Algebras of singular integral operators on nilpotent groups

(joint work with A. Nagel, E.M. Stein, S. Wainger)

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Dilations and CZ theory on nilpotent groups

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However, several situations have been encountered in the past where non-automorphic dilations must be considered, at least locally.
The typical case is that of the isotropic dilations on the Heisenberg group:

Phong-Stein, 1982
Müller, R., Stein, 1995
Nagel, Stein, 2006
Müller, Peloso, R., 2015

Product theory

More precisely, the common theme in the above mentioned papers is the simultaneous presence of isotropic dilations and the standard automorphic (parabolic) dilations and their combination in a (non-automorphic) two-parameter dilation structure calling for some adaptation of the *product theory* of singular integrals on $\mathbb{R}^n$.

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The limitations imposed by the compatibility with the group structure have produced a restricted form of product theory in which the two dilation parameters are subject to a one-sided limitation (*flag kernels*, Nagel, R., Stein, 2001, Nagel, R., Stein, Wainger, 2012).

General dilations on $\mathbb{R}^n$

For $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n_+$ define the $\mathbf{a}$-dilations

$$x = (x_1, x_2, \dots, x_n) \longmapsto (t^{1/a_1} x_1, t^{1/a_2} x_2, \dots, t^{1/a_n} x_n) = \delta_{\mathbf{a}}(t) x$$

Homogenous norm:

$$|x|_{\mathbf{a}} = |x_1|^{a_1} + |x_2|^{a_2} + \cdots + |x_n|^{a_n} \ .$$

Homogeneous dimension:

$$Q_{\mathbf{a}} = \frac{1}{a_1} + \frac{1}{a_2} + \cdot + \frac{1}{a_n}$$

Let $\mathbf{k} = (k_1, k_2, \dots, k_n) \in \mathbb{N}^n$ be a multiindex. We set

$$\partial_x^{\mathbf{k}} = \partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \cdots \partial_{x_n}^{k_n} \ .$$

Calderón-Zygmund kernels adapted to the **a**-dilations

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- differential inequalities: for any **k** and any $x \neq 0$,

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  with $[\mathbf{k}]_{\mathbf{a}} = \sum k_j / a_j$;

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  with $[\mathbf{k}]_{\mathbf{a}} = \sum k_j / a_j$;

- <u>cancellations</u>:

$$\left| \int K(x) \varphi(\delta_{\mathbf{a}}(t)x)\, dx \right| \leq C \|\varphi\|_{C^1} \ ,$$

  for all $\varphi \in C_c^1(B)$, B a fixed "unit ball", and every $t > 0$.

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$$\left|\partial_\xi^{\mathbf{k}} m(\xi) \leq C_{\mathbf{k}} |\xi|_{\mathbf{a}}^{-\sum k_j/a_j} ;\right.$$

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(iii)

$$K = \sum_{j\in\mathbb{Z}} 2^{Q_{\mathbf{a}}j} \varphi_j\big(\delta_{\mathbf{a}}(2^j)x\big) = \sum_{j\in\mathbb{Z}} \varphi_j^{(j)}(x) \ ,$$

where the φ_j are C^∞ functions supported on the unit ball B, bounded in any C^m-norm and with

$$\int \varphi_j(x)\, dx = 0 \ .$$

Proper kernels

In the rest of this talk we want to restrict our attention to “proper” kernels, which exhibit a CZ singularity at the origin, but combined with a Schwartz decay at infinity.

This involves modifying the previous conditions in the following way:

• differential inequalities of the kernel K:

$$\left|\partial_x^{\mathbf{k}} K(x)\right| \leq C_{\mathbf{k},N} |x|_{\mathbf{a}}^{-\sum(1+k_j)/a_j}$$

• differential inequalities of the multiplier m:

$$\left|\partial_\xi^{\mathbf{k}} m(\xi)\right| \leq C_{\mathbf{k}} |\xi|_{\mathbf{a}}^{-\sum k_j/a_j}$$

• dyadic decomposition:

$$K = \sum_{j \in \mathbb{Z}} \varphi_j^{(j)}$$

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• dyadic decomposition:

$$K = \eta + \sum_{j\geq 0} \varphi_j^{(j)} , \qquad \eta \in \mathcal{S}$$

Composition of CZ kernels with different homogeneities

Consider now two proper CZ kernels, K_1, K_2, adapted to dilations of type $\mathbf{a}$ and $\mathbf{b}$ respectively. We want to understand what kind of estimates are satisfied by the convolution $K_1 * K_2$.

