Improper coloring of sparse graphs with a given girth, II: Constructions

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Abstract

A graph G is (j,k)-*colorable* if $V(G)$ can be partitioned into two sets V_j and V_k so that the maximum degree of $G[V_j]$ is at most j and of $G[V_k]$ is at most k. While the problem of verifying whether a graph is $(0,0)$-colorable is easy, the similar problem with (j,k) in place of $(0,0)$ is NP-complete for all nonnegative j and k with $j+k \geq 1$.

Let $F_{j,k}(g)$ denote the supremum of all x such that for some constant c_g every graph G with girth g and $|E(H)| \leq x|V(H)| + c_g$ for every $H \subseteq G$ is (j,k)-colorable. It was proved recently that $F_{0,1}(3) = 1.2$. In a companion paper we find the exact value $F_{0,1}(4) = F_{0,1}(5) = \frac{11}{9}$. In this paper, we show that increasing g from 5 further on does not increase $F_{0,1}(g)$ much. Our constructions show that for every g, $F_{0,1}(g) \leq 1.25$. We also find exact values of $F_{j,k}(g)$ for all g and all $k \geq 2j+2$.

1 Introduction

A *proper k-coloring* of a graph G is a partition of $V(G)$ into k independent sets $V_1, \ldots, V_k$. A $(d_1, d_2, \cdots, d_k)$-*coloring* of a graph G is a partition of $V(G)$ into sets $V_1, V_2, \cdots, V_k$ such that for every $1 \leq i \leq k$, the subgraph $G[V_i]$ of G induced by V_i has maximum degree at most d_i. If $d_1 = \ldots = d_k = 0$, then a $(d_1, d_2, \cdots, d_k)$-coloring is simply a proper k-coloring. If at least one of the d_i is positive, then a $(d_1, d_2, \cdots, d_k)$-coloring is called *improper* or *defective*. Several papers on improper colorings of planar graphs with restrictions on girth and of sparse graphs have appeared.

In [11] and this paper we consider improper colorings with just two colors, the (j,k)-colorings. Even such colorings are not simple if $(j,k) \neq (0,0)$. In particular, Esperet, Montassier, Ochem and Pinlou [8] proved that the problem of verifying whether a given planar graph of girth 9 has a $(0,1)$-coloring is NP-complete. Since the problem is hard, it is natural to consider related extremal problems.

The maximum average degree, $\mathrm{mad}(G)$, of a graph G is the maximum of $\frac{2|E(H)|}{|V(H)|}$ over all subgraphs H of G. It measures sparseness of G. Kurek and Ruciński [12] called graphs with low maximum average degree *globally sparse*. In particular,

$$\text{if } G \text{ is a planar graph of girth } g, \text{ then } \mathrm{mad}(G) < \tfrac{2g}{g-2}. \tag{1}$$

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We will use the following slight refinement of the notion of $\mathrm{mad}(G)$. For $a, b \in \mathbf{R}$, a graph G is (a,b)-*sparse* if $|E(H)| < a|V(H)| + b$ for all $H \subseteq G$. For example, every forest is $(1,0)$-sparse, and every graph G with $\mathrm{mad}(G) < a$ is $(a/2, 0)$-sparse. We also say that G is *almost* (a,b)-*sparse* if $|E(G)| = a|V(G)| + b$ and $|E(H)| < a|V(H)| + b$ for all $H \subsetneq G$. For example, every connected k-regular connected graph G is almost $(k/2, 0)$-sparse. Note that every almost (a,b)-sparse graph is (a,b')-sparse for all $b' > b$. Almost (a,b)-sparse graphs could be considered as critical: they become (a,b)-sparse after deleting any edge.

Glebov and Zambalaeva [9] proved that every planar graph G with girth at least 16 is $(0,1)$-colorable. Then Borodin and Ivanova [1] proved that every graph G with $\mathrm{mad}(G) < \frac{7}{3}$ is $(0,1)$-colorable. By (1), this implies that every planar graph G with girth at least 14 is $(0,1)$-colorable. Borodin and Kostochka [2] proved that every graph G with $\mathrm{mad}(G) < \frac{12}{5}$ is $(0,1)$-colorable, and this is sharp. This implies that every planar graph G with girth at least 12 is $(0,1)$-colorable. As mentioned above, Esperet et al. [8] proved that the problem of verifying whether a given planar graph of girth 9 has a $(0,1)$-coloring is NP-complete. Dorbec, Kaiser, Montassier, and Raspaud [6] mention that because of these results, the remaining open question is whether all planar graphs with girth 10 or 11 are $(0,1)$-colorable. Our results in [11] yield the positive answer for planar graphs with girth 11.

In [11] and this paper, instead of considering planar graphs with given girth, we consider graphs with given girth that are (a,b)-sparse for small a. A recent result by Borodin and Kostochka [3] can be stated in the language of (a,b)-sparse graphs as follows.

