# On $r$-dynamic Coloring of Graphs

Sogol Jahanbekam*, Jaehoon Kim†, Suil O‡, Douglas B. West§

Last revised July, 2015

**Abstract**

An *$r$-dynamic proper $k$-coloring* of a graph $G$ is a proper $k$-coloring of $G$ such that every vertex in $V(G)$ has neighbors in at least $\min\{d(v), r\}$ different color classes. The *$r$-dynamic chromatic number* of a graph $G$, written $\chi_r(G)$, is the least $k$ such that $G$ has such a coloring. By a greedy coloring algorithm, $\chi_r(G) \le r\Delta(G) + 1$; we prove that equality holds for $\Delta(G) > 2$ if and only if $G$ is $r$-regular with diameter 2 and girth 5. We improve the bound to $\chi_r(G) \le \Delta(G) + 2r - 2$ when $\delta(G) > 2r \ln n$ and $\chi_r(G) \le \Delta(G) + r$ when $\delta(G) > r^2 \ln n$.

In terms of the chromatic number, we prove $\chi_r(G) \le r\chi(G)$ when $G$ is $k$-regular with $k \ge (3 + o(1))r \ln r$ and show that $\chi_r(G)$ may exceed $r^{1.377}\chi(G)$ when $k = r$. We prove $\chi_2(G) \le \chi(G) + 2$ when $G$ has diameter 2, with equality only for complete bipartite graphs and the 5-cycle. Also, $\chi_2(G) \le 3\chi(G)$ when $G$ has diameter 3, which is sharp. However, $\chi_2$ is unbounded on bipartite graphs with diameter 4, and $\chi_3$ is unbounded on bipartite graphs with diameter 3 or 3-colorable graphs with diameter 2. Finally, we study $\chi_r$ on grids and toroidal grids.

## 1 Introduction

A *$k$-coloring* of a graph $G$ is a map $c\colon V(G) \to S$, where $|S| = k$; it is *proper* if adjacent vertices receive different labels. An *$r$-dynamic $k$-coloring* is a proper $k$-coloring $c$ of $G$ such that on each vertex neighborhood $N(v)$ at least $\min\{r, d(v)\}$ colors are used. The *$r$-dynamic chromatic number*, introduced by Montgomery [16] and written as $\chi_r(G)$, is the minimum $k$ such that $G$ has an $r$-dynamic $k$-coloring.

*University of Colorado Denver, Denver, CO, sogol.jahanbekam@ucdenver.edu.

†University of Birmingham, Edgbaston, Birmingham B15 2TT, UK, kimJS@bham.ac.uk. Research partially supported by the Arnold O. Beckman Research Award of the University of Illinois at Urbana-Champaign and by the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013) / ERC Grant Agreements no. and 306349 (J. Kim)

‡Georgia State University, Atlanta, GA, 30303, suilo@gsu.edu.

§Zhejiang Normal University, Jinhua, China 321004, and University of Illinois, Urbana, IL 61801, dwest@math.uiuc.edu. Research supported by Recruitment Program of Foreign Experts, 1000 Talent Plan, State Administration of Foreign Experts Affairs, China.

The 1-dynamic chromatic number of a graph $G$ is its chromatic number $\chi(G)$. The 2-dynamic chromatic number was introduced as *dynamic chromatic number* by Montgomery [16]; he conjectured $\chi_2(G) \leq \chi(G)+2$ when $G$ is regular, which remains open. Alishahi [4] showed that for all $k$ there is a $k$-chromatic regular graph $G$ with $\chi_2(G) \geq \chi(G)+1$. Akbari et al. [1] proved Montgomery's conjecture for bipartite regular graphs. Lai, Montgomery, and Poon [11] proved $\chi_2(G) \leq \Delta(G)+1$ for $\Delta(G) \geq 3$ when no component is the 5-cycle $C_5$.

Akbari et al. [2] strengthened this to the list context: $\mathrm{ch}_2(G) \leq \Delta(G)+1$ under the same conditions, where $\mathrm{ch}_r(G)$ is the least $k$ such that an $r$-dynamic coloring can be chosen from any lists of size $k$ assigned to the vertices. Kim and Park [9] proved $\mathrm{ch}_2(G) \leq 4$ for planar $G$ with girth at least 7, and $\mathrm{ch}_2(G) \leq k$ when $k \geq 4$ and $G$ has maximum average degree at most $\frac{4k}{k+2}$ (both results are sharp). Kim, Lee, and Park [10] proved $\chi_2(G) \leq 4$ when $G$ is planar and no component is $C_5$; also, $\mathrm{ch}_2(G) \leq 5$ whenever $G$ is planar. Loeb, Mahoney, Reiniger and Wise [12] proved $\mathrm{ch}_3 \leq 10$ for planar and toroidal graphs and in general gave an upper bound on $\mathrm{ch}_r$ for graphs with genus $g$.

Given a graph $G$, form $G^2$ by adding edges joining nonadjacent vertices having a common neighbor in $G$. One motivation for the study of $r$-dynamic chromatic number is that it provides a spectrum of parameters between $\chi(G)$ and $\chi(G^2)$.

**Observation 1.1.** *Always* $\chi(G) = \chi_1(G) \leq \cdots \leq \chi_{\Delta(G)}(G) = \chi(G^2)$. *If* $r \geq \Delta(G)$, *then* $\chi_r(G) = \chi_{\Delta(G)}(G)$.

**Observation 1.2.** $\chi_r(G) \geq \min\{\Delta(G), r\}+1$ *(equality holds for trees).*

We begin in Section 2 with an analogue of Brooks' Theorem [7]: we prove $\chi_r(G) \leq r\Delta(G)+1$ for $r \geq 2$ and characterize the graphs achieving equality. The characterization does not reduce to Brooks' Theorem when $r=1$; for $r \geq 2$, equality holds if and only if $G$ is an $r$-regular graph with diameter 2 and girth 5. Such graphs, known as *Moore graphs*, are quite rare; they exist only when $r$ is 2, 3, 7, and possibly 57.