It is quite obvious that the convolution will be C^∞ away from the origin (pseudolocality), with Schwartz decay at infinity. A more refined question is what differential inequalities it satisfies near the origin.

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We take a model example in two variables.

On $\mathbb{R}^2$ we denote by $z = (x, y)$ the space variables and $\zeta = (\xi, \eta)$ the frequency variables.
$\mathbf{a} = (1, 1)$ (isotropic dilations), with $|z|_{\mathbf{a}} = |x| + |y|$, $Q_{\mathbf{a}} = 2$,
$\mathbf{b} = (2, 1)$ (parabolic dilations), with $|z|_{\mathbf{b}} = |x|^2 + |y|$, $Q_{\mathbf{b}} = 3/2$.

The multiplier side

It suffices to consider points $\zeta = (\xi, \eta)$ with $|\xi|, |\eta| > 1$.

Set $m_1 = \widehat{K}_1$, $\quad m_2 = \widehat{K}_2$, Then

$$\left|\partial^{\mathbf{j}} m_1(\zeta)\right| \lesssim \big(1 + |\xi| + |\eta|\big)^{-j_1 - j_2}$$
$$\left|\partial^{\mathbf{l}} m_2(\zeta)\right| \lesssim \big(1 + |\xi|^2 + |\eta|\big)^{-\frac{1}{2}l_1 - l_2} \, .$$

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We estimate the derivatives of $m = \widehat{K_1 * K_2} = m_1 m_2$.

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(I) For $|\eta| < |\xi|$, $\left|\partial^{\mathbf{k}} m(\zeta)\right| \lesssim |\xi|^{-k_1-k_2}$

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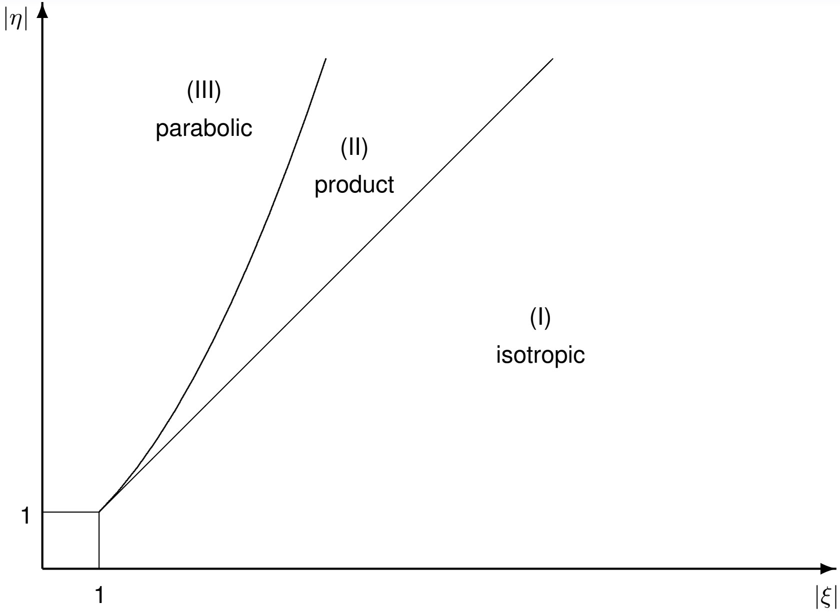
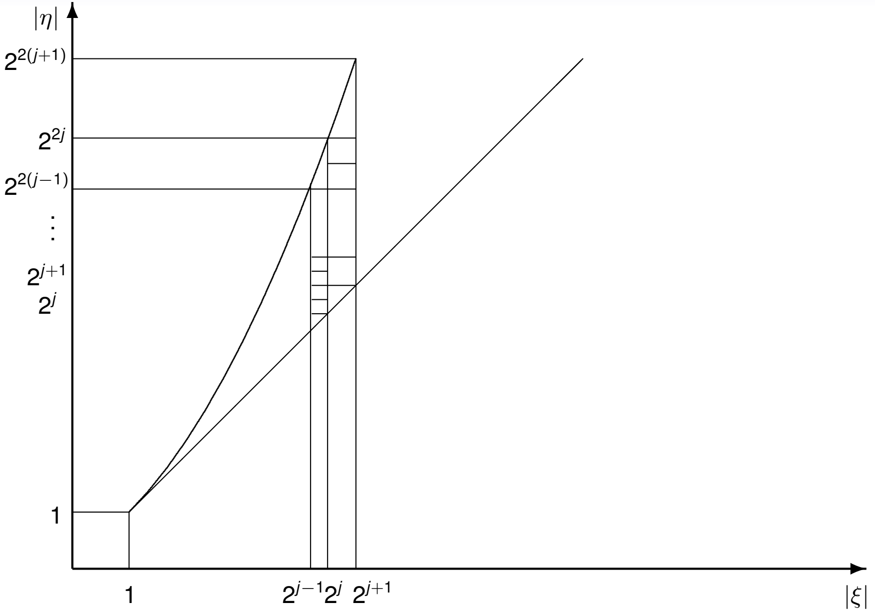
These inequalities are subsumed by the single formula