Theorem 1.1 ([3])**.** *Let $k \geq 2j+2$ and G be a graph. If G is $\left(2 - \frac{k+2}{(j+2)(k+1)}, \frac{1}{k+1}\right)$-sparse, then it is (j,k)-colorable. Moreover, the result is sharp in the sense that there are infinitely many almost $\left(2 - \frac{k+2}{(j+2)(k+1)}, \frac{1}{k+1}\right)$-sparse graphs that are not (j,k)-colorable.*

Our first result gives triangle-free sharpness examples for Theorem 1.1.

Theorem 1.2. *Let $j \geq 0$ and $k \geq j+1$. Then there are infinitely many triangle-free almost $\left(2 - \frac{k+2}{(j+2)(k+1)}, \frac{1}{k+1}\right)$-sparse graphs that are not (j,k)-colorable. Furthermore, for every $k \geq 1$, there are infinitely many almost $\left(2 - \frac{k+2}{2(k+1)}, \frac{1}{k+1}\right)$-sparse graphs of girth 5 that are not $(0,k)$-colorable.*

When $k \geq 2j+2$, the graphs we construct in Theorem 1.2 are (j,k)-*critical* in the sense that each proper subgraph of every such graph is (j,k)-colorable by Theorem 1.1 but the graphs themselves are not.

Let $F_{j,k}(g)$ denote the supremum of all positive a such that there is some (possibly negative) b with the property that every (a,b)-sparse graph G with girth g is (j,k)-colorable. The above mentioned result in [2] implies $F_{0,1}(3) = \frac{12}{5} = 1.2$. In [11] we prove the exact result that $F_{0,1}(4) = F_{0,1}(5) = \frac{11}{9}$ and also find the best possible value of b. In this paper we extend this result in two directions: to large girth and to (j,k)-colorings instead of $(0,1)$-colorings.

Since $F_{0,0}(4)$ and $F_{0,1}(4)$ are already known, with Theorem 1.2 we have the values of $F_{0,k}(4)$ for all $k \geq 0$.

Our second result concerns graphs with large girth.

Theorem 1.3. *For all $k \geq j \geq 0$ and $g \geq 3$, $F_{j,k}(g) \leq 2 - \frac{(k+2)}{(j+2)(k+1)}$.*

So, we have $F_{0,1}(3) = 1.2$, $F_{0,1}(4) = F_{0,1}(5) = \frac{11}{9} = 1.222\ldots$, $F_{0,1}(g) \leq 1.25$ for all g, and if $k \geq 2j+2$ then $F_{j,k}(g) = 2 - \frac{(k+2)}{(j+2)(k+1)}$ for all g.

Remark. The case $j = k$ seems to be quite different. Apart from the trivial equality $F_{0,0}(g) = 1$, the only known to us exact result is $F_{1,1}(3) = \frac{7}{5}$ [4]. The value $\frac{7}{5}$ does not fit the formula in Theorem 1.1 and differs from the lower bound by Havet and Sereni in [10]. Even $F_{2,2}(3)$ is not known.

2 On (j,k)-coloring of triangle-free graphs

For a graph G and $W \subseteq V(G)$, $0 \leq j \leq k$, let *the (j,k)-potential of W in G* be defined as

$$\phi(W,G) = \phi_{j,k}(W,G) = (2 - \frac{k+2}{(j+2)(k+1)})|W| - |E(G[W])|.$$

(We will drop the subscripts j,k and G if they are clear from the context.)

Note that for a graph G, the condition

$$\phi_{j,k}(W,G) > -\tfrac{1}{k+1} \text{ for all } W \subseteq V(G), \tag{2}$$

is equivalent to the statement that G is $\left(2 - \frac{k+2}{(j+2)(k+1)}, \frac{1}{k+1}\right)$-sparse.

In this section we prove Theorem 1.2, i.e. we show that for all $k \geq j+1$ there are infinitely many triangle-free graphs G with $\phi_{j,k}(W,G) \geq -\frac{1}{k+1}$ for all $W \subseteq V(G)$, but not (j,k)-colorable. We also show that for all $k \geq 2$ there are infinitely many graphs G of girth 5 with $\phi_{0,k}(W,G) \geq -\frac{1}{k+1}$ for all $W \subseteq V(G)$, and not $(0,k)$-colorable. Together with Theorem 1.1, this means that for all $k \geq 2j+2$, $F_{j,k}(4) = F_{j,k}(3)$. Recall that this is not the case for $(j,k) = (0,1)$ by our result in [11].

For $j \neq k$, we consider a (j,k)-coloring of a graph G as a 2-coloring of $V(G)$ with color j and color k such that the vertices of color j (respectively, k) induce a subgraph with maximum degree at most j (respectively, k). We remark that this convention does not apply to the case $j = k$.