When the minimum degree is not too small, we can greatly improve the upper bound $r\Delta(G)$. For an $n$-vertex graph $G$, we show that $\delta(G) > \frac{r+s}{s+1} r \ln n$ implies $\chi_r(G) \leq \Delta(G)+r+s$. In particular, $\delta(G) > 2r \ln n$ implies $\chi_r(G) \leq \Delta(G)+2r-2$ (setting $s=r-2$), and $\delta(G) > r^2 \ln n$ implies $\chi_r(G) \leq \Delta(G)+r$ (setting $s=0$).

In Section 3, we study bounds on $\chi_r$ for $k$-regular graphs in terms of the chromatic number, motivated by Montgomery's conjecture. Akbari et al. [1] proved $\chi_2(G) \leq 2\chi(G)$ for every $k$-regular graph $G$. Alishahi [3] proved $\chi_2(G) \leq \chi(G)+14.06 \ln k+1$ (and later $\chi_2(G) \leq \chi(G)+2\lceil 4 \ln k+1 \rceil$ [4]) for $k$-regular graphs without 4-cycles (when $k \geq 35$), which Taherkhani [18] improved to $\chi_2(G) \leq \chi(G)+\lceil 5.437 \ln k+2.721 \rceil$. For general $r$, we prove $\chi_r(G) \leq r\chi(G)$ for $k$-regular graphs $G$ with $k \geq (3+o(1))r \ln r$. The thesis of the third author [17] contains the same conclusion when $k \geq 7r \ln r$, and later Taherkhani [18] obtained a similar result by essentially the same method as ours. When $k$ is not sufficiently large in terms of $r$, the ratio $\chi_r(G)/\chi(G)$ can grow superlinearly in $r$; we provide an example using Kneser graphs where $k = r$ and $\chi_r(G) > r^{1.37744}\chi(G)$.

In Section 4 we consider $k$-chromatic graphs with small diameter (the *diameter* of $G$, written $\mathrm{diam}(G)$, is the maximum distance between any two vertices of $G$). We first give a short proof of $\chi_2(G) \le \chi(G) + 2$ when $\mathrm{diam}(G) = 2$. Furthermore, equality holds only for complete bipartite graphs and $C_5$. For regular graphs with $\chi(G) \ge 4$, Alishahi [4] proved $\chi_2(G) \le \chi(G) + \alpha(G^2)$, where $\alpha(G)$ is the maximum size of an independent set of vertices in $G$; note that $\alpha(G^2) = 1$ when $\mathrm{diam}(G) = 2$.

We also prove $\chi_2(G) \le 3\chi(G)$ when $\mathrm{diam}(G) \le 3$, and this is sharp. However, $\chi_2(G)$ is unbounded when $\mathrm{diam}(G) = 4$ even for $\chi(G) = 2$; form $G$ by subdividing every edge of the complete graph $K_n$. Also, $\chi_3$ is unbounded with $\mathrm{diam}(G) = 2$ and $\chi(G) = 2$.

Finally, in Section 5 we study $\chi_r$ on grids and toroidal grids, providing some partial results. For example, consider the $m$-by-$n$ grid graph $P_{m,n}$ with $m, n \ge 3$ (the vertex set is $\{0, \dots, m-1\} \times \{0, \dots, n-1\}$, with $(a, b)$ adjacent to $(a', b')$ when $|a - a'| + |b - b'| = 1$). When $r \ge 4$, always $\chi_r(P_{m,n}) = 5$. For $r = 3$, we have $\chi_3(P_{m,n}) = 4$ if $m$ and $n$ are both even, and otherwise $\chi_3(P_{m,n}) = 5$ except possibly when $mn \equiv 2 \mod 4$. The case $mn \equiv 2 \mod 4$ was resolved by Kang, Müller, and West [8], completing the proof that $\chi_3(P_m \Box P_n) = 5$ whenever $m$ and $n$ are not both even.

# 2 Bounds in Terms of Maximum Degree

We prove an upper bound on $\chi_r(G)$ in terms of $\Delta(G)$ and characterize (for $r \ge 2$) when equality holds. The idea is similar to a well-known proof of Brooks' Theorem by Lovász [13].

**Theorem 2.1.** *$\chi_r(G) \le r\Delta(G) + 1$, with equality for $r \ge 2$ if and only if $G$ is $r$-regular with diameter 2 and girth 5.*

*Proof.* When $G$ is $r$-regular, vertices with a common neighbor need distinct colors in an $r$-dynamic coloring. Thus $\chi_r(G) = |V(G)|$ when $\mathrm{diam}(G) = 2$. The maximum value of $|V(G)|$ is $r^2 + 1$, which occurs if and only if $G$ has girth 5. Hence equality holds for Moore graphs.

For the upper bound, it suffices to consider connected $G$. We use a vertex ordering $v_1, \dots, v_n$. First let $v_n$ be a vertex of minimum degree, and if $G$ is regular let $v_n$ lie on a shortest cycle. Complete the ordering so that each vertex before $v_n$ has a higher-indexed neighbor (this is an *increasing ordering to* $v_n$. Color the vertices in the order $v_1, \dots, v_n$. When coloring $v_i$, avoid each color used on its neighborhood $N(v_i)$ or on a neighbor of a vertex in $N(v_i)$ not yet having $r$ colors in its neighborhood. Note that the first $\min\{d(v), r\}$ colors used on $N(v)$ are distinct. At most $r$ colors must be avoided for vertex of $N(v_i)$.

For $i < n$, the uncolored higher-indexed neighbor of $v_i$ means that at most $r\Delta(G) - 1$ colors need to be avoided when coloring $v_i$. If $G$ is not regular, then at most $r(\Delta(G) - 1)$

colors need to be avoided at $v_n$. If $G$ is regular and $\Delta(G) > r$, then when $v_n$ is colored, its neighbors already have $r$ distinct colors in their neighborhoods, so only $\Delta(G)$ colors need to be avoided in coloring $v_n$. If $\Delta(G) < r$, then Observation 1.1 yields $\chi_r(G) = \chi(G^2) \le \Delta(G^2) + 1 \le \Delta(G)^2 + 1 < r\Delta(G)$.