$$\begin{aligned}\left|\partial^{\mathbf{k}} m(\zeta)\right| &\lesssim \left(1 + |\xi| + |\eta|^{\frac{1}{2}}\right)^{-k_1} \left(1 + |\xi| + |\eta|\right)^{-k_2} \\ &\cong \left(1 + |\zeta|_{\mathbf{b}}^{\frac{1}{2}}\right)^{-k_1} \left(1 + |\zeta|_{\mathbf{a}}\right)^{-k_2} .\end{aligned}$$



$|\eta|$
(III)
parabolic
(II)
product
(I)
isotropic
1
1
$|\xi|$


$|\eta|$
$2^{2(j+1)}$
2^{2j}
$2^{2(j-1)}$
$\vdots$
2^{j+1}
2^{j}
1
1
2^{j-1}
2^{j}
2^{j+1}
$|\xi|$

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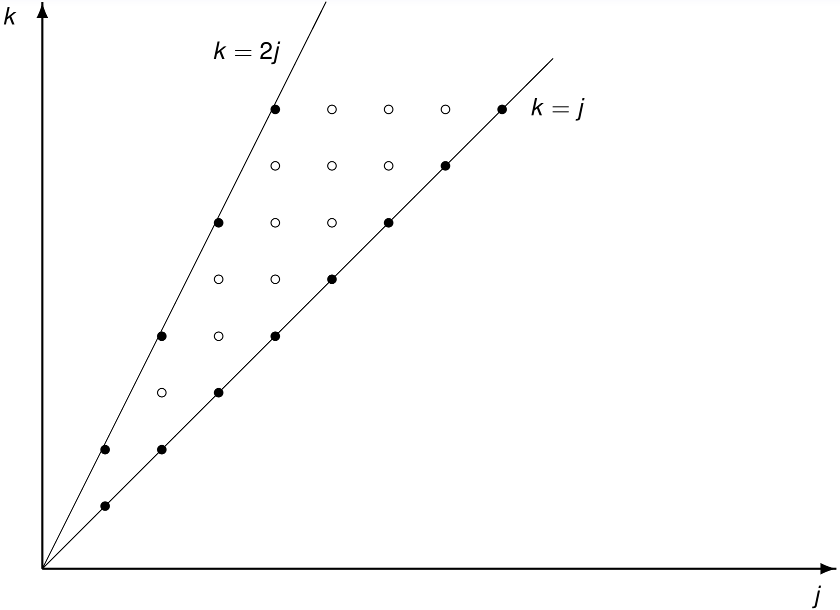
The rectangles in (II) do not intersect the axes.
Those in (I) and (III) intersect one of the two axes, but do not contain the origin.



