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Note

The difference and ratio of the fractional matching number and the matching number of graphs

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ABSTRACT

Given a graph G, the *matching number* of G, written $\alpha'(G)$, is the maximum size of a matching in G, and the *fractional matching number* of G, written $\alpha'_f(G)$, is the maximum size of a fractional matching of G. In this paper, we prove that if G is an n-vertex connected graph that is neither K_1 nor K_3, then $\alpha'_f(G) - \alpha'(G) \le \frac{n-2}{6}$ and $\frac{\alpha'_f(G)}{\alpha'(G)} \le \frac{3n}{2n+2}$. Both inequalities are sharp, and we characterize the infinite family of graphs where equalities hold.

1. Introduction

For undefined terms, see [5]. Throughout this paper, n will always denote the number of vertices of a given graph. A *matching* in a graph is a set of pairwise disjoint edges. A *perfect matching* in a graph G is a matching in which each vertex has an incident edge in the matching; its size must be $n/2$, where $n = |V(G)|$. A *fractional matching* of G is a function $\phi : E(G) \to [0, 1]$ such that for each vertex v, $\sum_{e\in\Gamma(v)} \phi(e) \le 1$, where $\Gamma(v)$ is the set of edges incident to v, and the *size of a fractional matching* ϕ is $\sum_{e\in E(G)} \phi(e)$. Given a graph G, the *matching number* of G, written $\alpha'(G)$, is the maximum size of a matching in G, and the *fractional matching number* of G, written $\alpha'_f(G)$, is the maximum size of a fractional matching of G.

Given a fractional matching ϕ, since $\sum_{e\in\Gamma(v)} \phi(e) \le 1$ for each vertex v, we have that $2\sum_{e\in E(G)} \phi(e) \le n$, which implies $\alpha'_f(G) \le n/2$. By viewing every matching as a fractional matching it follows that $\alpha'_f(G) \ge \alpha'(G)$ for every graph G, but equality need not hold. For example, the fractional matching number of a k-regular graph equals $n/2$ by setting weight $1/k$ on each edge, but the matching number of a k-regular graph can be much smaller than $n/2$. Thus it is a natural question to find the largest difference between $\alpha'_f(G)$ and $\alpha'(G)$ in a (connected) graph.

In Sections 3 and 4, we prove tight upper bounds on $\alpha'_f(G) - \alpha'(G)$ and $\frac{\alpha'_f(G)}{\alpha'(G)}$, respectively, for an n-vertex connected graph G, and we characterize the infinite family of graphs achieving equality for both results. As corollaries of both results, we have upper bounds on both $\alpha'_f(G) - \alpha'(G)$ and $\frac{\alpha'_f(G)}{\alpha'(G)}$ for an n-vertex graph G, and we characterize the graphs achieving equality for both bounds.

Our proofs use the famous Berge–Tutte Formula [1] for the matching number as well as its fractional analogue. We also use the fact that there is a fractional matching ϕ for which $\sum_{e\in E(G)} \phi(e) = \alpha'_f(G)$ such that $f(e) \in \{0, 1/2, 1\}$ for every edge e, and some refinements of the fact. We can prove both Theorems 6 and 8 with two different techniques, and for the sake of the readers we demonstrate each method in the proofs of Theorems 6 and 8.

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2. Tools

In this section, we introduce the tools we used to prove the main results. To prove Theorem 6, we use Theorems 1 and 2. For a graph H, let $o(H)$ denote the number of components of H with an odd number of vertices. Given a graph G and $S \subseteq V(G)$, define the *deficiency* $\mathrm{def}(S)$ by $\mathrm{def}(S) = o(G - S) - |S|$, and let $\mathrm{def}(G) = \max_{S \subseteq V(G)} \mathrm{def}(S)$. Theorem 1 is the famous Berge–Tutte formula, which is a general version of Tutte's 1-factor Theorem [4].

Theorem 1 *([1]). For any n-vertex graph G,* $\alpha'(G) = \frac{1}{2}(n - \mathrm{def}(G))$.

For the fractional analogue of the Berge–Tutte formula, let $i(H)$ denote the number of isolated vertices in H. Given a graph G and $S \subseteq V(G)$, let $\mathrm{def}_f(S) = i(G - S) - |S|$ and $\mathrm{def}_f(G) = \max_{S \subseteq V(G)} \mathrm{def}_f(S)$. Theorem 2 is the fractional version of the Berge–Tutte Formula. This is also the fractional analogue of Tutte's 1-Factor Theorem saying that G has a fractional perfect matching if and only if $i(G - S) \le |S|$ for all $S \subseteq V(G)$ (implicit in Pulleyblank [2]), where a fractional perfect matching is a fractional matching f such that $2\sum_{e \in E(G)} f(e) = n$.