Hence we may assume that $G$ is $r$-regular; at most $r^2$ colors are used in coloring $v_1, \ldots, v_{n-1}$. Recall that $v_n$ lies on a shortest cycle $C$. If $C$ has length at most 4, then when coloring $v_n$, the two neighbors of $v_n$ on $C$ generate at least one common color to be avoided (on a vertex of $C$), leaving at most $r^2 - 1$ colors to be avoided. Thus $r^2$ colors suffice.

Hence we may assume that $G$ has girth at least 5, which yields $r^2$ distinct other vertices within distance 2 of each vertex. If $\mathrm{diam}(G) = 2$, then we are finished, so assume $\mathrm{diam}(G) \ge 3$. Let $u$ and $w$ be vertices at distance 3, with $\langle u, x, y, w \rangle$ an induced path. Let $T_1, \ldots, T_k$ be the components of $G - \{u, w\}$, with $x, y \in T_k$.

Color $u$ and $w$ first (each get color 1), and then use an increasing ordering in each $T_i$ to a neighbor of $u$ or $w$, leaving $T_k$ last with an increasing ordering to $x$. As usual, at most $r^2 - 1$ colors must be avoided on any vertex of $T_i$ before the last. For $i < k$, the last vertex $v$ in $T_i$ has an uncolored vertex at distance 2 (it is $x$ or $y$), so it needs to avoid at most $r^2 - 1$ colors. Finally, when coloring $x$, the two vertices $u$ and $w$ have the same color, so again at most $r^2 - 1$ colors need to be avoided on $x$. $\square$

We believe that also there is no graph $G$ with $\chi_r(G) = r\Delta(G)$ other than cycles whose length is not divisible by 3 (when $r = 2$); that is, when $\Delta(G) > 2$ and Moore graphs are excluded, the upper bound should improve further. It is known for example that $\chi_2(G) \le \Delta(G) + 1$ when $\Delta(G) \ge 3$ and no component of $G$ is $C_5$ [11]. We present a restricted construction for special $\Delta(G)$ where the bound cannot be improved by much.

**Example 2.2.** *Graphs with $\chi_r(G) = r\Delta(G) - 1$ when $r = \Delta(G)$ and $\Delta(G) \in \{3, 7\}$.* When $r = \Delta(G)$, deleting an edge $uv$ from a Moore graph with $\Delta(G) > 2$ yields a graph $G$ with $\chi_r(G) = r\Delta(G) - 1 = r^2 - 1$. For the lower bound, any two vertices in $V(G) - \{u, v\}$ are adjacent or have a common neighbor, so they must have distinct colors. For the upper bound, give distinct colors to $V(G) - \{u, v\}$, give $u$ a color in $N(v)$, and give $v$ a color in $N(u)$; now no color is on two vertices with a common neighbor in $G$.

Let $G_i$ for $i \in \mathbb{Z}_k$ be a copy of this graph $G$. Add the edges joining the copy of $v$ in $G_i$ to the copy of $u$ in $G_{i+1}$, for all $i \in \mathbb{Z}_k$. Since $\Delta(G) > 2$, the duplicated colors in successive copies of $G$ can be chosen to be distinct. Thus infinitely many 2-connected graphs are constructed with $r$-dynamic chromatic number $r\Delta(G) - 1$, but only for $\Delta(G) = r$ and $r \in \{3, 7\}$ (and possibly 57), since those are the only degrees of Moore graphs.

**Question 2.3.** *For fixed $r$ and $k$, what is the best bound on $\chi_r(G)$ that holds for all but finitely many graphs $G$ with maximum degree $k$?*

Related to this question, Taherkhani [18] proved that if $2 \le r \le \delta(G)/\log(2er(\Delta^2+1))$, then $\chi_r(G) \le \chi(G) + (r-1)\lceil e\frac{\Delta(G)}{\delta(G)}\log(2er(\Delta(G)^2+1))\rceil$. This gives an upper bound for $\chi_r(G)$ when $r$ is substantially smaller than $\delta(G)$.

We show next that if the minimum degree is not too small relative to the number of vertices (for fixed $r$), then the bound in Theorem 2.1 can be improved by replacing the product with a sum involving $\Delta(G)$ and $r$. The idea is to modify the greedy coloring algorithm used in Theorem 2.1. There we ensured that the first $r$ neighbors of a vertex would have distinct colors. Now we allow $r$ distinct colors to be obtained at any time.

**Theorem 2.4.** *If $G$ is an $n$-vertex graph, and $\delta(G) > \frac{r+s}{s+1} r \ln n$, then $\chi_r(G) \le \Delta(G) + r + s$. In particular, $\delta(G) > 2r \ln n$ implies $\chi_r(G) \le \Delta(G) + 2r - 2$, and $\delta(G) > r^2 \ln n$ implies $\chi_r(G) \le \Delta(G) + r$.*

*Proof.* The special cases arise by setting $s = r-2$ and $s = 0$ in the general statement.

If $n \le \Delta(G) + r + s$, then we can give the vertices distinct colors, so we may assume $n > \Delta(G) + r + s$. Let $v_1, \ldots, v_n$ be any vertex ordering of $G$. Color $v_1, \ldots, v_n$ in order using $\Delta(G) + r + s$ colors. Give $v_i$ a color chosen uniformly at random among those not used on its earlier neighbors; at least $r+s$ colors are available. This produces a proper coloring.

We claim that with positive probability the coloring is also $r$-dynamic. This fails at a vertex $v$ only if the colors in $N(v)$ are confined to a particular set of $r-1$ colors. The probability that this happens with a particular set of $r-1$ colors is bounded by $(\frac{r-1}{r+s})^{\delta(G)}$, which in turn is bounded by $e^{-\delta(G)\frac{s+1}{r+s}}$. There are $\binom{\Delta(G)+r+s}{r-1}$ choices of a set of $r-1$ colors, which is less than $n^{r-1}$ since $\Delta(G) + r + s < n$.

Since $G$ has $n$ vertices, the probability of having a bad vertex is less than $n^r e^{-\delta(G)\frac{s+1}{r+s}}$. The constraint on $\delta(G)$ bounds this by $n^r n^{-r}$, which equals 1. $\square$

# 3 Regular Graphs and Chromatic Number

To strengthen $\chi_r(G) \le r\Delta(G)$ for non-Moore graphs, we want to replace $\Delta(G)$ with a value no larger. In general, $r\chi(G)$ would be a better upper bound, since $\chi(G) \le \Delta(G)$ by Brooks' Theorem when $\Delta(G) \ge 3$ and $G$ is not complete. We prove $\chi_r(G) \le r\chi(G)$ for regular graphs with sufficiently large degree in terms of $r$. The Petersen graph shows that the inequality does not hold for all $G$ when $r = 3$.