k
$k = 2j$
$k = j$
j

Dyadic decomposition of the multiplier

The multiplier m can be decomposed

$$\begin{aligned} m &= m^0 + m^{(\mathrm{I})} + m^{(\mathrm{II})} + m^{(\mathrm{III})} \\ &= m^0 + \sum_{j\geq 0} m_j^{(\mathrm{I})}(2^{-j}\xi, 2^{-j}\eta) \\ &\quad + \sum_{j\leq k\leq 2j} m_{jk}^{(\mathrm{II})}(2^{-j}\xi, 2^{-k}\eta) \\ &\quad + \sum_{j\geq 0} m_j^{(\mathrm{III})}(2^{-j}\xi, 2^{-2j}\eta)\ , \end{aligned}$$

where all the summands are smooth functions supported in the unit ball B, uniformly bounded in all C^k-norms and

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- the terms $m_j^{(\mathrm{I})}, m_j^{(\mathrm{III})}$ vanish at 0,
- the $m_{jk}^{(\mathrm{II})}$ vanish on the coordinate axes.

The kernel in dyadic form

On the other side of the Fourier transform,

$$K_1 * K_2 = \eta + K^{(\mathrm{I})} + K^{(\mathrm{II})} + K^{(\mathrm{III})} ,$$

where $\eta \in \mathcal{S}$, $K^{(\mathrm{I})}, K^{(\mathrm{III})}$ are proper CZ kernels, the first isotropic and the second parabolic, while

$$K^{(\mathrm{II})} = \sum_{j \le k \le 2j} 2^{j+k} \varphi_{jk}(2^j x, 2^k y) . \tag{1}$$

The φ_{jk} are uniformly bounded in every Schwartz norm and

$$\int \varphi_{jk}(x,y)\,dx = 0 \quad \forall\, y , \qquad \int \varphi_{jk}(x,y)\,dy = 0 \quad \forall\, x .$$

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As a further refinement, each φ_{jk} can be replaced by a dyadic sum, with rapidly decreasing coefficients, of smooth functions with compact support and the same cancellations.

In this way the summands in (1) can be assumed to be supported in B and uniformly bounded in each C^k-norm.

Outside of CZ theory

The resulting kernel $K = K^{(\mathrm{I})} + K^{(\mathrm{II})} + K^{(\mathrm{III})}$ is *not* a CZ kernel. It satisfies the differential inequalities

$$\left|\partial_x^p \partial_y^q K(z)\right| \le C_{pqN} |z|_{\mathbf{a}}^{-1-p} |z|_{\mathbf{b}}^{-1-q} \big(1 + |z|\big)^{-N} \tag{2}$$

for all p, q, N, plus cancellations in each variable separately.

Condition (2) is weaker than any type of CZ condition. In fact it gives the estimate

$$\left|\{z : |K(z)| > \alpha\}\right| \lesssim \frac{\log \alpha}{\alpha} ,$$

and no better in general.

New classes of kernels

We consider n different homogeneities on $\mathbb{R}^n$, with homogeneous norms

$$
\begin{aligned}
|x|_{\mathbf{e}_1} &= |x_1|^{e(1,1)} + |x_2|^{e(1,2)} + \cdots + |x_n|^{e(1,n)} \\
\cdots \quad & \quad \cdots \\
|x|_{\mathbf{e}_n} &= |x_1|^{e(n,1)} + |x_2|^{e(n,2)} + \cdots + |x_n|^{e(n,n)} \ .
\end{aligned}
$$

It will be necessary to also consider the situation where each x_j is itself a multivariate component in $\mathbb{R}^{d_j}$, in a higher dimensional space $\mathbb{R}^d = \mathbb{R}^{d_1+d_2+\cdots+d_n}$. In this case $|x_j|$ denotes a homogeneous norm adapted to given dilations on $\mathbb{R}^{d_j}$.