Let graph $L(j,k)$ be defined as follows. Let

$$V(L(j,k)) = \{x,w\} \cup \{u_1,\ldots,u_{j+1}\} \cup \bigcup_{i=1}^{k+1} \{y_{i,1},\ldots,y_{i,j+1},y_i\}.$$

Vertex x is adjacent to all vertices in $\{u_1,\ldots,u_{j+1}\} \cup \{y_1,\ldots,y_{k+1}\}$, vertex w is adjacent to all vertices in $\{u_1,\ldots,u_{j+1}\} \cup \bigcup_{i=1}^{k+1}\{y_{i,1},\ldots,y_{i,j+1}\}$, for every $i \in [1,k+1]$, vertex y_i is adjacent to all vertices in $\{y_{i,1},\ldots,y_{i,j+1}\}$, and there are no other edges (See Fig 1). We will call x *the base* and w *the top* of $L(j,k)$.

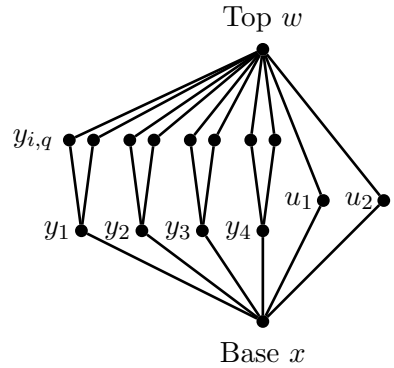

Figure 1 : Graph $L(1,3)$.

By construction, $L(j,k)$ is triangle-free and $L(0,k)$ has girth 5. We need the following simple property of $L(j,k)$.

Claim 2.1. *In every (j,k)-coloring f of $L(j,k)$, x has a neighbor of color k.*

Proof. Suppose $f(y_1) = \ldots = f(y_{k+1}) = f(u_1) = \ldots = f(u_{j+1}) = j$. Then for every $1 \le i \le k+1$ at least one of $y_{i,1}, \ldots, y_{i,j+1}$ must be colored with k. So, w has at least $k+1$ neighbors of color k and $j+1$ neighbors $u_1, \ldots, u_{j+1}$ of color j, a contradiction to the definition of (j,k)-coloring. $\square$

A (j,k)-*flag* in a graph G is a pendant block isomorphic to $L(j,k)$ whose unique cut vertex is the base vertex x in $L(j,k)$. Claim 2.1 immediately implies the following.

Claim 2.2. *In every (j,k)-coloring f of a graph G, for any $x \in V(G)$,*
(a) if x is the base of $k+1$ distinct (j,k)-flags, then $f(x) = j$;
(b) if x is the base of k distinct (j,k)-flags and $f(x) = k$, then x has no neighbors of color k outside of these k blocks.

Another helpful property of (j,k)-flags is that they are sparse:

Claim 2.3. *Let graph G consist of q distinct (j,k)-flags, $W_1, W_2, \cdots, W_q$, with a common base x, and for $i = 1, \ldots, q$, let w_i be the top of W_i.*
(a) If $\emptyset \ne W \subseteq W_i$, then $\phi(W) \ge \phi(\{x\}) - \frac{1}{k+1}$, and equality holds only for $W = W_i$.
(b) If $\emptyset \ne W \subseteq V(G)$, then $\phi(W) \ge \phi(\{x\}) - \frac{q}{k+1}$ and equality holds only for $W = V(G)$.

Proof. To prove (a), choose among the nonempty subsets of W_i a set W of the smallest potential $\phi(W)$. Since deleting an isolated or pendant vertex from a set decreases the potential and the claim holds for a 1-element W, we may assume

$$\delta(G[W]) \ge 2. \tag{3}$$

If $\emptyset \ne W \subset W_i$ and $w_i \notin W$, then W induces a forest, a contradiction to (3). So $w_i \in W$.

Since adding to a set U of vertices a vertex with at least two neighbors in U decreases the potential, by (3),

$$\begin{gathered}\text{for all } 1 \le h \le j+1 \text{ and } 1 \le h' \le k+1,\ u_h \in W \text{ if and only if } x \in W \\ \text{and } y_{h',h} \in W \text{ if and only if } y_{h'} \in W.\end{gathered} \tag{4}$$