When $G$ is $k$-regular with $k$ sufficiently large in terms of $r$, some $r$-coloring of $V(G)$ puts $r$ distinct colors into each vertex neighborhood. Giving each vertex a pair consisting of its color under such an $r$-coloring and its color under a proper $\chi(G)$-coloring produces an $r$-dynamic coloring with $r\chi(G)$ colors. The dynamic part follows from a standard probabilistic computation: a random $r$-coloring succeeds with high probability. The computation appears for example in [15].

**Lemma 3.1.** *Let $H$ be a hypergraph in which each edge has size at least $k$ and each vertex appears in at most $D$ edges. If* $\mathrm{e}p(k(D-1)+1) \le 1$*, where* $p = r\mathrm{e}^{-k/r}$*, then some $r$-coloring of $V(H)$ puts all $r$ colors into each edge of $H$.*

**Theorem 3.2.** *If $G$ is a $k$-regular graph with $k \ge (3+x)r\ln r$, where $x - \frac{2\ln\ln r}{\ln r}$ is a small positive constant, then $\chi_r(G) \le r\chi(G)$.*

*Proof.* Let $H$ be the hypergraph with vertex set $V(G)$ whose edges are the vertex neighborhoods in $G$; $H$ is $k$-uniform, with each vertex in $k$ edges. Let $p = r\mathrm{e}^{-k/r}$. If $\mathrm{e}p(k(k-1)+1) \le 1$, then by Lemma 3.1 there is an $r$-coloring of $V(H)$ such that every edge in $E(H)$ has $r$ colors on it. Pairing this coloring with a proper coloring of $G$ yields an $r$-dynamic coloring of $H$ with $r\chi(G)$ colors.

For the needed inequality, note that $k \ge (3+x)r\ln r$ with $x > \frac{2\ln\ln r}{\ln r}$ implies $\mathrm{e}r\mathrm{e}^{-k/r}k^2 \le 1$, which suffices. □

Taherkhani [18] independently used essentially the same argument as Lemma 3.1 but did not simplify the resulting threshold on $k$ in terms of $r$. The thesis [17] gave the threshold $k \ge 7r\ln r$ by a similar method. We do not know the least $k$ to guarantee $\chi_r(G) \le r\chi(G)$ when $G$ is $k$-regular, but we can show that when $k = r$ the ratio $\chi_r(G)/\chi(G)$ can grow superlinearly in $r$. Let $[n]$ denote $\{1,\ldots,n\}$.

**Theorem 3.3.** *For infinitely many $r$, there is an $r$-regular graph $G$ such that $\chi_r(G) > r^{1.37744}\chi(G)$.*

*Proof.* The *Kneser graph* $K(n,t)$ is the graph whose vertices are the $t$-element subsets of $[n]$, with two sets adjacent when they are disjoint. Each vertex is adjacent to $\binom{n-t}{t}$ other vertices. Given $t \in \mathbb{N}$, let $G = K(3t-1,t)$ and $r = \binom{n-t}{t}$. Any two nonadjacent vertices in $G$ have a common neighbor, since two intersecting $t$-sets in $[3t-1]$ omit at least $t$ elements, so $\mathrm{diam}(G) = 2$. Since $G$ is $r$-regular, we thus have $\chi_r(G) = |V(G)| = \binom{3t-1}{t}$. On the other hand, Lovász [14] and Bárany [6] proved $\chi(K(n,t)) = n - 2t + 2$, so $\chi(G) = t+1$.

It remains to express $\chi_r(G)$ in terms of $r$ and $\chi(G)$. In terms of $t$, we have $r = \binom{2t-1}{t} = \frac{1}{2}\binom{2t}{t}$ and $\chi_r(G) = \binom{3t-1}{t} = \frac{2}{3}\binom{3t}{t}$. What is important is the ratio between $\chi_r(G)$ and $r$ as a function of $r$. For $c \in \{\frac{1}{2}, \frac{1}{3}\}$, we use the approximation $\binom{m}{cm} \approx \frac{(c^c(1-c)^{1-c})^{-m}}{\sqrt{c(1-c)2\pi m}}$ from Stirling's Formula to compute

$$\frac{\chi_r(G)}{r\chi(G)} \approx \frac{\frac{2}{3}\binom{3t}{t}}{(t+1)\frac{1}{2}\binom{2t}{t}} \approx \frac{4}{3t}\frac{(\frac{27}{4})^t\sqrt{\pi t}}{4^t\sqrt{(4/3)\pi t}} \approx \frac{1}{t}\sqrt{\frac{4}{3}}\left(\frac{27}{16}\right)^t.$$

Setting this ratio to be $r^x$, where $r \approx \frac{1}{2}4^t/\sqrt{\pi t}$, we take logarithms to obtain $t\lg(27/16) = (1+o(1))tx\lg 4$, which simplifies to $x = \frac{1}{2}(\lg 27 - 4) > .37744$. Thus $\chi_r(G) > r^{1.37744}\chi(G)$. □

Theorems 3.2 and 3.3 suggest the following question.

**Question 3.4.** *As a function of $r$, what is the least $k$ such that $\frac{\chi_r(G)}{\chi(G)}$ is at most linear in $r$ when $G$ is $k$-regular? What is the least $k$ such that $\frac{\chi_r(G)}{\chi(G)}$ is bounded by $r(\ln r)^c$ for some $c$?*

# 4 Diameter and Chromatic Number

In this section we study the relationship between $\chi_r(G)$ and $\chi(G)$ when $G$ has small diameter. We first prove $\chi_2(G) \le \chi(G)+2$ when $\mathrm{diam}(G) = 2$, regardless of whether $G$ is regular, and we characterize when equality holds. Alishahi [4] proved that $\chi_2(G) \le \chi(G)+1$ when $G$ has diameter 2 and chromatic number at least 4 and satisfies $\mathrm{e}(\Delta^2(G) - \Delta(G) + 1) \le 2^{\delta(G)}$.