Multi-norm inequalities

We want to consider kernels K which satisfy the following inequalities:

(*) $$\left|\partial_x^{\mathbf{k}} K(x)\right| \leq C_{\mathbf{k}N} |x|_{\mathbf{e}_1}^{-1-k_1} |x|_{\mathbf{e}_2}^{-1-k_2} \cdots |x|_{\mathbf{e}_n}^{-1-k_n} \big(1+|x|\big)^{-N} .$$

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It is natural to impose the following *basic assumptions*:

$$e(j,j) = 1 , \qquad e(j,k) \le e(j,l)e(l,k)$$

for all j, k, l.

Under the basic assumptions, the inequalities (*) hold, e.g., for proper CZ kernels of type $\mathbf{a} = \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

The classes $\mathcal{P}_0(\mathbf{E})$

Denote by $\mathbf{E}$ the matrix $\big(e(j,k)\big)_{jk}$. We will always assume that the basic assumptions are satisfied.

Definition

The class $\mathcal{P}_0(\mathbf{E})$ consists of the distributions K which are smooth away from the origin, satisfy the differential inequalities

$$(*)\qquad \left|\partial_x^{\mathbf{k}} K(x)\right| \le C_{\mathbf{k}N} |x|_{\mathbf{e}_1}^{-1-k_1} |x|_{\mathbf{e}_2}^{-1-k_2} \cdots |x|_{\mathbf{e}_n}^{-1-k_n} \big(1+|x|\big)^{-N},$$

for all $\mathbf{k}$, N, plus appropriate cancellations in each variable x_j.

Roughly speaking, the cancellations are defined inductively by stating that, once we integrate K against a scaled bump function in a single variable x_j, we obtain a kernel of the same kind in the remaining variables.

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We will give two characterizations of the kernels in $\mathcal{P}_0(\mathbf{E})$, one in terms of their Fourier transforms, the other in terms of dyadic decompositions.

Characterization via multipliers

In the frequency space we introduce a new family of norms:

$$
\begin{aligned}
|\xi|_{\widehat{\mathbf{e}}_1} &= |\xi_1|^{1/e(1,1)} + |\xi_2|^{1/e(2,1)} + \cdots + |\xi_n|^{1/e(n,1)} \\
\cdots \quad & \quad \cdots \\
|\xi|_{\widehat{\mathbf{e}}_n} &= |\xi_1|^{1/e(1,n)} + |\xi_2|^{1/e(2,n)} + \cdots + |\xi_n|^{1/e(n,n)} \ .
\end{aligned}
$$

Lemma

A distribution K is in $\mathcal{P}_0(\mathbf{E})$ if and only if $m = \widehat{K}$ is a smooth function satisfying the inequalities

$$
\left|\partial_\xi^{\mathbf{k}} m(\xi)\right| \leq C_{\mathbf{k}N} \left(1 + |\xi|_{\widehat{\mathbf{e}}_1}\right)^{-k_1} \cdots \left(1 + |\xi|_{\widehat{\mathbf{e}}_n}\right)^{-k_n} \ .
$$

The cone $\Gamma(\mathbf{E})$

Denote by $\Gamma(\mathbf{E})$ the cone in the positive orthant $\mathbb{R}^n_+$ defined by the inequalities

$$\frac{1}{e(l,k)} t_l \le t_k \le e(k,l) t_l \qquad \forall\, k, l$$

Let

$$m(\xi) = \sum_{J \in \Gamma(\mathbf{E}) \cap \mathbb{N}^n} m_J(2^{-j_1}\xi_1, \dots, 2^{-j_n}\xi_n)\ ,$$

where the functions m_J^{pr} are supported in B, vanish on a neighborhood of the coordinate hyperplanes and are uniformly bounded in every C^k-norm.
Then m satisfies the above inequalities, hence $\mathcal{F}^{-1} m \in \mathcal{P}_0(\mathbf{E})$.