Theorem 2 *([3] See Theorem 2.2.6). For any n-vertex graph G,* $\alpha'_f(G) = \frac{1}{2}(n - \mathrm{def}_f(G))$.

When we characterize the equalities in the bounds of Theorems 6 and 8, we need the following proposition. Recall that $G[S]$ is the graph induced by a subset of the vertex set S.

Proposition 3 *([3] See Proposition 2.2.2). The following are equivalent for a graph G.*

(a) *G has a fractional perfect matching.*

(b) *There is a partition* $\{V_1, \ldots, V_n\}$ *of the vertex set* $V(G)$ *such that, for each i, the graph* $G[V_i]$ *is either* K_2 *or Hamiltonian.*

(c) *There is a partition* $\{V_1, \ldots, V_n\}$ *of the vertex set* $V(G)$ *such that, for each i, the graph* $G[V_i]$ *is either* K_2 *or Hamiltonian graph on an odd number of vertices.*

Theorem 4 and Observation 5 are used to prove Theorem 8.

Theorem 4 *([3] See Theorem 2.1.5). For any graph G, there is a fractional matching f for which*

$$\sum_{e \in E(G)} f(e) = \alpha'_f(G)$$

such that $f(e) \in \{0, 1/2, 1\}$ *for every edge e.*

Given a fractional matching f, an *unweighted* vertex v is a vertex with $\sum_{e \in \Gamma(v)} f(e) = 0$, and a *full* vertex v is a vertex with $f(vw) = 1$ for some vertex w. Note that w is also a full vertex. An *i-edge* e is an edge with $f(e) = i$. Note that the existence of an 1-edge guarantees the existence of two full vertices. A vertex subset S of a graph G is *independent* if $E(G[S]) = \emptyset$, where $G[S]$ is the graph induced by S.

Observation 5. *Among all the fractional matchings of an n-vertex graph G satisfying the conditions of Theorem 4, let f be a fractional matching with the greatest number of edges e with* $f(e) = 1$. *Then we have the following:*

(a) *The graph induced by the* $\frac{1}{2}$*-edges is the union of odd cycles. Furthermore, if C and C′ are two disjoint cycles in the graph induced by* $\frac{1}{2}$*-edges, then there is no edge uu′ such that* $u \in V(C)$ *and* $u' \in V(C')$.

(b) *The set S of the unweighted vertices is independent. Furthermore, every unweighted vertex is adjacent only to a full vertex.*

(c) $\alpha'(G) \ge w_1 + \sum_{i=1}^{\infty} i c_i$, $\alpha'_f(G) = w_1 + \sum_{i=1}^{\infty} (\frac{2i+1}{2}) c_i$, *and* $n = w_0 + 2w_1 + \sum_{i=1}^{\infty} (2i+1) c_i$, *where* w_0, w_1, *and* c_i *are the number of unweighted vertices, the number of 1-edges, and the number of odd cycles of length* $2i + 1$ *in the graph induced by* $\frac{1}{2}$*-edges in G, respectively.*

Proof. (a) The graph induced by the $\frac{1}{2}$-edges cannot have a vertex with degree at least 3 since $\sum_{e \in \Gamma(v)} f(e) \le 1$ for each vertex v. Thus the graph must be a disjoint union of paths or cycles. If the graph contains a path or an even cycle, then by replacing weight 1/2 on each edge on the path or the even cycle with weight 1 and 0 alternatively, we can have a fractional matching with the same fractional matching number and more edges with weight 1, which contradicts the choice of f. Thus the graph induced by the $\frac{1}{2}$-edges is the union of odd cycles. If there is an edge uv such that $u \in V(C)$ and $v \in V(C')$, where C and C' are two different odd cycles induced by some $\frac{1}{2}$-edges, then $f(uv) = 0$, since $\sum_{e \in \Gamma(x)} f(e) \le 1$ for each vertex x. By replacing weights 0 and 1/2 on the edge uv and the edges on C and C' with weight 1 on uv, and 0 and 1 on the edges in $E(C)$ and $E(C')$ alternatively, not violating the definition of a fractional matching, we have a fractional matching with the same fractional matching number with more edges with weight 1, which is a contradiction. Thus we have the desired result.