Suppose $x \notin W$. Then by (4), $W \cap \{u_1, \ldots, u_{j+1}\} = \emptyset$. Also, if in this case $y_h \in W$ then by (4), all $y_{h,1}, \ldots, y_{h,j+1}$ are in W and

$$\phi(W) - \phi(W - \{y_{h,1}, \ldots, y_{h,j+1}, y_h\}) \geq \left(2 - \frac{k+2}{(j+2)(k+1)}\right)(j+2) - (2j+2) = \frac{k}{k+1},$$

a contradiction to the choice of W. Thus $x \in W$. Then by (4), $\{u_1, \ldots, u_{j+1}\} \subset W$. Also adding each y_h together with $y_{h,1}, \ldots, y_{h,j+1}$ decreases the potential by exactly $\frac{1}{k+1}$. So, the unique subset of W_i with the minimum possible potential is W_i itself and

$$\phi(W_i) - \phi(\{x\}) = \left(2 - \frac{k+2}{(j+2)(k+1)}\right)(|W_i| - 1) - |E(G[W_i])|$$

$$= \left(2 - \frac{k+2}{(j+2)(k+1)}\right)(j+2)(k+2) - ((2j+3)(k+2) - 1) = -\frac{1}{k+1},$$

as claimed. This proves (a).

To prove (b), suppose that W intersects exactly $r > 0$ of $W_1, \ldots, W_q$. If $x \notin W$, then

$$\phi(W) = \sum_{i=1}^{q} \phi(W \cap W_i) > r(\phi(\{x\}) - \frac{1}{k+1}) \geq \phi(\{x\}) - \frac{r}{k+1}.$$

If $x \in W$, then $r = q$ and

$$\phi(W) = \sum_{i=1}^{q} \phi(W \cap W_i) - (q-1)\phi(\{x\}) \geq \phi(\{x\}) - \frac{q}{k+1}. \tag{5}$$

By (a), equality in (5) holds only when $W \cap W_i = W_i$ for all i, which means $W = V(G)$. $\square$

Basic construction. We construct a graph $H_0 = H_0(j,k)$ from a star $K_{1,j+1}$ with the center x_0 and leaves $x_1, \ldots, x_{j+1}$ by adding $k+1$ (j,k)-flags to each of $x_0, x_1, \ldots, x_{j+1}$. (When we say "add (j,k)-flags to a vertex x", we mean that x will be the base of the added flags.)

By construction, $H_0(j,k)$ is triangle-free and $H_0(0,k)$ has girth 5. If H_0 has a (j,k)-coloring f, then by Claim 2.2(a), $f(x_0) = \ldots = f(x_{j+1}) = j$, and vertex x_0 of color j has $j+1$ neighbors $x_1, \ldots, x_{j+1}$ of color j, a contradiction. Thus

$$H_0 \text{ is not } (j,k)\text{-colorable.} \tag{6}$$

Now we want to prove that H_0 satisfies (2).

Claim 2.4. *If $W \subseteq V(H_0)$, then $\phi(W) \geq -\frac{1}{k+1}$, and equality holds only for $W = V(H_0)$.*

Proof. Choose a largest $W \subset V(H_0)$ among the sets with minimum $\phi(W)$. As in the proof of Claim 2.3, $\delta(H_0[W]) \geq 2$. By Claim 2.3(a), if L is any (j,k)-flag in H_0 and $W \cap L \neq \emptyset$, then $L \subseteq W$ otherwise $\phi(W \cup L) < \phi(W)$.

It follows that if we know which vertices in $X = \{x_0, \ldots, x_{j+1}\}$ are in W, then we know W. Similarly, if $x_0 \in W$ and $x_i \notin W$ for some i, then by Claim 2.3(b), adding to W vertex x_i and all the $k+1$ (j,k)-flags containing x_i we get a set W' with

$$\phi(W') \leq \phi(W) + \phi(\{x_i\}) - \frac{k+1}{k+1} - 1 < \phi(W),$$

a contradiction to the minimality of $\phi(W)$. So, $W = V(H_0)$ is the unique set of minimum potential among the sets containing x_0.

If $x_0 \notin W$, then every component of $H_0[W]$ is a subgraph of a graph G described in Claim 2.3 and so has a nonnegative potential. So in this case $\phi(W) \geq 0$. $\square$

Thus H_0 is the first in the series of examples proving Theorem 1.2.

In order to generalize H_0, we need one more notion. A vertex v in a graph G is a *remote* (j,k)*-base* if it is the base of $k+1$ (j,k)-flags $W_1, \ldots, W_{k+1}$ in G and has exactly one neighbor outside of $W_1 \cup \ldots \cup W_{k+1}$. This unique neighbor of v will be called *the main neighbor* of v.

Claim 2.5. *Suppose a graph H has no (j,k)-colorings, and $v \in V(H)$ is a remote (j,k)-base contained in (j,k)-flags $W_1, \ldots, W_{k+1}$ with the main neighbor x.*
(a) for any (j,k)-coloring f' of $H' = H - (W_1 - v)$ (if it exists), $f'(v) = k$ and v has k neighbors of color k in H';
(b) for any (j,k)-coloring f'' of $H'' = H - \bigcup_{i=1}^{k+1} W_i$ (if it exists), $f''(x) = j$ and x has j neighbors of color j in H''.