Proving the desired bound is easy; characterizing equality is harder. Given a coloring $f$ of a graph, say that a vertex is *$f$-monochromatic* if its neighbors have the same color under $f$.

**Theorem 4.1.** *If $\mathrm{diam}(G) = 2$, then $\chi_2(G) \le \chi(G) + 2$, with equality only when $G$ is a complete bipartite graph or $C_5$.*

*Proof.* The claim holds for stars. For a non-star $G$ with minimum degree 1, deleting all vertices of degree 1 yields a graph $G'$ with $\chi(G') = \chi(G)$, and a 2-dynamic coloring of $G'$ extends to a 2-dynamic coloring of $G$. Hence we may assume $\delta(G) \ge 2$. We claim

> $(*)$ If $f$ is a proper coloring of a graph with diameter 2, and $v$ is $f$-monochromatic with neighborhood of color $a$, then $N(v) = \{u\colon\ f(u) = a\}$. In particular, nonadjacent $f$-monochromatic vertices have the same neighborhood.

To prove $(*)$, note that the set of vertices with color $a$ is independent, so a vertex outside $N(v)$ with color $a$ cannot have a common neighbor with $v$. For the second statement, nonadjacent vertex must have a common neighbor, and then being $f$-monochromatic makes both adjacent to all vertices of that color.

We next prove the upper bound. Let $f$ be a proper $\chi(G)$-coloring of a graph $G$ with diameter 2; note that $G$ is connected. If $f$ is not 2-dynamic, then some vertex $v$ is $f$-monochromatic, with color $a$ on all of $N(v)$. Modify $f$ by giving a new color $\alpha$ to $v$ and another new color $\beta$ to one vertex $x$ in $N(v)$.

The resulting coloring $f'$ is a proper $(\chi(G)+2)$-coloring. If $f'$ is not 2-dynamic, then some vertex $z$ is $f'$-monochromatic. By construction, $z$ cannot be $v$ or a neighbor of $v$ (each neighbor of $v$ has color $\alpha$ on exactly one neighbor). If $z$ is not a neighbor of $v$, then by $(*)$ we have $N(z) = N(v)$, and colors $a$ and $\beta$ both appear in $N(z)$.

For the characterization of equality, note first that if no vertex of $N(v)$ is $f$-monochromatic, then we do not need to introduce a new color on $v$, and we obtain $\chi_2(G) \le \chi(G)+1$. Hence we may assume that two adjacent vertices $v$ and $u$ are $f$-monochromatic. If no vertex lies outside $N(v) \cup N(u)$, then $G$ is the complete bipartite graph with parts $N(v)$ and $N(u)$, since those sets are independent and $\mathrm{diam}(G) = 2$.

Hence if $G$ is not a complete bipartite graph, then there is a vertex $w$ outside $N(v) \cup N(u)$. Let $a$ and $b$ be the colors on $N(v)$ and $N(u)$, respectively. Since $\mathrm{diam}(G) = 2$, vertex $w$ has neighbors in both $N(v)$ and $N(u)$ and hence has a third color, $c$. Let $W$ be the set of all vertices with color $c$ under $f$. Let $U$ and $V$ be the set of all $f$-monochromatic vertices having the same color as $u$ and $v$, respectively; note that $U \subseteq N(v)$ and $V \subseteq N(u)$. By $(*)$, there are no other $f$-monochromatic vertices.

All vertices in $U$ have the same neighborhood, as do all vertices in $V$. If $|U| > 1$, then we change the color of $u$ to $c$ and use a new color $d$ on $v$. This produces a 2-dynamic coloring with $\chi(G)+1$ colors, since the neighbors of $u$ now have colors $a$ and $c$ in their neighborhood. Similarly, if $|V| > 1$ we can change the color of $v$ to $c$ and use $d$ on $u$.

Hence we may assume $|U| = |V| = 1$. The change still works unless $N(u)$ contains a vertex $x$ whose neighbors other than $u$ all have color $c$, and similarly $N(v)$ contains a vertex $y$ whose neighbors other than $v$ all have color $c$. In particular, $x$ and $y$ are not adjacent. Now changing $x$ and $y$ to a new color $d$ produces the desired 2-dynamic coloring of $G$ with $\chi(G) + 1$ colors, unless there is a vertex $w \in W$ whose only neighbors are $x$ and $y$.

In this case, reaching $N(u) - \{v, x\}$ in two steps from $w$ requires $N(u) - \{v, x\} \subseteq N(y)$, and similarly $N(v) - \{u, y\} \subseteq N(x)$. Since vertices of $N(x) - \{u\}$ and $N(y) - \{v\}$ all have color $c$, we have $N(u) = \{v, x\}$ and $N(v) = \{u, y\}$. Now $G = C_5$ unless $G$ has a vertex outside the 5-cycle induced by $w, x, u, v, y$ in order. Since $u$ and $v$ have no other neighbors, and all other neighbors of $x$ and $y$ have color $c$, reaching $u$ and $v$ in two steps requires that all other vertices have neighborhood $\{x, y\}$. Now we use colors $1, 2, 3, 4$ in order on the path $\langle x, u, v, y\rangle$ and use colors 2 and 3 on the remaining independent set, each at least once. $\square$

The graph obtained by subdividing every edge of an $n$-vertex complete graph has diameter 4. Its chromatic number is 2, but its 2-dynamic chromatic number is $n$. Hence for diameter 4 there is no bound in terms of the chromatic number, while for diameter 2 the bound is very tight. For diameter 3 we determine the best bound.

To study $\chi_2$ on graphs with diameter 3 we use a related notion from hypergraph coloring. A vertex coloring of a hypergraph is *c-strong* if every edge $e$ has at least $\min\{c, |e|\}$ distinct colors; this concept was introduced by Blais, Weinstein, and Yoshida [5]. Like $\chi_r$, this concept yields a spectrum of parameters, from ordinary hypergraph coloring to "strong coloring", where all vertices in an edge have distinct colors.