The cone $\Gamma(\mathbf{E})$

Denote by $\Gamma(\mathbf{E})$ the cone in the positive orthant $\mathbb{R}^n_+$ defined by the inequalities

$$\frac{1}{e(l,k)} t_l \leq t_k \leq e(k,l) t_l \qquad \forall\, k, l$$

Let

$$m(\xi) = \sum_{J \in \Gamma(\mathbf{E}) \cap \mathbb{N}^n} m_J(2^{-j_1}\xi_1, \ldots, 2^{-j_n}\xi_n) \ ,$$

where the functions m_J^{pr} are supported in B, vanish on a neighborhood of the coordinate hyperplanes and are uniformly bounded in every C^k-norm.
Then m satisfies the above inequalities, hence $\mathcal{F}^{-1} m \in \mathcal{P}_0(\mathbf{E})$.

Up to "boundary terms", it is also true that every $m \in \mathcal{F}\mathcal{P}_0(\mathbf{E})$ can be decomposed as a dyadic sum of this kind. We will explain where the "boundary terms" come from.

Dominant variables

We decompose the complement of B in the ξ-space into regions, considering, for each j, which variable is *dominant* in the norm $|\xi|_{\widehat{\mathbf{e}}_k}$.
We say that ξ_j is dominant in $|\xi|_{\widehat{\mathbf{e}}_k}$ at a point ξ if

$$|\xi_j|^{1/e(j,k)} > |\xi_l|^{1/e(l,k)} \qquad \forall l \neq j$$

After removing a finite union of hypersurfaces, at each point there is a unique dominant variable in each norm (regular points).

Lemma

If ξ_k is dominant in $|\xi|_{\widehat{\mathbf{e}}_j}$ at a regular point ξ, with $j \neq k$, then ξ_k is also dominant in $|\xi|_{\widehat{\mathbf{e}}_k}$ at ξ.

Marked partitions

To each regular point ξ we can associate a *marked partition*, i.e., a partition $\{I_1, \ldots, I_s\}$ of $\{1, 2, \ldots, n\}$, together with a distinguished element $k_r \in I_r$ for each $r = 1, \ldots, s$.

For instance, if $\xi \in \mathbb{R}^5$ is associated to the marked partition

$$\{1, \bar{3}\}, \{2, \bar{4}, 5\}$$

this means that ξ_3 is dominat in $|\xi|_{\widehat{\mathbf{e}}_1}$ and in $|\xi|_{\widehat{\mathbf{e}}_3}$, while ξ_4 is dominat in $|\xi|_{\widehat{\mathbf{e}}_2}$, $|\xi|_{\widehat{\mathbf{e}}_4}$ and $|\xi|_{\widehat{\mathbf{e}}_5}$.

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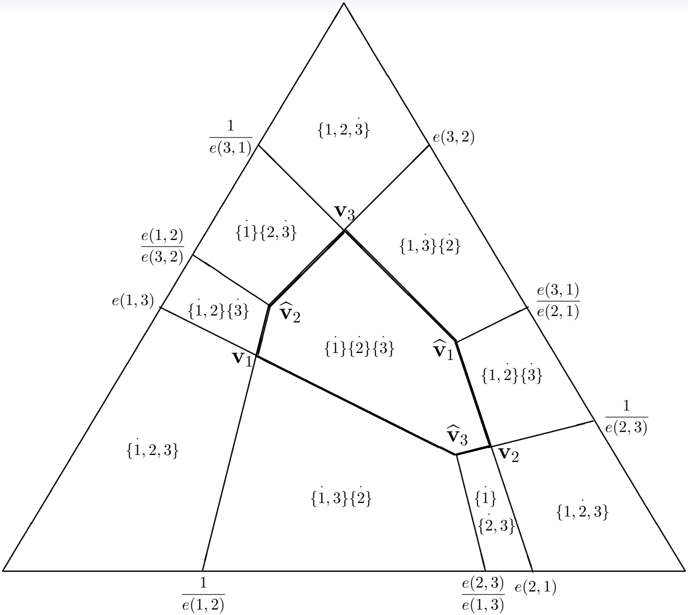
To each marked partition $S = \{I_1, \ldots, I_s; k_1, \ldots, k_s\}$ we associate the (possibly empty) set E_S of those regular ξ such that, for all r, ξ_{k_r} is dominant in $|\xi|_{\widehat{\mathbf{e}}_j}$ for all $j \in I_r$.