Proof. If H' has a (j,k)-coloring f' with $f'(v) = j$, then f' can be extended to W_1 by coloring all neighbors of v in W_1 and the top vertex of W_1 with k and the remaining vertices with j. But H has no (j,k)-colorings. Thus if a (j,k)-coloring f' of H' exists, then $f'(v) = k$, and by Claim 2.1 each of $W_2, \ldots, W_{k+1}$ contains a neighbor of v of color k. This proves (a).

Similarly, if H'' has a (j,k)-coloring f'' with either $f''(x) = k$ or with $f''(x) = j$ and at most $j-1$ neighbors of color j, then we can extend f'' to the whole H by letting $f''(v) = j$, coloring all its neighbors in $W_1 \cup \ldots \cup W_{k+1}$ and the tops of $W_1, \ldots, W_{k+1}$ with k, and the remaining vertices in $W_1 \cup \ldots \cup W_{k+1}$ with j. $\square$

General construction. Recall that $H = H_0$ has the following properties:
(P1) H is not (j,k)-colorable;
(P2) H has no triangles and if $j = 0$, then H has girth 5;
(P3) $\phi(W) \geq -\frac{1}{k+1}$ for each $W \subseteq V(H)$, and equality holds only for $W = V(H)$;
(P4) H has at least two remote bases (if $j = 0$, then x_0 also is a remote base in $H_0(0,k)$).

We now show how to use a graph H satisfying (P1)–(P4) to construct a larger graph satisfying (P1)–(P4). Take two copies, H_1 and H_2 of H. For $h = 1, 2$, choose in H_h a remote base v_h contained in (j,k)-flags $W_{h,1}, \ldots, W_{h,k+1}$ with the main neighbor x_h. Let $H' = H_1 - (W_{1,1} - v_1)$ and $H'' = H_2 - \bigcup_{i=1}^{k+1} W_{2,i}$. We get the new graph $\widetilde{H}$ by adding to $H' \cup H''$ a new vertex z adjacent to v_1 in $V(H')$ and to x_2 in $V(H'')$.

Property (P2) for $\widetilde{H}$ directly follows from (P2) for H_1 and H_2. Since $H_1 \cup H_2$ had at least four remote bases and we destroyed only two of them when creating H' and H'', (P4) holds for $\widetilde{H}$.

Suppose $\widetilde{H}$ has a (j,k)-coloring f. Then by Claim 2.5(a), $f(v_1) = k$ and v_1 has k neighbors of color k in $V(H')$. Thus we need $f(z) = j$. But by Claim 2.5(b), $f(x_2) = j$ and x_2 has j neighbors of color j in $V(H'')$. This contradiction proves (P1) for $\widetilde{H}$.

To prove (P3), consider a set W of minimum potential in $\widetilde{H}$. If $z \notin W$, then by (P3) for H, $\phi(W) = \phi(W \cap V(H')) + \phi(W \cap V(H'')) \geq 0 + 0 = 0$ since each of $W \cap V(H')$ and $W \cap V(H'')$ is proper subset of $V(H')$ and $V(H'')$ respectively. Suppose $z \in W$. Then, similarly to (3), $v_1, x_2 \in W$.

Let $W' = W \cap V(H')$ and $W'' = W \cap V(H'')$. Since adding to W'' vertex v_2 together with all $k+1$ (j,k)-flags containing v_2 would decrease the potential of W'' by $\frac{k+2}{(j+2)(k+1)}$, we conclude that $\phi(W'') \geq \frac{k+2}{(j+2)(k+1)} - \frac{1}{k+1}$ with equality only when $W'' = V(H'')$. Similarly, $\phi(W') \geq 0$ with equality only when $W' = V(H')$. Thus

$$\phi(W) \geq \phi(W') + \phi(W'') + \phi(\{z\}) - 2 \geq 0 + \frac{k+2}{(j+2)(k+1)} - \frac{1}{k+1} + (2 - \frac{k+2}{(j+2)(k+1)}) - 2 \geq \frac{-1}{k+1},$$

with equality only when $W = V(\widetilde{H})$.

This construction yields Theorem 1.2.

3 On (j,k)-coloring of graphs with large girth

In this section, we prove Theorem 1.3.

