Singular integrals on NA groups

joint work with Alessio Martini and Alessandro Ottazzi

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LMS Midlands Regional Meeting and Workshop on
Interactions of Harmonic Analysis and Operator Theory,
Birmingham, 13 - 16 September 2016

Singular integrals on $\mathbb{R}^n$

If $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a convolution operator with kernel $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ s.t.

$$\boxed{\sup_{y \neq 0} \int_{|x| \geq 2|y|} |k(x-y) - k(x)| \, \mathrm{d}x < \infty \ ,}$$

then T is of weak type $(1,1)$ and bounded on $L^p(\mathbb{R}^n)$, $p \in (1, \infty)$.

The key ingredient is the classical Calderón–Zygmund theory which relies on the doubling property, i.e.

$$V(2B) \leq C\, V(B) \qquad \text{for every ball } B\,.$$

Singular integrals on $\mathbb{R}^n$

TWO EXAMPLES: $\Delta = -\sum_{j=1}^n \partial_j^2 = \int_0^\infty \lambda \, \mathrm{d}E(\lambda)$

- Riesz transforms

  $R_j = \partial_j \circ \Delta^{-1/2} \qquad R_j f = c_n \, f * p.v. \frac{x_j}{|x|^{n+1}}$

  R_j is of weak type $(1,1)$ and bounded on $L^p(\mathbb{R}^n)$, $p \in (1,\infty)$

- Spectral multipliers of Mihlin-Hörmander type

  $F : \mathbb{R}^+ \to \mathbb{C}$ bounded

  $F(\Delta) = \int_0^\infty F(\lambda) \, \mathrm{d}E(\lambda) \qquad F(\Delta)f = f * \mathcal{F}^{-1}F(|\cdot|^2)$

  If $\sup_{t>0} \|F(t\cdot)\phi(\cdot)\|_{W_2^s} < \infty$ for some $s > \frac{n}{2}$ and $0 \neq \phi \in C_c^\infty(0,\infty)$,

  then $F(\Delta)$ is of weak type $(1,1)$ and bounded on $L^p(\mathbb{R}^n)$, $p \in (1,\infty)$

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  then $F(\Delta)$ is of weak type $(1,1)$ and bounded on $L^p(\mathbb{R}^n)$, $p \in (1,\infty)$

  (Ex.: $F \in C^\infty(\mathbb{R}^+)$ s.t. $|D^k F(\lambda)| \leq C_k \, \lambda^{-k} \quad \forall \lambda > 0, k \in \mathbb{N}$)

Singular integrals on Lie groups

G connected noncompact Lie group

μ right Haar measure

$X_0, \ldots, X_q$ left-invariant vector fields satisfying Hörmander's condition

$\Delta = -\sum_{j=0}^{q} X_j^2$ (sub)Laplacian s.a. on $L^2(G) = L^2(G,\mu)$
hypoelliptic and positive

QUESTIONS:

- boundedness of Riesz transforms $R_j = X_j \Delta^{-1/2}$
- boundedness of spectral multipliers of Δ, i.e. find conditions on

  $F : \mathbb{R}^+ \to \mathbb{C}$ bounded s.t.

  $F(\Delta) = \int_0^{+\infty} F(\lambda) \mathrm{d}E(\lambda)$ is bounded on $L^p(G)$ for some $p \neq 2$

- G Lie group of polynomial growth
  - R_j is of weak type $(1,1)$ and bounded on $L^p(G)$, $p \in (1,\infty)$
    [Christ-Geller (1984), Lohoué-Varopoulos (1985), Alexopoulos (1992)]
  - a Mihlin-Hörmander type theorem holds
    [Mauceri-Meda (1990), Christ (1991), Alexopoulos (1994),...]