Then

$$m = m^0 + \sum_S m^S ,$$

where m^0 is supported on B and each m^S near E_S.



$\{1,2,\dot{3}\}$
$\frac{1}{e(3,1)}$
$e(3,2)$
$\mathbf{v}_3$
$\{\dot{1}\}\{2,\dot{3}\}$
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Dyadic decomposition of K

Putting all together,

$$m(\xi) = m^0(\xi) + \sum_{S} \sum_{J \in \Gamma(\mathbf{E}_S) \cap \mathbb{N}^s} m_J^S(2^{-j_1} \cdot \xi_{I_1}, \dots, 2^{-j_s} \cdot \xi_{I_s})$$

where, for each S, the functions m_J^S are supported in B, vanish on a neighborhood of the subspaces $\{\xi : \xi_{I_1} = 0\}, \ \dots \ \{\xi : \xi_{I_s} = 0\}$ and are uniformly bounded in every C^k-norm.

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Undoing the Fourier transform,

$$K(x) = \eta(x) + \sum_S \sum_{J \in \Gamma(\mathbf{E}_S) \cap \mathbb{N}^s} 2^{Q_1 j_1 + \cdots + Q_s j_s} \varphi_J^S(2^{j_1} \cdot x_{I_1}, \dots, 2^{j_s} \cdot x_{I_s})$$

where $\eta \in \mathcal{S}$ and, for each S, the functions φ_J^S are uniformly bounded in every Schwartz norm and satisfy the cancellations

$$\int \varphi_J^S(x_{I_1}, \dots, x_{I_r}, \dots, x_{I_s})\, dx_{I_r} = 0 \qquad \forall\, x_{I_1}, \dots, x_{I_{r-1}}, x_{I_{r+1}}, \dots, x_{I_s}$$

Equivalent characterizations of $\mathcal{P}_0(\mathbf{E})$

Theorem

Let $\mathbf{E}$ *be a matrix satisfying the basic assumptions. The following are equivalent for* $K \in \mathcal{S}'(\mathbb{R}^n)$*:*

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- $K \in \mathcal{P}_0(\mathbf{E})$,
- $m = \widehat{K}$ *is a smooth function satisfying the inequalities*

$$\left|\partial_\xi^{\mathbf{k}} m(\xi)\right| \leq C_{\mathbf{k}N}\left(1 + |\xi|_{\widehat{\mathbf{e}}_1}\right)^{-k_1} \cdots \left(1 + |\xi|_{\widehat{\mathbf{e}}_n}\right)^{-k_n},$$

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- $K(x) = \eta(x) + \sum_S \sum_{J \in \Gamma(\mathbf{E}_S) \cap \mathbb{N}^s} 2^{Q_1 j_1 + \cdots + Q_s j_s} \varphi_J^S(2^{j_1} \cdot x_{l_1}, \ldots, 2^{j_s} \cdot x_{l_s})$ *as above, or even with the additional condition* $\operatorname{supp} \varphi_J^S \subset B$.