First, we inductively define the tree $T'_d(j,k)$ which will be a gadget to construct graphs we want. For $i = 0, 1, \ldots, k$, let S_i be a copy of the star $K_{1,j+1}$ with the center c_i. We subdivide each of the $j+1$ edges of each star S_i once and add edges c_0c_i for $i = 1, 2, 3, \cdots, k$. The resulting tree is $T_1(j,k)$ and c_0 is called *the center of* $T_1(j,k)$. Note that $T_1(j,k)$ has $(k+1)(j+1)$ leaves. Assume we already have defined the tree $T_{d-1}(j,k)$ and it has $(k+1)^{d-1}(j+1)^{d-1}$ leaves. Let T^0 be a copy of $T_1(j,k)$ with the center c_0 and $T^1, \ldots, T^{(k+1)(j+1)}$ be disjoint copies of $T_{d-1}(j,k)$ with the centers $c_1, \ldots, c_{(k+1)(j+1)}$. Let $x_1, \ldots, x_{(k+1)(j+1)}$ be the leaves of T^0. The tree $T_d(j,k)$ with *the center* c_0 is obtained by gluing c_i with x_i for all $i = 1, \ldots, (k+1)(j+1)$. Finally, the tree $T'_d(j,k)$ is obtained from two disjoint copies of $T_d(j,k)$ by adding an edge connecting their centers. The example of $T'_1(2,3)$ is in Fig. 2.

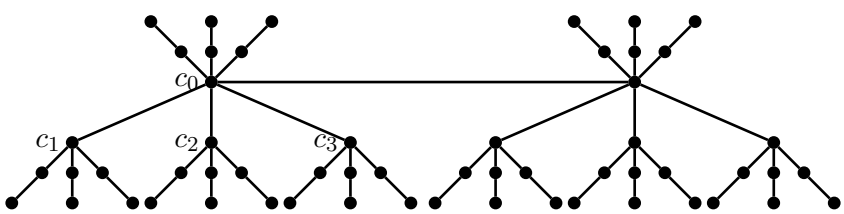

Figure 2 : $T'_1(2,3)$.

Claim 3.1. *For $d \geq 1$, let f be a (j,k)-coloring of $T_d(j,k)$ with the center c_0 such that every neighbor of a leaf has color j. Then $f(c_0) = k$ and c_0 has k neighbors of color k.*

Proof. We use induction on d.

Let L be the set of all leaves of $T_1(j,k)$. If all the neighbors of L are all colored with the color j, then each of the remaining non-leaf vertices is adjacent to $j+1$ vertices of color j, and thus has color k. These vertices form a star $K_{1,k}$ with the center c_0, which yields the claim for $d = 1$.

Assume the statement holds for $d-1$. Let $T^0, T^1, \ldots, T^{(k+1)(j+1)}$ be the trees from the definition of $T_d(j,k)$ and $c_0, c_1, \ldots, c_{(k+1)(j+1)}$ be their centers. Let f be a (j,k)-coloring of $T_d(j,k)$ such that every neighbor of a leaf has color j. By the induction assumption, for each $i = 1, \ldots, (k+1)(j+1)$, $f(c_i) = k$ and c_i has k neighbors of color k in T^i. It follows that the neighbor of c_i in T^0 has color j. Again by the induction assumption, the conclusion holds for c_0. $\square$

Claim 3.2. *For $k \geq j$ and $d \geq 1$, in every (j,k)-coloring of $T'_d(j,k)$, some neighbor of a leaf has color k.*

Proof. Tree $T'_d(j,k)$ contains two disjoint copies T_1 and T_2 of $T_d(j,k)$ with centers c_1, c_2 connected by edge c_1c_2. If f is a (j,k)-coloring of $T'_d(j,k)$ such that every neighbor of a leaf has color j, then by the Claim 3.1, for $i = 1, 2$ the center c_i of T^i has color k and has k neighbors of color k in T^i. Since c_1 and c_2 are adjacent, each of them has $k+1$ neighbors of the color k, a contradiction. □

Claim 3.3. *Let $k \geq j$. Let L be the set of leaves in $T_d(j,k)$ and $B = V(T_d(j,k)) - L$. Then for every subgraph T of $T_d(j,k)$,*

$$|E(T)| \leq (2 - \frac{(k+2)}{(j+2)(k+1)})|B \cap V(T)|. \tag{7}$$

Proof. First, suppose that $d = 1$. Recall that in this case, $B = C \cup D$, where D is the set of vertices of degree 2 adjacent to L, $|D| = |L| = (j+1)(k+1)$, $C = \{c_1, \ldots, c_{k+1}\}$ is the set of centers of the original stars, each c_i is adjacent to $j+1$ vertices in D, and in addition c_1 is adjacent to each vertex in $C - c_1$. Thus there are three types of edges: Type 1 — the edges connecting D with L, Type 2 — the edges connecting D with C, and Type 3 — the edges connecting c_1 with $C - c_1$. We will prove (7) using discharging. Let every $e \in E(T)$ have charge $ch(e) = 1$ so that $\sum_{e \in E(T)} ch(e) = |E(T)|$. Now each $e \in E(T)$ distributes its charge to its endvertices according to the following rules.

Rule 1: Each edge $d\ell$ of Type 1 gives all its charge to the end $d \in D$.

Rule 2: Each edge c_id of Type 2 gives charge $1 - \frac{(k+2)}{(j+2)(k+1)}$ to the end $d \in D$ and charge $\frac{(k+2)}{(j+2)(k+1)}$ to the end $c_i \in C$.