(b) If two unweighted vertices u and v are adjacent, then we can put a positive weight on the edge uv, which contradicts the choice of f. If there exists an unweighted vertex x, which is not incident to any full vertex, then x must be adjacent to a vertex y such that $f(yy_1) = 1/2$ and $f(yy_2) = 1/2$ for some vertices y_1 and y_2. By replacing the weights 0, 1/2, and 1/2 on xy, yy_1, and yy_2 with 1, 0, 0, respectively, we have a fractional matching with the same fractional matching number with more edges with weight 1, which is a contradiction.

(c) By the definitions of w_0, w_1, and c_i, we have the desired result. □

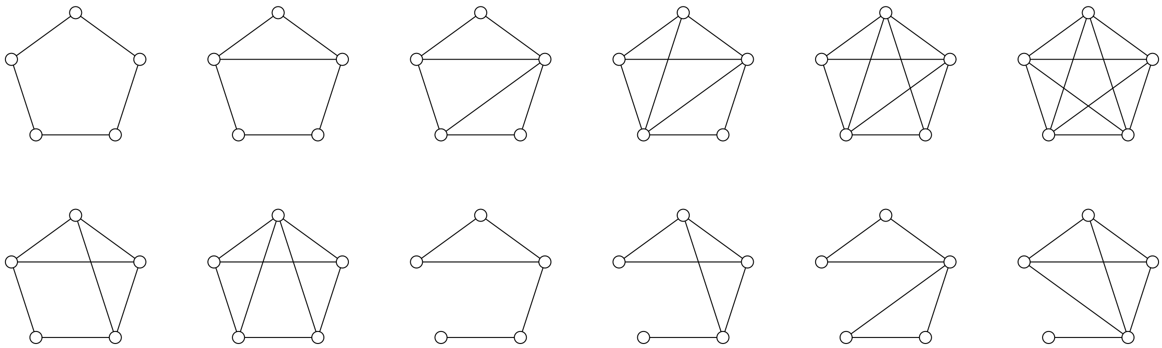

Fig. 1. All 5-vertex graphs in Theorems 6(i) and 8(i).

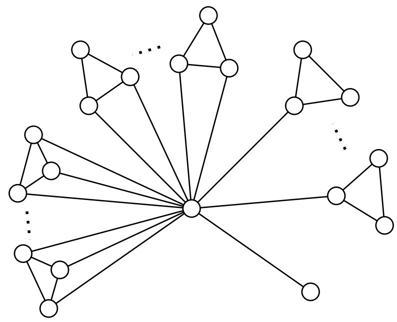

Fig. 2. All graphs in Theorems 6(ii) and 8(ii).

3. Sharp upper bound for $\alpha'_f(G) - \alpha'(G)$

What are the structures of the graphs having the maximum difference between the fractional matching number and the matching number in an n-vertex connected graph? The graphs may have big fractional matching number and small matching number. So, by the Berge–Tutte Formula and its fractional version, they may have a vertex subset S such that almost all of the odd components of $G - S$ have at least three vertices in order to get S to have small fractional deficiency and big deficiency. This is our idea behind the proof of Theorem 6.

Theorem 6. *For $n \geq 5$, if G is a connected graph with n vertices, then $\alpha'_f(G) - \alpha'(G) \leq \frac{n-2}{6}$, and equality holds only when either*
(i) *$n = 5$ and either C_5 is subgraph of G or $K_2 + K_3$ is a subgraph of G (see Fig. 1), or*
(ii) *G has a vertex v such that the components of $G - v$ are all K_3 except one single vertex (see Fig. 2).*

Proof. Among all the vertex subsets with maximum deficiency, let S be the largest set. By the Berge–Tutte Formula, $\alpha'(G) = \frac{1}{2}(n - \mathrm{def}(S))$, and by the choice of S, all components of $G - S$ have an odd number of vertices. Let x be the number of isolated vertices of $G - S$, and let y be the number of other components of $G - S$. This implies $n \geq |S| + x + 3y$. If $S = \emptyset$, then $\alpha'(G) \in \{\frac{n}{2}, \frac{n-1}{2}\}$, depending on the parity of n. In this case, $\alpha'_f(G) - \alpha'(G) \leq \frac{n}{2} - \frac{n-1}{2} = \frac{1}{2} \leq \frac{n-2}{6}$, since $n \geq 5$. Now, assume that S is non-empty.