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$\Rightarrow$ the classical CZ theory applies

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  In these cases G is unimodular and G is a doubling measured space
  $\Rightarrow$ the classical CZ theory applies

- $G = \mathbb{R}^{n-1} \rtimes \mathbb{R} \simeq \mathbb{H}^n(\mathbb{R})$

  $X_0, \ldots, X_{n-1}$ basis of $\mathfrak{g} = Lie(G)$ $\qquad \Delta = -\sum_{j=0}^{n-1} X_j^2$ complete Laplacian

  G is a nonunimodular group of exponential growth $\Rightarrow$ G is nondoubling $\Rightarrow$
  Hebisch and Steger (2003) developed a new CZ theory to prove that
  - R_j is of weak type $(1,1)$ and bounded on $L^p(G)$, $p \in (1,2]$
    (see also Sjögren (1999))
  - a Mihlin-Hörmander type theorem holds
    (see also Hebisch (1993), Cowling-Giulini-Hulanicki-Mauceri (1994))

NA groups

$G = N \rtimes A$

- N stratified group
  $Lie(N) = \mathfrak{n} = V_1 \oplus \cdots \oplus V_s \quad [V_i, V_j] \subseteq V_{i+j} \quad (V_k = 0 \text{ if } k > s)$
  $\delta_t \in Aut(\mathfrak{n})$ automorphic dilations s.t. $\delta_t|_{V_j} = t^j Id_{V_j} \quad \forall t > 0$
  $\det \delta_t = t^Q \qquad Q = \sum_{j=1}^s j \dim V_j$ homogeneous dimension of N
- $A = \mathbb{R}$ acting on N via automorphic dilations

$(z, u) \cdot (z', u') = (z \cdot \delta_{e^u} z', u + u') \qquad \forall z, z' \in N, u, u' \in \mathbb{R}$

$\mathrm{d}\mu(z, u) = \mathrm{d}z\,\mathrm{d}u$ right Haar measure

$m(z, u) = e^{-Qu}$ modular function

$E_0 = \partial_u$ basis of $\mathfrak{a} \simeq \mathbb{R} \qquad E_1, \ldots, E_q$ basis of the first layer V_1 of $\mathfrak{n}$

$X_0 = \partial_u \qquad X_j = e^u E_j \quad j = 1, \ldots, q$ left invariant v.f. on G

$\Delta = -\sum_{j=0}^q X_j^2$ subLaplacian

$G = N \rtimes A$

- N stratified group of homogeneous dimension Q
- $A = \mathbb{R}$ acting on N via automorphic dilations

THEOREM 1. [Martini-Ottazzi-V.]

$R_j = X_j \Delta^{-1/2}$ is of weak type $(1,1)$ and bounded on $L^p(G)$, $p \in (1,2]$.

THEOREM 2. [Martini-Ottazzi-V.]
If $F : \mathbb{R}^+ \to \mathbb{C}$ bounded s.t.

$$\sup_{0<t\leq 1} \|F(t\cdot)\phi(\cdot)\|_{W_2^{s_0}} < \infty \qquad s_0 > \frac{3}{2}$$

$$\sup_{t\geq 1} \|F(t\cdot)\phi(\cdot)\|_{W_2^{s_\infty}} < \infty \qquad s_\infty > \frac{Q+1}{2},$$

then $F(\Delta)$ is of weak type $(1,1)$ and bounded on $L^p(G)$, $p \in (1,\infty)$.

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REMARKS:

- when N is abelian, $G = N \rtimes A \simeq \mathbb{H}^n(\mathbb{R})$, Δ is a complete Laplacian related with the hyperbolic Laplacian: THEOREMS 1 and 2 were proved by Hebisch and Steger using spherical harmonic analysis and explicit formulae for the heat kernel
- if $F \in C_c(\mathbb{R}) \cap W_2^s(\mathbb{R})$ with $s > 2$, then $F(\Delta)$ is bounded on $L^1(G)$ [Mustapha (1998), Gnewuch (2002)]

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- if $F \in C_c(\mathbb{R}) \cap W_2^s(\mathbb{R})$ with $s > 2$, then $F(\Delta)$ is bounded on $L^1(G)$ [Mustapha (1998), Gnewuch (2002)]
- boundedness of R_j for $p > 2$?