Rule 3: Each edge c_1c_i of Type 3 gives charge $\frac{k}{k+1}$ to $c_i \in C - c_1$ and charge $\frac{1}{k+1}$ to c_1.

By the rules, only vertices of $V(T) \cap B$ may receive a positive charge and total charge on them will be exactly $|E(T)|$. Thus it is enough to prove that for every $v \in V(T) \cap B$,

$$ch(v) \leq 2 - \frac{(k+2)}{(j+2)(k+1)}. \tag{8}$$

If $v \in D$, then it gets at most 1 by Rule 1 and at most $1 - \frac{(k+2)}{(j+2)(k+1)}$ by Rule 2, so (8) holds for v. If $v = c_i$ for some $2 \leq i \leq k+1$, then it gets at most $(j+1)\frac{(k+2)}{(j+2)(k+1)}$ by Rule 2 and at most $\frac{k}{k+1}$ by Rule 3, so

$$ch(v) \leq (j+1)\frac{(k+2)}{(j+2)(k+1)} + \frac{k}{k+1} = 2 - \frac{(k+2)}{(j+2)(k+1)}.$$

Finally, if $v = c_1$, then it again gets at most $(j+1)\frac{(k+2)}{(j+2)(k+1)}$ by Rule 2 and at most $k\frac{1}{k+1}$ by Rule 3, so again (8) holds for v. This proves Case $d = 1$.

Suppose now that $d \geq 2$. Then $T_d(j,k)$ is obtained from several copies of $T_1(j,k)$ by gluing leaves of some copies with the centers of some others. So if we do the discharging from $E(T)$ to $V(T) \cap B$ in each copy of $T_1(j,k)$ forming $T_d(j,k)$ by the Rules 1–3 above, then again only vertices of $V(T) \cap B$ may receive a positive charge and the total charge on them will be exactly $|E(T)|$. Moreover, since by Rule 1 the leaves of each copy of $T_1(j,k)$ will get zero charge from this copy, as we have checked above, (8) will hold for every $v \in V(T) \cap B$. This proves the claim. □

Proof of Theorem 1.3. Our goal is to show that for any $\epsilon > 0$, $g \geq 3$ and $k \geq j \geq 0$,

$$\text{there is an } (2 - \tfrac{(k+2)}{(j+2)(k+1)} + \epsilon, 0)\text{-sparse non-}(j,k)\text{-colorable graph } G \text{ of girth } g. \tag{9}$$

Recall that G is $(2 - \frac{(k+2)}{(j+2)(k+1)} + \epsilon, 0)$-sparse if and only if $\mathrm{mad}(G) < 4 - \frac{2(k+2)}{(j+2)(k+1)} + \epsilon$. We use induction on $j + k$. If $j = k = 0$, then any odd cycle of length at least g is almost $(1, 0)$-sparse and not $(0, 0)$-colorable. Assume that $k \geq 1$ and (9) is proved for all pairs (j', k') with $j' + k' < j + k$ and $j' \leq k'$.

CASE 1: $j < k$. Then there is a graph G_0 with girth g which is not $(j, k-1)$-colorable and with

$$\mathrm{mad}(G_0) < 4 - \frac{2(k+1)}{(j+2)k} + 2\epsilon \leq 4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon. \tag{10}$$

Let $V(G_0) = \{v_1, v_2, \cdots, v_n\}$. Fix an integer $d > \frac{1}{\epsilon}$. Let M be the number of leaves in $T'_d(j, k)$. By an old result of Erdős and Hajnal [7], there exists a non-n-colorable nM-uniform hypergraph H with girth g. We construct our graph G using H and many copies of G_0 and $T'_d(j, k)$ as follows:

(i) Partition each $e \in E(H)$ into n subsets $e_1, \ldots, e_n$ of size M;

(ii) Replace each vertex x in H with a copy $G_0(x)$ of G_0;

(iii) For each $e \in H$ and $1 \leq i \leq n$, if $e_i = \{x_1, \ldots, x_M\}$, we take a copy $T(e, i)$ of $T'_d(j, k)$ with the set of leaves, say, $L(e, i) = \{\ell_1, \ldots, \ell_M\}$ and for $h = 1, \ldots, M$, glue ℓ_h with the vertex v_i in the copy $G_0(x_h)$ of G_0. We will say that $T(e, 1), \ldots, T(e, n)$ *belong to* e and denote $B(e, i) = V(T(e, i)) - L(e, i)$.

Let us check that the obtained graph G has the properties we need: (a) the girth of G is at least g, (b) G is not (j, k)-colorable, and (c) $\mathrm{mad}(G) < 4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon$.