A coloring of a graph $G$ is $r$-dynamic if it is proper and is an $r$-strong coloring of the "neighborhood hypergraph" on $V(G)$ whose edges are the vertex neighborhoods in $G$. Furthermore, combining a proper coloring of $G$ with an $r$-strong coloring of the neighborhood hypergraph yields an $r$-dynamic coloring of $G$. Thus $s \le \chi_r(G) \le s\chi(G)$, where $s$ is the minimum number of colors in an $r$-strong coloring of the neighborhood hypergraph of $G$.

**Theorem 4.2.** *If $G$ is a $k$-chromatic graph with diameter at most 3, then $\chi_2(G) \le 3k$, and this bound is sharp when $k \ge 2$.*

*Proof.* We begin with sharpness. Let $F$ be a subgraph of $K_{3k}$ consisting of disjoint triangles $T_1, \ldots, T_k$. Form $G$ from $K_{3k}$ by subdividing each edge of $F$. In $G$ the vertices belonging to $T_i$ can receive color $i$, and their neighbors of degree 2 can receive another color. In a 2-dynamic coloring, all $3k$ original vertices must have distinct colors, so $3k$ colors are needed. The diameter is 3, because any two vertices of degree 2 have neighbors that are adjacent.

Now let $G$ be any $k$-chromatic graph with diameter at most 3. We have noted that $\chi_r(G) \le sk$, where $s$ is the minimum number of colors in an $r$-strong coloring of the neighborhood hypergraph $H$ of $G$. More precisely, we give each vertex $v$ a color pair $(f(v), h(v))$, where $f$ is a proper $k$-coloring of $G$ and $h$ is an $r$-strong coloring of the subhypergraph $H'$ of $H$ whose edges are the edges of $H$ that do not already have $r$ colors under $f$.

Thus it suffices to show that $H'$ has a 2-strong 3-coloring when $\mathrm{diam}(G) \le 3$. Call a hypergraph with such a coloring *good*. A hypergraph is good if it has a minimal edge $e$ that intersects all other edges; use colors 1 and 2 on $e$ and use color 3 on all vertices not in $e$.

Given the proper $k$-coloring $f$ of $G$, we define $k$ subhypergraphs of $H'$. Let $V_i = \{v \in V(G) \colon f(v) = i\}$. The subhypergraph $H_i$ has vertex set $V_i$, and its edges are the vertex neighborhoods in $G$ in which $f$ colors every vertex with $i$. Since each vertex in $H_i$ has color $i$ under $f$, the vertex sets of these subhypergraphs are disjoint. Hence if each is good, then their union is good, with an $r$-strong coloring $h$. Extend $h$ arbitrarily for $v \notin V(H')$; these vertices are not needed to make neighborhoods $r$-dynamic.

It suffices to show that the edges of $H_i$ are pairwise intersecting when $\mathrm{diam}(G) \le 3$. Consider $x, y \in V(G)$ such that $N(x), N(y) \in E(H_i)$ and $N(x) \cap N(y) = \varnothing$. If $xy \in E(G)$, then $y \in N(x)$ and $x \in N(y)$, and hence $f(x) = f(y) = i$, contradicting that $f$ is a proper coloring of $G$. Similarly, no edge of $G$ can join $N(x)$ and $N(y)$, since all of $N(x) \cup N(y)$ has color $i$ under $f$. Hence a shortest path in $G$ from $x$ to $y$ visits $N(x)$ and $N(y)$ and some other vertex between them. Such a path has length at least 4, contradicting $\mathrm{diam}(G) \le 3$. We conclude that the edges of $H_i$ are pairwise intersecting and $H_i$ is good, as desired. □

These results are sharp in various ways. For larger $r$, there is no bound, not even on bipartite graphs with diameter 3 or on 3-chromatic graphs with diameter 2. Note that the only bipartite graphs with diameter 2 are complete bipartite graphs, where $r$-dynamic chromatic number does not exceed $2r$.

**Theorem 4.3.** *For $3 \le k < r$, the $r$-dynamic chromatic number is unbounded on the graphs with minimum degree $k+1$ that are bipartite and have diameter 3, and also on those that are 3-colorable and have diameter 2.*

*Proof.* Let $H$ be the incidence graph of the $k$-subsets of $[n]$, where $n > k+1$. That is, $H$ is bipartite, with one part being $[n]$ and the other being the family of $k$-subsets, and element $j$ is adjacent to set $A$ if $j \in A$. Form $G$ by adding a single vertex $v$ adjacent to all the $k$-sets, giving them degree $k+1$. The graph $G$ is bipartite, with the added vertex in the same part as $[n]$. Any two elements of $[n]$ lie in a common $k$-set, so the distance between them is 2, and the distance between any $k$-set and an element not in it is 3. The added vertex has distance 2 from all of $[n]$ and ensures distance 2 between any two $k$-sets. Hence $\mathrm{diam}(G) = 3$.

In an $r$-dynamic coloring, the neighbors of any vertex with degree at most $r$ receive distinct colors. In $G$, any two vertices of $[n] \cup \{v\}$ have a common neighbor with degree at most $r$, so $\chi_r(G) \geq n + 1$ (equality holds).

For a construction with diameter 2, let the added vertex $v$ be adjacent also to all of $[n]$. Now $v$ is a dominating vertex, so the diameter is 2, but the chromatic number increases to 3. However, each $k$-set vertex still has $k + 1$ neighbors, so the argument for $\chi_r(G) \geq n + 1$ remains the same. $\square$

# 5 Grids

In this final section, we study $r$-dynamic coloring of grids and toroidal grids. Restricting to $m, n \geq 3$, set $V(P_{m,n}) = [m] \times [n]$ and $V(C_{m,n}) = \mathbb{Z}_m \times \mathbb{Z}_n$; vertices are adjacent when they have equal values in one coordinate and consecutive values in the other. (Indeed, $P_{m,n}$ is the cartesian product of paths $P_m$ and $P_n$, and $C_{m,n}$ is the product of cycles $C_m$ and $C_n$.)

