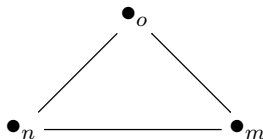


1 Eulerian Circuits in Directed Graphs

Recall that an *Eulerian circuit* in a graph G is a circuit that contains every edge exactly once.

Example 1.1. Consider the graph below.



The path $n - o - m - n$ is an Eulerian Circuit.

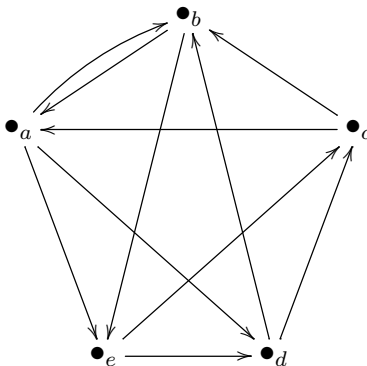
Theorem 1.2. A graph G has an Eulerian circuit if and only if it is connected and its vertices all have even valence.

Definition 1.3. A *directed graph* D is a graph with vertices V and edges E that are arrows. If uv is an edge in a directed graph, then u is the tail of the edge and v is the head. We say that uv is directed from u into v .



Example 1.4. Directed graph Z has vertex set $V = \{a, b, c, d, e\}$ and edge set $E = \{ab, ad, ae, ba, be, ca, cb, db, dc, ec, ed\}$.

Let's draw the graph!

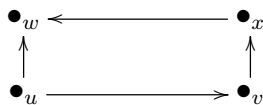


- Notice that ab and ba are distinct edges.
- Edge ab has a as its tail and b as its head.
- Notice that vertex a is the tail of three edges and the head of two edges. We say that vertex a has *out-valence* three and *in-valence* two in graph Z .

Definition 1.5. An *Eulerian circuit* in a directed graph is a circuit that

1. contains every edge exactly once and
2. only travels in the direction the edges are pointing.

Example 1.6. Consider the directed graph D .

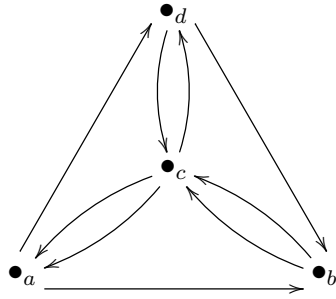


The circuit $u - v - x - w - u$ is not a valid Eulerian circuit in the graph D .

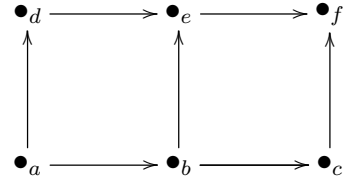
Exercises

1. Determine which of these directed graphs has an Eulerian circuit. If so, write an Eulerian circuit down. If not, explain why none exists.

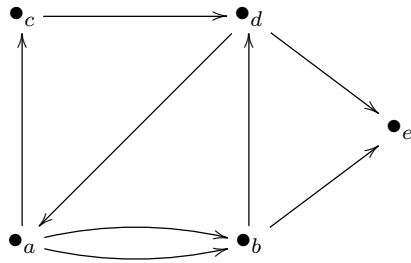
1.



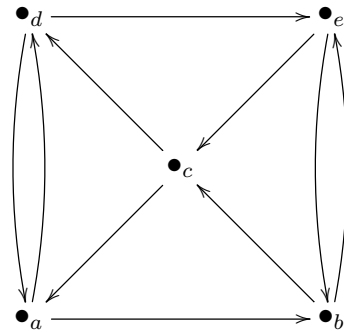
2.



3.



4.



2. Use your intuition from the problem above to complete the conjecture below.

Conjecture 1.7. *A directed graph D has an Eulerian circuit if and only if*

2 de Bruijn Sequences: An Application of Eulerian Circuits

Imagine you are a computer scientist needing to store the set of all length three binary words.

Exercise: What is the set of all length three binary words? Hint: there will be $2^3 = 8$ words.

$$\{111, 110, 101, 011, 001, 010, 100, 000\}.$$

You want to save as much room as possible on your computer. How could we store the words and minimize space used?

1. We could store the list as is, but we are using up 24 spots.
2. What if we layer pairs of words together?

$$\begin{array}{cccc} \begin{array}{c} 111 \\ \underbrace{1\ 1\ 1\ 0} \\ 110 \end{array} & , & \begin{array}{c} 101 \\ \underbrace{1\ 0\ 1\ 1} \\ 011 \end{array} & , & \begin{array}{c} 001 \\ \underbrace{0\ 0\ 1\ 0} \\ 010 \end{array} & , & \begin{array}{c} 100 \\ \underbrace{1\ 0\ 0\ 0} \\ 000 \end{array} \end{array}$$

This only uses up 16 spots! Could we do even better?

3. What if each word took one spot? The cyclic sequence

$$0\ 0\ 1\ 1\ 1\ 0\ 1\ 0$$

contains each length three binary word as a subword exactly once. This is the most compact list of this information within a string of characters.

$$\begin{array}{ccc} & 0 & \\ 0 & & 0 \\ 1 & \curvearrowright & 1 \\ 0 & & 1 \\ & 1 & \end{array}$$

The first three characters in the sequence form the word 001. If we shift our length three window by one, the characters in positions two through four form the word 011. By continuing this process we find 111, 110, 101, 010, 100, and 000. Note that the word 100 is formed by taking the last two digits in our sequence followed by the first digit. The word 000 is formed in a similar fashion by taking the last digit followed by the first two digits in our sequence. Thus, each length three binary word appears exactly once as a consecutive subsequence.

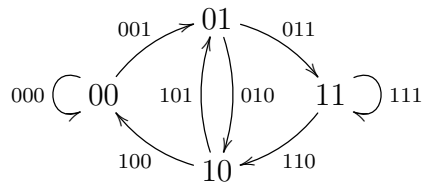
Definition 2.1. A *de Bruijn Sequence* is a cyclic sequence from the binary alphabet $\{0, 1\}$ such that every possible length k binary word appears as a consecutive subsequence exactly once.

A Historical Note: Dutch Mathematician Nicolaas Govert de Bruijn, for whom this type of universal cycle is named, proved certain results about what he called “circular arrangements of 2^n zeros and ones that show each n -letter word exactly once.” However, de Bruijn did not introduce the idea nor was he the first mathematician to find the number of solutions. De Bruijn attributes the problem of how many de Bruijn sequences exist for length n binary words to A. de Rivière. De Rivière proposed the question in the French journal “L’Intermédiaire des Mathématiciens” in 1894. The problem was promptly solved by C. Flye Sainte-Marie in the same year.

While Sainte-Marie implicitly proved many of the results that appear in de Bruijn’s proof, the explicit graph theoretic approach that we will employ wasn’t developed until the 1940’s. This method was developed simultaneously and independently by de Bruijn and British mathematician Irving John Good. Again, the relevant object is named after de Bruijn.

A de Bruijn graph D_k is a directed graph that represents the overlap between sequences of characters.

Example 2.2. The graph D_3 of all length 3 words on a binary alphabet.



Notice that all length three binary words appear as an edge in D_3 . This graph is Eulerian, meaning it contains a directed circuit that uses each edge exactly once. The de Bruijn sequence 00111010 corresponds to the Eulerian circuit 001, 011, 111, 110, 101, 010, 100, 000 in D_3 .

We need a way to identify an Eulerian graph G without having to find an Eulerian circuit of G .

Theorem 2.3. *A directed graph D has an Eulerian circuit if and only if*

1. *D is connected and*
2. *for each vertex, the in-valence is equal to the out-valence.*