Ingredients of the proofs

- $X_0, \ldots, X_q$ defines a sub-Riemannian structure on G with associated left-invariant Carnot-Carathéodory distance ϱ

  $$\cosh |x|_\varrho = \cosh \varrho(x, e_G) = \cosh u + e^u |z|_N^2/2 \qquad \forall x = (z, u) \in N \rtimes A\,,$$

  where $|\cdot|_N$ is the Carnot-Carathéodory distance on N
  - formula known when N is abelian and stated without proof for stratified N by Hebisch (1999)
  - solutions to Hamilton's equations on N can be re-parametrized and lifted to solutions to Hamilton's equations on G
  - length-minimizing curves need not be solutions to to Hamilton's equations . But points that can be joined to the identity via length-minimizing solution to Hamilton's equations are dense [Agrachev (2009)]
- $\mu(B_\varrho(e_G, r)) \lesssim \begin{cases} r^{Q+1} & r \leq 1 \\ e^{Qr} & r > 1 \end{cases} \quad \Rightarrow \quad (G, \varrho, \mu)$ is nondoubling

  We develop a Calderón–Zygmund theory on the nondoubling subRiemannian space (G, ϱ, μ)

Ingredients of the proofs

- Boundedness result for convolution operators on G

T right convolution operator bounded on $L^2(G)$, i.e., $Tf = f * k$
$\Rightarrow$ the integral kernel of T is $K(x,y) = k(y^{-1}x)m(y)$

If $k = \sum_{n\in\mathbb{Z}} k^n$ with

$$(i) \int_G |k^n(x)| \left(1 + 2^{-n/2}|x|_\varrho\right) \mathrm{d}\mu(x) \lesssim 1$$

$$(ii) \int_G |k^n(y^{-1}x)m(y) - k^n(x)| \,\mathrm{d}\mu(x) \lesssim 2^{-n/2}\,|y|_\varrho \quad \forall y \in G\,,$$

then T is of weak type $(1,1)$ and bounded on $L^p(G)$, $p \in (1,2]$.

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$$(i) \int_G |k^n(x)|\,(1 + 2^{-n/2}|x|_\varrho)\,\mathrm{d}\mu(x) \lesssim 1$$
$$(ii) \int_G |K^n(x,y) - K^n(x,e_G)|\,\mathrm{d}\mu(x) \lesssim 2^{-n/2}\,|y|_\varrho \quad \forall y \in G\,,$$

then T is of weak type $(1,1)$ and bounded on $L^p(G)$, $p \in (1,2]\,.$

Ingredients of the proofs

- Gradient estimates of the heat kernel h_t of Δ

LEMMA. For every $t > 0$

$$\boxed{\int_G |X_j h_t(x)| \, \mathrm{d}\mu(x) \leq C\, t^{-1/2}\,.}$$

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$$\boxed{\int_G |X_j h_t(x)| \, \mathrm{d}\mu(x) \leq C\, t^{-1/2} .}$$

It follows from the formula [Mustapha 1998, Gnewuch 2006]

$$h_t(z,u) = \int_0^\infty \Psi_t(\xi) \, \exp\left(-\frac{\cosh u}{\xi}\right) h^N_{e^u \xi/2}(z) \mathrm{d}\xi \, ,$$

where

$$\Psi_t(\xi) = \frac{\xi^{-2}}{\sqrt{4\pi^3 t}} \exp\left(\frac{\pi^2}{4t}\right) \int_0^\infty \sinh\theta \, \sin\frac{\pi\theta}{2t} \, \exp\left(-\frac{\theta^2}{4t} - \frac{\cosh\theta}{\xi}\right) \mathrm{d}\theta \, ,$$

estimates for the derivative of the heat kernel on N and various integration by parts (delicate cancellation occurs)

Ingredients of the proofs

- Gradient estimates of the heat kernel h_t of Δ

LEMMA. For every $\varepsilon \geq 0$ and $t > 0$

$$\boxed{\int_G |X_j h_t(x)| \exp(\varepsilon t^{-1/2} |x|_\varrho)\, \mathrm{d}\mu(x) \lesssim C_\varepsilon\, t^{-1/2}\,.}$$

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Sketch of the proof of THEOREM 1

$$R_j = X_j \Delta^{-1/2} = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} X_j e^{-t\Delta}\, \mathrm{d}t$$

$$k_j = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} X_j h_t \mathrm{d}t = \sum_{n \in \mathbb{Z}} k_j^n$$

$$\boxed{k_j^n = \tfrac{1}{\Gamma(1/2)} \int_{2^n}^{2^{n+1}} t^{-1/2} X_j h_t \, \mathrm{d}t}$$

