ any more. However, as we can see in Section 3, $|H| \leq (1+o(1))\binom{n-1}{k-1}$ still holds if $k \geq 2^{\lceil \log r \rceil - 1} + 2$. Thus, for $r = 2^l$ and $k \geq \frac{r}{2} + 3 = 2^{l-1} + 2$, the asymptotics of the number of edges in hypergraphs with no r-regular subgraphs is still $(1+o(1))\binom{n-1}{k-1}$ even though $r \geq k$. However, the following example shows that this becomes false if r is much bigger.

Example 6.4. *For an integer $c > 1$, take an n-vertex k-uniform hypergraph H such that $E(H) = \{e : e \in \binom{V(H)}{k}, |e \cap \{x_1, x_2, \cdots, x_c\}| = 1\}$. Then $|H| = c\binom{n-c}{k-1} \sim c\binom{n-1}{k-1}$. However, H does not contain any r-regular subgraph when $r > c\binom{c(k-1)}{k-2}$.*

Proof. Suppose H contains an r-regular subgraph R, then R must cover some vertices in $\{x_1, x_2, \cdots, x_c\}$. Assume it covers $\{x_1, x_2, \cdots, x_{c'}\}$. Since it must cover those vertices exactly r-times, $|R| = c'r$. Then $V(R) = \frac{k|R|}{r} = c'k$. Then a vertex x in $V(R)$ can be covered only by edges e with $|e \cap (V(R) \setminus \{x_1, \cdots, x_c\}| = k-1$. So, degree of x is at most $c'\binom{c'(k-1)}{k-2} < r$, a contradiction. □

Hence, it is natural to ask the following question. Note that, such $r(k)$ must exist and $k \leq r(k) \leq 2\binom{2k-2}{k-2} + 1$ by Theorem 3.1 and Example 6.4.

Question 6.5. *What is the minimum $r = r(k)$ such that*

$$\limsup_{n \to \infty} \frac{\max |H|}{\binom{n-1}{k-1}} > 1$$

where the maximum is taken over all n-vertex k-uniform hypergraphs with no r-regular subraphs.

Now we consider the case where r does not divide k while k is bigger than r. In [12], Mubayi and Verstraëte conjectured the following.

Conjecture 6.6. [12] *For every integer k with $2 \nmid k$, there exists an integer n_k such that for $n \geq n_k$, if H is an n-vertex k-uniform hypergraph with no 2-regular subgraphs then $|H| \leq \binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$. Equality holds if and only if H is a full k-star together with a maximal matching disjoint from the full k-star.*

In the same spirit, we may add more edges to full k-star when $r \geq 3$, $r < k$, $r \nmid k$. In order to construct an example, we need the following concept.

In 1973, Brown, Erdős and Sós [2] proposed a study for a new parameter, $f_k(n, a, b)$, the largest number of edges in a k-uniform hypergraph on n vertices that contains no b edges spanned by a vertices. Determining $f_k(n, a, b)$ for general (k, a, b) is very difficult. Note that finding value of $f_3(n, 6, 3)$ is known as the famous $(6, 3)$-problem. In [2], they showed the following.

Theorem 6.7. [2] *If $a > k$ and $b > 1$, then $f_k(n, a, b) > c_{a,b} n^{\frac{kb-a}{b-1}}$.*

Now we consider the following construction.

Construction 6.8. *Let $k = k'd, r = r'd$ be integers with $k \geq 3, r' \geq 3$ such that k' and r' are relatively prime. Consider a $(k-1)$-uniform $(n-2)$-vertex hypergraph H' with $f_{k-1}(n-2, 2k-2, r')$ edges such that H' does not contain any r' edges spanning at most $2k-2$ vertices. Especially, $|H'|$ contains at least $c_{k,r'} n^{\frac{r'-2}{r'-1}(k-1)}$ edges.*

Now we consider two vertices x, y disjoint from $V(H')$ and the hypergraph $H_{k,r}$ with

$$V(H_{k,r}) = V(H') \cup \{x, y\},$$

$$E(H_{k,r}) = \{e \in \binom{V(H)}{k} : x \in e\} \cup \{e \cup \{y\} : e \in E(H')\}.$$

Then $H_{k,r}$ contains at least $\binom{n-1}{k-1} + c_{k,r} n^{\frac{r'-2}{r'-1}(k-1)}$ edges, and $H_{k,r}$ contains no r-regular subgraphs.

Proof. Assume that $H_{k,r}$ contains an r-regular subgraph R. Let H_x be the full k-star in $H_{k,r}$ and H^* be the hypergraph consisting edges not containing x. Since both H_x and H^* are subgraphs of two distinct full k-star, each of them does not contain any r-regular subgraph. Thus R must intersect both H_x and H^*, thus R must cover both x and y. Since R covers x exactly r times,

$$|R| = |R \cap H_x| + |R \cap H^*| = r + |R \cap H^*| \geq r + 1.$$

However, because R induces an r-regular subgraph,

$$r|V(R)| = k|R| = kr + k|R \cap H^*|.$$

Since k', r' are relatively prime, $|R \cap H^*|$ must be a multiple of r', moreover $|R \cap H^*| \leq r$ because y must be covered exactly r-times. Hence $|V(R)| = \frac{k|R|}{r} \leq 2k$. Now we consider $\{e - y : e \in R \cap H^*\}$. It is a set of at least r' edges of H' covering at most $2k-2$ vertices, a subset of $V(R) - \{x, y\}$. It is a contradiction to the definition of H'. Thus $H_{k,r}$ does not contain any r-regular subgraph. □

Hence, there is an n-vertex k-uniform hypergraph H with no r-regular subgraphs which contains quite more edges than $\binom{n-1}{k-1}$ if r does not divide k. Hence we propose the following question.

Question 6.9. *Determine the least value of $h(k,r)$ such that there exists a constant $c_{k,r}$ so that every n-vertex k-uniform hypergraph H with no r-regular subgraphs satisfies*

$$|H| \leq \binom{n-1}{k-1} + c_{k,r} n^{h(k,r)}.$$

The author suspects that $h(k,r)$ is related to the value of $gcd(k,r)$ based on the fact that the value we get from Construction 6.8 is related to k, r, and $gcd(k,r)$.

Also, considering linear hypergraphs is another direction of studying regular subgraphs. The following question is proposed in [3].

Question 6.10. [3] *For an integer r, let $f_{k,r}(n)$ be the maximum number of edges in a linear n-vertex k-uniform hypergraphs with no r-regular subgraphs. Is $f_{3,3}(n) = o(n^2)$?*

Especially, authors of [3] asked if sufficiently large Steiner triple system contains a 3-regular subgraph. In [16], Verstraëte observed that Lemma 2.7 together with the fact that all linear k-uniform hypergraphs have maximum degree at most $\frac{n-1}{k-1}$ trivially imply the following.

Corollary 6.11. *For any integers $k, r \geq 3$ and sufficiently large n,*

$$f_{k,r}(n) < 6n^2(\log\log n)^{-\frac{1}{2(k-1)}}.$$

Thus this answers Question 6.10 and it implies that for an integer r, every n-vertex Steiner system contains an r-regular subgraph if n is sufficiently large.

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