For an edge $e \in E(H)$, let $G(e)$ denote the subgraph of G formed by the copies $G_0(x)$ of G_0 for all nM vertices $x \in e$ plus all the copies $T(e, i)$ of $T'_d(j, k)$ for $i = 1, \ldots, n$. If G has a cycle C of length less than g, then C is not contained in a copy of G_0 since G_0 has girth g. Moreover, then C is not contained in any $G(e)$, since all edges of $G(e)$ in $\bigcup_{i=1}^{n} T(e, i)$ are cut-edges in $G(e)$. Since H is a linear hypergraph, C yields a (hypergraph) cycle in H, and any such cycle has at least g edges, a contradiction to the choice of C. This proves (a).

Suppose we have a (j, k)-coloring f of G. Since G_0 is not $(j, k-1)$-colorable, each graph $G_0(x)$ has a vertex v_i of color k with k neighbors in $G_0(x)$ of color k in f. Let $i(x)$ be the minimum i such that $G_0(x)$ has a vertex v_i of color k with k neighbors in $G_0(x)$ of color k in f. We define a coloring ϕ of H as follows: for each $x \in V(H)$, let $\phi(x) = i(x)$. Then ϕ is an n-coloring of H, and H has no proper n-colorings. Thus there is a monochromatic $e \in E(H)$. Suppose $f(x) = i$ for each $x \in e$. By construction, all the leaves of the copy $T(e, i)$ of $T'_d(j, k)$ are in e_i; each of these leaves is of color k and has k neighbors of color k in $\bigcup_{x \in e_i} G_0(x)$. Thus none of these leaves has a neighbor of color k in $T(e, i)$. This contradicts Claim 3.2. Thus (b) holds.

In order to prove (c), consider some $W \subseteq V(G)$ with the largest $\frac{|E(G[W])|}{|W|}$. If this ratio is at most 1, then (c) holds; otherwise by the maximality of the average degree, $G[W]$ has no isolated vertices and no leaves. Let $W' = \bigcup_{x \in V(H)} (W \cap V(G_0(x)))$. Then $W - W' = \bigcup_{e \in E(H)} \bigcup_{i=1}^{n} (W \cap B(e, i))$. Since each component of $G[W']$ is contained in some $G_0(x)$, by (10), the average degree of $G[W']$ is less than $4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon$. We can obtain W from W' by a sequence of adding the sets $W \cap B(e, i)$, one by one. We will show that after every such step,

$$\text{the average degree of the obtained subgraph remains less than } 4 - \tfrac{2(k+2)}{(j+2)(k+1)} + 2\epsilon. \tag{11}$$

Indeed, suppose it is the turn to add to a current set W'' the set $W \cap B(e,i)$. Let c_1c_1' be the edge in $T(e,i)$ connecting the centers c_1 and c_1' of the two disjoint copies of $T_d(j,k)$. If $\{c_1, c_1'\} \not\subset W$, then by Claim 3.3, adding $W \cap B(e,i)$ to W'' adds at most $(2 - \frac{(k+2)}{(j+2)(k+1)})|W \cap B(e,i)|$ edges, as claimed. So let $\{c_1, c_1'\} \subset W$. Since $G[W]$ has no leaves, W contains the vertices of disjoint paths from c_1 and c_1' to $L(e,i)$ and thus $|W \cap B(e,i)| \geq 6d$. Again by Claim 3.3, adding $W \cap B(e,i)$ to W'' adds at most $1 + (2 - \frac{(k+2)}{(j+2)(k+1)})|W \cap B(e,i)|$ edges. Since $d > 1/\epsilon$ and $|W \cap B(e,i)| \geq 6d$, the last expression is less than $(2 - \frac{(k+2)}{(j+2)(k+1)} + \epsilon)|W \cap B(e,i)|$, as claimed. This proves (c).

CASE 2: $0 < j = k$. Then there is a graph G_0 with girth g which is not $(k-1,k)$-colorable and with

$$\mathrm{mad}(G_0) < 4 - \frac{2(k+2)}{(k+1)^2} + 2\epsilon \leq 4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon. \tag{12}$$

Now we simply repeat the proof of Case 1 with the only twist that using $j = k$, we consider G_0 as not $(k, k-1)$-colorable instead of not $(k-1,k)$-colorable. $\square$

Concluding remark. Studying improper colorings with more colors, one can consider the function $F_{a_1,a_2,\cdots,a_t}(g)$ generalizing $F_{j,k}(g)$. Using similar techniques, we can prove the following extension of Theorem 1.3.

Theorem 3.4. *Let $a_1 \leq a_2 \leq \cdots \leq a_t$, $t \geq 2$ and $g \geq 3$. Then $F_{a_1,a_2,\cdots,a_t}(g) \leq t - \frac{(a_2+2)}{(a_1+2)(a_2+1)}$.*

Since we do not know how sharp is this bound, we do not supply a proof of Theorem 3.4.

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