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$$\begin{aligned}
&\int_G |k_j^n(x)| \left(1 + 2^{-n/2} |x|_\varrho\right) \mathrm{d}\mu(x) \\
&\lesssim \int_{2^n}^{2^{n+1}} t^{-1/2} \int_G |X_j h_t(x)| \left(1 + t^{-1/2} |x|_\varrho\right) \mathrm{d}\mu(x) \mathrm{d}t \\
&\lesssim \int_{2^n}^{2^{n+1}} t^{-1/2}\, t^{-1/2} \mathrm{d}t \lesssim 1\,.
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Using

$$h_t = h_{t/2} * h_{t/2} \qquad X_j h_t = h_{t/2} * X_j h_{t/2} \qquad h_{t/2}(x^{-1}) = h_{t/2}(x)m(x^{-1})$$

$$\begin{aligned}
&\int_G |k_j^n(y^{-1}x)m(y) - k_j^n(x)|\,\mathrm{d}\mu(x)\\
&\lesssim \int_{2^n}^{2^{n+1}} t^{-1/2}\int_G |X_j h_t(y^{-1}x)m(y) - X_j h_t(x)|\,\mathrm{d}\mu(x)\mathrm{d}t\\
&\lesssim |y|_\varrho \int_{2^n}^{2^{n+1}} t^{-1/2}\sum_{i=0}^{q} \|X_i h_{t/2}\|_1 \|X_j h_{t/2}\|_1 \mathrm{d}t\\
&\lesssim |y|_\varrho \int_{2^n}^{2^{n+1}} t^{-1/2} t^{-1/2}\, t^{-1/2}\mathrm{d}t \lesssim 2^{-n/2}|y|_\varrho\,.
\end{aligned}$$

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$$\boxed{k_j^n = \tfrac{1}{\Gamma(1/2)} \int_{2^n}^{2^{n+1}} t^{-1/2} X_j h_t\, \mathrm{d}t}$$

$\Rightarrow$ k_j^n satisfy the integral estimates of the boundedness theorem for convolution operators

$\Rightarrow$ R_j is of weak type $(1,1)$ and bounded on $L^p(G)$, $p \in (1,2]$

Related problems

- we also introduce a Hardy type space $H^1(G)$ and prove an endpoint result on $H^1(G)$ both for Riesz transforms and MH spectral multipliers of Δ
- $R_j : L^p(G) \to L^p(G) \quad p \in (2, \infty)$?
  $$\Updownarrow$$
  $R_j^* = -\Delta^{-1/2} X_j : L^p(G) \to L^p(G) \quad p \in (1, 2)$?
  - if $G = \mathbb{R} \rtimes \mathbb{R}$:
    $R_1^* : L^p(G) \to L^p(G) \quad p \in (1, 2)$ and is of weak type $(1, 1)$
    [Gaudry-Sjögren (1999)]
    $R_0^* : L^p(G) \to L^p(G) \quad p \in (1, 2)$ and is NOT of weak type $(1, 1)$ [Hebisch]
  - if $G = \mathbb{R}^2 \rtimes \mathbb{R}$, then R_j^* is NOT bounded from $H^1(G)$ to $L^1(G)$
    [Sjögren-V. (2008)]

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- we also introduce a Hardy type space $H^1(G)$ and prove an endpoint result on $H^1(G)$ both for Riesz transforms and MH spectral multipliers of Δ

- $R_j : L^p(G) \to L^p(G) \quad p \in (2,\infty)$?

$$\Updownarrow$$

$R_j^* = -\Delta^{-1/2} X_j : L^p(G) \to L^p(G) \quad p \in (1,2)$?

  - if $G = \mathbb{R} \rtimes \mathbb{R}$:
    $R_1^* : L^p(G) \to L^p(G) \quad p \in (1,2)$ and is of weak type $(1,1)$ [Gaudry-Sjögren (1999)]
    $R_0^* : L^p(G) \to L^p(G) \quad p \in (1,2)$ and is NOT of weak type $(1,1)$ [Hebisch]
  - if $G = \mathbb{R}^2 \rtimes \mathbb{R}$, then R_j^* is NOT bounded from $H^1(G)$ to $L^1(G)$ [Sjögren-V. (2008)]