Akbari, Ghanbari, and Jahanbekam [2] showed that $\chi_2(P_{m,n}) = 4$ and that $\chi_2(C_{m,n}) = 3$ when $3 \mid mn$, and that otherwise $\chi_2(C_{m,n}) = 4$. Since these graphs have maximum degree 4, by Observation 1.1 the $r$-dynamic chromatic number equals the 4-dynamic chromatic number when $r \geq 4$. Hence we consider $r \in \{3, 4\}$.

**Theorem 5.1.** *If $m$ and $n$ are at least 2, then*

$$\chi_4(P_{m,n}) = \begin{cases} 4 & \textit{if } \min\{m,n\} = 2 \\ 5 & \textit{otherwise} \end{cases} \quad \textit{and} \quad \chi_3(P_{m,n}) = \begin{cases} 4 & \min\{m,n\} = 2 \\ 4 & m \textit{ and } n \textit{ are both even.} \end{cases}$$

*Proof.* The lower bounds follow from Observation 1.2: $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$. For the upper bounds, index the vertices as $\{(i, j)\colon\ 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\}$, adjacent when they differ by 1 in one coordinate.

First consider $\chi_4(P_{m,n})$ with $\min\{m, n\} > 2$. Define a coloring $c$ on $V(P_{m,n})$ by $c(i, j) = i + 2j \mod 5$. By construction, $c$ is a proper 5-coloring of $P_{m,n}$. It is 4-dynamic because the neighbors of any vertex have distinct colors. Thus $\chi_4(P_{m,n}) = 5$.

For $\min\{m, n\} = 2$ and $r \geq 3$, we have $\Delta(P_{m,n}) = 3$, so $\chi_r(P_{m,n}) \geq 4$; the coloring shown below for $m = 2$ achieves equality and illustrates the coloring $h$ defined below.

$$\begin{matrix} 0 & 1 & 2 & 3 & 0 & \cdots \\ 2 & 3 & 0 & 1 & 2 & \cdots \end{matrix}$$

Now consider $\chi_3(P_{m,n})$ when both $m$ and $n$ are even; at least four colors are needed. Define $g$ by $g(4k) = 0$, $g(4k + 1) = 2$, $g(4k + 2) = 1$, and $g(4k + 3) = 3$ for $k \in \mathbb{Z}$. Define $h$

by setting $h(i,j) = g(i) + j \mod 4$ when $i$ or $i-1$ is divisible by 4 and $h(i,j) = g(i) - j \mod 4$ otherwise. By construction, $h$ is a proper 4-coloring of $P_{m,n}$. Also $h$ is 3-dynamic. $\square$

The case $r = 3$ with $m, n \geq 3$ and $m, n$ not both even remains. Theorem 5.1 yields $\chi_3(P_{m,n}) = 5$, since always $\chi_r(G) \leq \chi_{r+1}(G)$. In the next lemma we prove equality when also $mn \not\equiv 2 \mod 4$. The more difficult case $mn \equiv 2 \mod 4$ is completed in [8].

**Lemma 5.2.** *A 3-dynamic 4-coloring of $P_{m,n}$ when $m, n \geq 3$ and $m, n$ are not both even requires $mn \equiv 2 \mod 4$.*

*Proof.* Let $A$ be the matrix with color $a_{i,j}$ in position $(i,j)$, representing the coloring. By symmetry, we let $m$ (the number of rows) be odd. The four upper-left vertices have distinct colors, since the vertices on the edges of the matrix have degree at most 3. Let $a = a_{1,1}$, $b = a_{1,2}$, $c = a_{2,1}$, and $d = a_{2,2}$. The neighbors of a vertex of degree 3 must have the other three colors. This determines the first two rows and first two columns. Once the argument for the first two rows and columns reaches their ends, the same argument determines the elements of the last two rows and columns.

The matrices below, in the two cases $m \equiv 1 \mod 4$ and $m \equiv 3 \mod 4$, exhibit all the cases for $m$. In the bottom row the first two elements agree with the top row when $m \equiv 1 \mod 4$ and reverse those two elements when $m \equiv 3 \mod 4$. The diagram shows that the last two columns exhibit the same behavior.

$$
\begin{array}{cccccccccccccc}
a & b & c & d & a & b & c & d & a & b & c & d & a & b \\
c & d & a & b & c & d & a & b & c & d & a & b & c & d \\
b & a & & & & & & & & & & & b & a \\
d & c & a & b & d & c & a & b & d & c & a & b & d & c \\
a & b & d & c & a & b & d & c & a & b & d & c & a & b \\
 & & & 1 & 1 & & 4 & 4 & & 3 & 3 & & 2 & 2
\end{array}
\qquad
\begin{array}{cccccccccccccc}
a & b & c & d & a & b & c & d & a & b & c & d & a & b \\
c & d & a & b & c & d & a & b & c & d & a & b & c & d \\
b & a & & & & & & & & & & & b & a \\
d & c & & & & & & & & & & & d & c \\
a & b & & & & & & & & & & & a & b \\
c & d & b & a & c & d & b & a & c & d & b & a & c & d \\
b & a & c & d & b & a & c & d & b & a & c & d & b & a \\
 & & & 1 & 1 & & 4 & 4 & & 3 & 3 & & 2 & 2
\end{array}
$$

This property of the last two columns occurs for each congruence class of $n$ modulo 4. The numbers below the grid designate where the rows end when $n$ is congruent to 1, 4, 3, or 2, respectively. In the first three cases, the relationship between the top row and bottom row is not as would be required by the last two columns if the rows ended there. Hence in those cases no 3-dynamic 4-coloring can exist. $\square$

The proof of Lemma 5.2 shows that consistency around the borders can be achieved when $m$ is odd and $n \equiv 2 \mod 4$, leaving the possibility of a 3-dynamic 4-coloring in such cases. Using other structural arguments, Kang, Müller, and West [8] found a proof for all $(m,n)$ with $mn \equiv 2 \mod 4$ that $\chi_3(P_{m,n}) > 4$. Their result completes the following theorem.

**Theorem 5.3.** *When $m$ and $n$ are not both even, $\chi_3(P_{m,n}) = 5$.*

Finally, consider $\chi_r(C_{m,n})$ for $r \geq 3$. Since $C_{m,n} \cong C_{n,m}$, we may assume that the remainder upon dividing $m$ by 4 is no larger than when dividing $n$ by 4.