Case 1: $x = 0$. Since $\mathrm{def}_f(G) \geq 0$, $|S| \geq 1$, and $n \geq |S| + 3y$, we have

$$\begin{aligned}\alpha'_f(G) - \alpha'(G) &= \frac{1}{2}(n - \mathrm{def}_f(G)) - \frac{1}{2}(n - \mathrm{def}(S)) = \frac{1}{2}(\mathrm{def}(S) - \mathrm{def}_f(G)) \\ &\leq \frac{1}{2}(y - |S| - 0) \leq \frac{1}{2}\left(\frac{n - |S|}{3} - |S|\right) = \frac{n - 4|S|}{6} \leq \frac{n-4}{6} < \frac{n-2}{6}.\end{aligned}$$

Case 2: $x \geq 1$. Since $n \geq |S| + x + 3y$, $|S| \geq 1$, and $x \geq 1$, we have

$$\begin{aligned}\alpha'_f(G) - \alpha'(G) &= \frac{1}{2}(n - \mathrm{def}_f(G)) - \frac{1}{2}(n - \mathrm{def}(S)) = \frac{1}{2}(\mathrm{def}(S) - \mathrm{def}_f(G)) \\ &\leq \frac{1}{2}\left(x + y - |S| - (x - |S|)\right) \leq \frac{y}{2} = \frac{n - x - |S|}{6} \leq \frac{n-2}{6}.\end{aligned}$$

Equality in the bound requires equality in each step of the computation. When $n = 5$, we conclude that (i) follows by Proposition 3. In Case 1, we cannot have equality, and in Case 2, we have $|S| = 1$, $x = 1$, and $n = |S| + x + 3y = 2 + 3y$.

Since G is connected, the components of $G-S$ are P_3 or K_3 except only one single vertex. If a component of $G-S$ is a copy of P_3, then by choosing the central vertex u of the path, we have $\text{def}(S\cup\{u\}) = o(G-(S\cup\{u\})) - |S\cup\{u\}| = o(G-S)-|S|$, yet $|S\cup\{u\}| > |S|$, which contradict the choice of S. Thus we have the desired result. $\square$

Corollary 7. *For any n-vertex graph G, we have* $\alpha'_f(G)-\alpha'(G) \le \frac{n}{6}$*, and equality holds only when G is the disjoint union of copies of* K_3*.*

Proof. First, we show that if $n \le 4$ and G is connected, then $\alpha'_f(G)-\alpha'(G) \le \frac{n}{6}$, and equality holds only when $G = K_3$. If $n \le 2$, then $G \in \{K_1, K_2\}$, which implies that $\alpha'_f(G)-\alpha'(G) = 0 < n/6$. If $n = 3$, then $G \in \{P_3, K_3\}$. Note that $\alpha'_f(P_3)-\alpha'(P_3) = 1-1 = 0 < 3/6$ and $\alpha'_f(K_3)-\alpha'(K_3) = 3/2-1 = 1/2 \le 3/6$. Furthermore, equality holds only when $G = K_3$. If $n = 4$, then either $G = K_{1,3}$ or G contains P_4 as a subgraph. Since $\alpha'_f(K_{1,3})-\alpha'(K_{1,3}) = 1-1 = 0 < 4/6$ and $\alpha'_f(P_4)-\alpha'(P_4) = 2-2 = 0 < 4/6$, we conclude that for any positive integer n, $\alpha'_f(G)-\alpha'(G) \le \frac{n}{6}$. In fact, if $n \ge 5$, then by Theorem 6, the difference must be at most $\frac{n-2}{6}$. Thus, for connected graphs, equality holds only when $G = K_3$.

Now, if we assume that G is disconnected, then G is the disjoint union of connected graphs $G_1,\dots,G_k$. Let $|V(G_i)| = n_i$ for $i \in [k]$. Since

$$\begin{aligned}\alpha'_f(G)-\alpha'(G) &= \left[\alpha'_f(G_1)+\cdots+\alpha'_f(G_k)\right] - \left[\alpha'(G_1)+\cdots+\alpha'(G_k)\right]\\ &= [\alpha'_f(G_1)-\alpha'(G_1)]+\cdots+[\alpha'_f(G_k)-\alpha'(G_k)] \le \frac{n_1}{6}+\cdots+\frac{n_k}{6} = \frac{n}{6},\end{aligned}$$

equality holds only when each G_i is a copy of K_3 for $i \in [k]$. $\square$

4. Sharp upper bound for $\frac{\alpha'_f(G)}{\alpha'(G)}$

To prove the upper bound of Theorem 8, we still can use the Berge–Tutte formula and its fractional analogue. However, we provide an alternative way to prove the theorem.

Theorem 8. *For* $n \ge 5$*, if G is a connected graph with n vertices, then* $\frac{\alpha'_f(G)}{\alpha'(G)} \le \frac{3n}{2n+2}$*, and equality holds only when either*
(i) $n = 5$ *and either* C_5 *is a subgraph of G or* K_2+K_3 *is a subgraph of G (see Fig. 1), or*
(ii) *G has a vertex v such that the components of* $G-v$ *are all* K_3 *except one single vertex (see Fig. 2).*

Proof. Among all the fractional matchings of an n-vertex graph G with the size equal to $\alpha'_f(G)$, let f be a fractional matching such that the number of edges e with $f(e) = 1$ is maximized. We follow the notation in Observation 5.