- Riesz transforms of the second order
  - $X_i \Delta^{-1} X_j$ are of weak type $(1,1)$ and bounded on $L^p(G, \forall (i,j) \neq (0,0)$ [Martini-Ottazzi-V.]
  - *If* Δ is a distinguished Laplacian on NA groups arising from the Iwasawa decomposition of a semisimple Lie group of rank one, then $X_i \Delta^{-1} X_j$ are bounded on $L^p(G)$, $p \in (1,\infty)$, and of weak type $(1,1)$ but $X_i X_j \Delta^{-1}$ and $\Delta^{-1} X_i X_j$ are NOT bounded on $L^p(G)$, $p \in [1,\infty)$ [Gaudry-Sjögren (1996)]

Related problems

- we also introduce a Hardy type space $H^1(G)$ and prove an endpoint result on $H^1(G)$ both for Riesz transforms and MH spectral multipliers of Δ
- $R_j : L^p(G) \to L^p(G) \quad p \in (2, \infty)$?
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  $R_j^* = -\Delta^{-1/2} X_j : L^p(G) \to L^p(G) \quad p \in (1, 2)$?
- Riesz transforms of the second order
- L^p-spectral multipliers for subLaplacian with drift $\Delta - X$ (holomorphic functional calculus)
- more general NA groups

Sketch of the proof of THEOREM 2

$$F(\Delta) = \sum_{n\in\mathbb{Z}} F(\Delta)\,\phi(2^{-n}\Delta) = \sum_{n\in\mathbb{Z}} F_n(2^{-n}\Delta)$$

$$\phi \in C_c^\infty(\mathbb{R}^+),\ \sum_n \phi(2^n\lambda) = 0,\ \mathrm{supp}(\phi) \subset [1/4, 4]$$

The kernel k^n of $F_n(2^{-n}\Delta)$ satisfy the conditions of the boundedness theorem. This follows from various facts:

- Δ and $\tilde{\Delta}$ on $\tilde{G} = \mathbb{R}^Q \rtimes \mathbb{R}$ have the same Plancherel measure, i.e.

$$\int_G |k_{F(\Delta)}(z,u)|^2 \mathrm{d}z\mathrm{d}u = \int_0^\infty |F(\lambda)|^2 |\mathbf{c_Q}(\lambda)|^{-2}\mathrm{d}\lambda \sim \int_0^\infty |F(\lambda)|^2(\lambda^2+\lambda^Q)\mathrm{d}\lambda$$

  $\mathbf{c_Q}$ Harish-Chandra function on $\mathbb{H}^{Q+1}(\mathbb{R})$

- weighted L^2-estimates on G for $k_{F(\Delta)}$ can be reduced to weighted L^2-estimates for $k_{F(\tilde{\Delta})}$ on $\tilde{G}$

$$\int_G |k_{F(\Delta)}(z,u)|^2\, |z|_N^{2a} \mathrm{d}z\mathrm{d}u \leq \int_{\tilde{G}} |k_{F(\tilde{\Delta})}(z,u)|^2\, |z|_{\mathbb{R}^Q}^{2a} \mathrm{d}z\mathrm{d}u\,,$$

  where $\tilde{m}^{-1/2} k_{F(\tilde{\Delta})}(z,u)$ is radial (and one can use spherical analysis on $\tilde{G}$)

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- Δ and $\tilde{\Delta}$ on $\tilde{G} = \mathbb{R}^Q \rtimes \mathbb{R}$ have the same Plancherel measure
- weighted L^2-estimates on G for $k_{F(\Delta)}$ can be reduced to weighted L^2-estimates for $k_{F(\tilde{\Delta})}$ on $\tilde{G}$
- if the Fourier transform of M is supported in $[-R, R]$, then

$$\|k_{M(\Delta)}\|_1 \lesssim \min\{R^{3/2}, R^{(Q+1)/2}\}\, \|k_{M(\Delta)}\|_2$$

- $F_n(2^{-n}\cdot)$ can be decomposed as the sum of functions whose Fourier transform is compactly supported in order to apply previous estimate

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$$\phi \in C_c^\infty(\mathbb{R}^+), \sum_n \phi(2^n\lambda) = 1, \operatorname{supp}(\phi) \subset [1/4, 4]$$

The kernel k^n of $F_n(2^{-n}\Delta)$ satisfy the conditions of the boundedness theorem

$\Rightarrow F(\Delta)$ is of weak type $(1,1)$ and bounded on $L^p(G)$, $p \in (1, \infty)$

THANK YOU!