Properties (on $\mathbb{R}^n$)

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- In particular, proper CZ kernels of type $\mathbf{a}$ are in $\mathcal{P}_0(\mathbf{E})$ if and only if $\mathbf{a} \in \Gamma(\mathbf{E})$.
- Given finitely many proper CZ kernels K_i of type $\mathbf{a}^i$, their convolution belongs to $\mathcal{P}_0(\mathbf{E})$, where

$$e(j, k) = \max_i \frac{a_k^i}{a_j^i} .$$

Homogeneous nilpotent groups

Let $\mathfrak{g}$ be a homogeneous Lie algebra with (automorphic) dilations $\delta(t)$, $t > 0$. Let

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_m \ ,$$

be a decomposition into homogeneous subspaces, with the property that $[\mathfrak{g}, \mathfrak{h}_j] \subset \mathfrak{h}_{j+1} \oplus \cdots \oplus \mathfrak{h}_m$ for every j.

Call G the associated homogenous group, parametrized by $\mathfrak{g}$ with the Campbell-Hausdorff product. Denote by $|\ |$ a homogeneous norm on G.

Other homogeneities

Besides the standard CZ kernels adapted to the given dilations, we consider CZ kernels adapted to other (non-automorphic) dilation

$$\delta_{\mathbf{a}}(t)x = \left(\delta(t^{1/a_1})x_1, \dots, \delta(t^{1/a_m})x_m\right) .$$

It is known (R.-Stein) that such kernels give bounded operators on L^p for $1 < p < \infty$, but in general they form a class which is not closed under convolution.

Theorem

Suppose that $a_1 \leq a_2 \leq \cdots \leq a_m$. Then the class of proper *CZ kernels adapted to the dilations $\delta_{\mathbf{a}}$ is closed under convolution.*

Remarks

1. Let

$$K = \eta + \sum_{j \in \mathbb{N}} \varphi_j^{(j)} \ , \qquad K' = \eta' + \sum_{j \in \mathbb{N}} \psi_j^{(j)}$$

be the dyadic decompositions of two proper CZ kernels adapted to the $\delta_{\mathbf{a}}$-dilations. Here $\varphi_j^{(j)}$ denotes the bump function φ_j scaled by a factor 2^{-j}, i.e. supported where

$$|x_1| < 2^{-j/a_1} \ , \quad |x_2| < 2^{-j/a_2} \ , \quad \dots \quad |x_m| < 2^{-j/a_m} \ .$$

The monotonicity condition on the a_j guarantees that the convolution of two terms $\varphi_j^{(j)}$ and $\psi_{j'}^{(j')}$ is scaled by a factor $\max\{2^{-j}, 2^{-j'}\}$, like in $\mathbb{R}^N$.

2. For a specific homogeneous group, weaker assumptions may suffice. E.g., on H_1 with the standard parabolic dilations and $\mathfrak{h}_1 = \mathbb{R}_x$, $\mathfrak{h}_2 = \mathbb{R}_y$, $\mathfrak{h}_3 = \mathbb{R}_t$, the condition $2/a_3 \leq 1/a_1 + 1/a_2$ is enough.

Convolution within $\mathcal{P}_0(\mathbf{E})$ - L^p-boundedness

We say that **E** is doubly monotonic if

$$e(j,k) \leq e(j-1,k)\ , \qquad e(j,k) \leq e(j,k+1)$$

for all j, k.

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Assume that **E** *satisfies the basic assumptions and is doubly monotonic.*

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Theorem

Assume that **E** *satisfies the basic assumptions and is doubly monotonic. Then*

- $\mathcal{P}_0(\mathbf{E})$ *is closed under convolution;*
- *convolution by kernels in* $\mathcal{P}_0(\mathbf{E})$ *is bounded on* L^p*,* $1 < p < \infty$*.*

Convolution of dyadic terms

Double monotonicity of **E** guarantees that the group convolution of two dyadic terms at scales $2^{-J}, 2^{-J'}$ with $J, J' \in \Gamma(\mathbf{E})$ is at scale $2^{-J} \vee 2^{-J'}$.

But the major problem in the proof is that the product cancellations are not preserved under convolution.
The hypothesis of double monotonicity gives however a form of "weak cancellation" of the convolution of two dyadic terms, which is sufficient to prove that the whole resulting dyadic sum converges to a kernel in $\mathcal{P}_0(\mathbf{E})$.