Case 1: $w_0 = w_1 = 0$. Since G is connected and $n \ge 5$, there exists only one i such that $i \ge 2$ and c_i is not zero, and $\alpha'(G) = ic_i \ne 0$. Then we have

$$\frac{\alpha'_f(G)}{\alpha'(G)} \le \frac{\left(\frac{2i+1}{2}\right)c_i}{ic_i} = 1+\frac{1}{2i} \le \frac{5}{4}.$$

Case 2: $w_0 \ge 1$ *and* $w_1 = 0$. By part (b) of Observation 5, this cannot happen.

Case 3: $w_0 = 0$ *and* $w_1 \ge 1$. Since $\sum_{i=1}^{\infty} c_i \le \frac{n-2w_1}{3}$, by part (c) of Observation 5, we have

$$\frac{\alpha'_f(G)}{\alpha'(G)} \le \frac{w_1+\sum\limits_{i=1}^{\infty}\left(\frac{2i+1}{2}\right)c_i}{w_1+\sum\limits_{i=1}^{\infty} ic_i} = \frac{\frac{n-w_0}{2}}{\frac{n-w_0-\sum\limits_{i=1}^{\infty} c_i}{2}} = \frac{n}{n-\sum\limits_{i=1}^{\infty} c_i} \le \frac{n}{n-\frac{n-2w_1}{3}} = \frac{3n}{2n+2w_1} \le \frac{3n}{2n+2}.$$

Case 4: $w_0 \ge 1$ *and* $w_1 \ge 1$. Since $\sum_{i=1}^{\infty} c_i \le \frac{n-2w_1-w_0}{3}$, by part (c) of Observation 5, we have

$$\frac{\alpha'_f(G)}{\alpha'(G)} \le \frac{\frac{n-w_0}{2}}{\frac{n-w_0-\sum\limits_{i=1}^{\infty} c_i}{2}} \le \frac{n-w_0}{n-w_0-\frac{n-2w_1-w_0}{3}} = \frac{3(n-w_0)}{2(n+w_1-w_0)} < \frac{3n}{2(n+w_1)} \le \frac{3n}{2(n+1)}.$$

Equality in the bound requires equality in each step of the computation; we only need to check Case 1 and Case 2. In Case 1, we have $i = 2$, which means that $n = 5$ and G contains a copy of C_5. In Case 3, we have $w_1 = 1$ and $\sum_{i=1}^{\infty} c_i = \frac{n-2}{3}$, which means that the graph induced by the $\frac{1}{2}$-edges is the union of K_3. Thus G has K_2+kK_3 as a subgraph for some positive integer k. Note that there is an edge between the copy of K_2 and any copy of K_3 by part (b) of Observation 5. Also, there are no edges between any pair of two triangles by part (a) of Observation 5. Let u and v be the two vertices corresponding to the copy of K_2. If there are two different triangles C and C' in G such that u and v are incident to C and C', respectively, then we have $\alpha'(G) > w_1+c_1$, which implies that we cannot have equality in the first inequality in Case 3. Thus, we conclude that G contains a copy of either K_2+K_3 as a subgraph or a vertex v such that the components of $G-v$ are all K_3 except only one single vertex. $\square$

Corollary 9. *For any n-vertex graph G with at least one edge, we have* $\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{3}{2}$*, and equality holds only when G is the disjoint union of copies of* K_3.

Proof. By the proof of Corollary 7, if $n \leq 4$ and G is connected, then $\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{3}{2}$, and equality holds only when $G = K_3$. If we assume that G is disconnected, then G is the disjoint union of connected graphs $G_1, \ldots, G_k$. Let $|V(G_i)| = n_i$ for $i \in [k]$. Without loss of generality, we may assume that $\frac{\alpha'_f(G_1)}{\alpha'(G_1)} \geq \frac{\alpha'_f(G_i)}{\alpha'(G_i)}$ for all $i \in [k]$. Then we have

$$\frac{\alpha'_f(G)}{\alpha'(G)} = \frac{\alpha'_f(G_1) + \cdots + \alpha'_f(G_k)}{\alpha'(G_1) + \cdots + \alpha'(G_k)} \leq \frac{\alpha'_f(G_1)}{\alpha'(G_1)} \leq \frac{3}{2},$$

and equality holds only when each G_i is a copy of K_3 for $i \in [k]$. □

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