**Theorem 5.4.** *Always $\chi_3(C_{m,n}) \geq 4$. For $m \equiv s \mod 4$ and $n \equiv t \mod 4$ with $0 \leq s \leq t \leq 3$, equality holds when $s = 0$ and $t \neq 3$.*

*Proof.* Since $C_{m,n}$ is 4-regular, Observation 1.1 yields $\chi_3(C_{m,n}) \geq 4$. For the upper bound, we construct colorings. Write $V(C_{m,n})$ as $\mathbb{Z}_m \times \mathbb{Z}_n$, with vertices adjacent when they agree in one coordinate and differ by 1 in the other.

Recall $g$ and $h$ from Theorem 5.1: $g(4k) = 0$, $g(4k+1) = 2$, $g(4k+2) = 1$, and $g(4k+3) = 3$ for $k \in \mathbb{Z}$. Also $h(i,j) = g(i) + j \mod 4$ when $i$ or $i-1$ is divisible by 4, and $h(i,j) = g(i) - j \mod 4$ otherwise. Indexing of the rows and columns begins with 0. Below we illustrate $h$ and two modifications of $h$ used when $s = 0$. Also let $A$ denote the 4-by-4 matrix appearing in the first four columns of the first matrix below. Note that $h$ is a tiling by copies of $A$ when $s = t = 0$, and otherwise it uses portions of $A$ in the rows and columns after the last multiple of 4.

$$
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 0 & 1 \\
2 & 3 & 0 & 1 & 2 & 3 \\
1 & 0 & 3 & 2 & 1 & 0 \\
3 & 2 & 1 & 0 & 3 & 2
\end{array}
\qquad
\begin{array}{cccc|c}
0 & 1 & 2 & 3 & 1 \\
2 & 3 & 0 & 1 & 3 \\
1 & 0 & 3 & 2 & 0 \\
3 & 2 & 1 & 0 & 2
\end{array}
\qquad
\begin{array}{cccc|ccc}
0 & 1 & 2 & 3 & 0 & 1 & 2 \\
4 & 3 & 0 & 1 & 2 & 3 & 0 \\
1 & 0 & 3 & 2 & 1 & 0 & 3 \\
3 & 2 & 1 & 0 & 3 & 2 & 4
\end{array}
$$

As shown above on the left, each column is periodic, so each vertex has vertically neighboring colors distinct and different from its own. When $t = 0$, the coloring is 4-dynamic (for the same reason) and hence also 3-dynamic. When $t = 2$, the coloring is still proper, but the vertices in the last column have horizontal neighbors with the same color; the coloring is still 3-dynamic.

When $t = 1$, modify $h$ by changing the colors on column $n-1$ (the last column) to agree with those on column 1 (the second column). Colors on vertices in the last column now differ by 2 from the color to their left, so the coloring is proper. Vertices in the last two columns have three distinct colors in their neighborhoods; other vertices have four. $\square$

In the remaining cases, explicit constructions yield $\chi_3(C_{m,n}) \leq 6$ (see [17]), but we have not determined the optimal values. Similarly, for $r \geq 4$ we have $\chi_r(C_{m,n}) = \chi_4(C_{m,n}) \geq 5$, with equality when $m$ and $n$ are both divisible by 5, and explicit constructions yield $\chi_4(C_{m,n}) \leq 9$ (see [17]). Note that $\chi_4(C_{3,3}) = 9$, since $(C_{3,3})^2$ is a complete graph.

# References


[1] S. Akbari, M. Ghanbari, S. Jahanbekam, On the dynamic chromatic number of graphs, in *Combinatorics and graphs. Contemp. Math.* 531 (Amer. Math. Soc. 2010), 1–18.

[2] S. Akabari, M. Ghanbari, S. Jahanbekam, On the list dynamic coloring of graphs. *Discrete Applied Math.* 157 (2009), 3005–3007.

[3] M. Alishahi, On the dynamic coloring of graphs, *Discrete Applied Math.* 159 (2011), 152–156.

[4] M. Alishahi, Dynamic chromatic number of regular graphs, *Discrete Applied Math.* 160 (2012), 2098–2103.

[5] E. Blais, A. Weinstein, and Y. Yoshida, Semi-strong coloring of intersecting hypergraphs. *Combin. Probab. Comput.* 23 (2014), 1–7.

[6] I. Bárány, A short proof of Kneser's conjecture. *J. Combin. Theory Ser. A* 25 (1978), 325–326.

[7] R.L. Brooks, On colouring the nodes of a network. *Proc. Cambridge Philos. Soc.* 37 (1941), 194–197.

[8] R. Kang, T. Muller, and D. B. West, On $r$-dynamic coloring of grids, *Discrete Applied Math.* 186 (2015), 286–290.

[9] S.-J. Kim and W.-J. Park, List dynamic coloring of sparse graphs, in *Combinatorial optimization and applications. Lect. Notes Comput. Sci.* 6831 (Springer, 2011), 156–162.

[10] S.-J. Kim, S. J. Lee, and W.-J. Park, Dynamic coloring and list dynamic coloring of planar graphs. *Discrete Applied Math.* 161 (2013), 2207–2212.

[11] H.J. Lai, B. Montgomery, and H. Poon, Upper bounds of dynamic chromatic number. *Ars Combin.* 68 (2003), 193–201.

[12] S. Loeb, T. Mahoney, B. Reiniger, and J.I. Wise, Dynamically coloring graphs with given genus. In preparation.

[13] L. Lovász, Three short proofs in graph theory. *J. Combin. Theory Ser. B* 19 (1975), 269–271.

[14] L. Lovász, Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A* 25 (1978), 319–324.

[15] C. McDiarmid, Hypergraph colouring and the Lovász local lemma, *Discrete Math.* 167/268 (1997), 481–486.

[16] B. Montgomery, Dynamic Coloring of Graphs. Ph.D Dissertation (West Virginia University, 2001).

[17] S. O, Matchings, Connectivity, and Eigenvalues in Regular Graphs, Ph.D Dissertation (University of Illinois, 2011).

[18] A. Taherkhani, $r$-Dynamic Chromatic Number of Graphs, *Discrete Applied Math.* doi:10.1016/j.dam.2015.07.019 .