Fourier L_p summability with frequencies in nonabelian groups

Javier Parcet

LMS Midlands Regional Meeting
Interactions of Harmonic Analysis and Operator Theory

Birmingham – September 13-16, 2016

Plan

1. Main questions
2. Basic operator algebra
3. Basic geometric group theory
4. Smooth Fourier multipliers
5. Nonsmooth Fourier multipliers
6. Incidence of Kazhdan property (T)
7. Fourier L_p summability over $\mathrm{SL_n}(\mathbb{R})$

Based on joint work with... and independent results by...

M. Caspers, A. González-Pérez, M. Junge, T. Mei
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Main questions

Fourier summability over $\mathbb{Z}^n$

MAIN PROBLEM. Determine those families of bounded, compactly supported symbols $m_R : \mathbb{Z}^n \to \mathbb{C}$ converging pointwise to 1 as $R \to \infty$, for which the limit below

$$\lim_{R\to\infty} \Big(\int_{\mathbb{T}^n} \Big| f(x) - \sum_{k\in\mathbb{Z}^n} m_R(k) \widehat{f}(k) e^{2\pi i k\cdot x} \Big|^p dx \Big)^{\frac{1}{p}} = 0 \quad \text{for} \quad f \in L_p(\mathbb{T}^n).$$

BASIC CONSTRUCTION

K. de Leeuw's restriction $\mathbb{R}^n \to \mathbb{Z}^n$ + dilation invariance in $\mathbb{R}^n$ $\Rightarrow$ compactly suppported Fourier multipliers in $\mathbb{R}^n$

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$$\boxed{\begin{array}{c}\text{K. de Leeuw's} \\ \text{restriction } \mathbb{R}^n \to \mathbb{Z}^n\end{array}} + \boxed{\begin{array}{c}\text{dilation} \\ \text{invariance in } \mathbb{R}^n\end{array}} \Rightarrow \boxed{\begin{array}{c}\text{compactly suppported} \\ \text{Fourier multipliers in } \mathbb{R}^n\end{array}}$$

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IMP. $\mathbb{Z}^n$ is an **abelian** group and it admits a **flat isometric embedding** into $\mathbb{R}^n$.

Nonabelian discrete frequencies

OBJECTIVE. Study Fourier L_p summability with frequencies in locally compact unimodular groups. Of course, our results highly depend on the **geometry** of the frequency group. We assume the group to be **discrete** for simplicity.

IMP REMARK. Dual to Müller-Ricci-Stein approach for nilpotent Lie groups... Pioneered by Haagerup, Cowling, de Canniere, etc. The L_p-theory is brand new.

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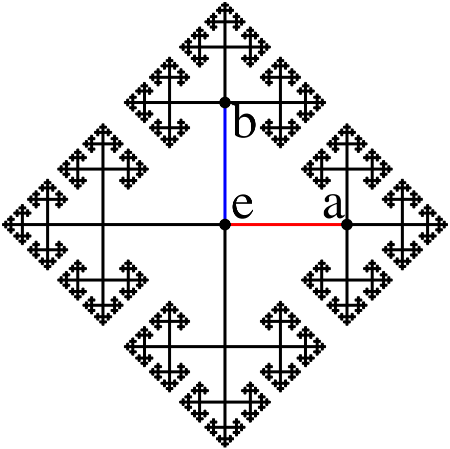

Cayley graph of the free group $\mathbb{F}_2$

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WHY SHOULD WE CARE? Our primary **motivations** are

NC HA Euclidean applications	+	**Operator Algebra** Classification of vNas

Basic operator algebra

The group von Neumann algebra

Given $f \in L_\infty(\mathbb{T})$, let $\Lambda_f(g) = fg$ so that

$$\operatorname*{ess\,sup}_{x \in \mathbb{T}} |f(x)| = \big\| \Lambda_f : L_2(\mathbb{T}) \to L_2(\mathbb{T}) \big\| = \big\| \Phi \circ \Lambda_f \circ \Phi^{-1} : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}) \big\|$$

for the Fourier transform $\Phi : L_2(\mathbb{T}) \ni \exp_k \mapsto \delta_k \in \ell_2(\mathbb{Z})$. Then, the **left regular representation** $\lambda : \mathbb{Z} \to \mathcal{B}(\ell_2(\mathbb{Z}))$ defined by $\lambda(k) = \Phi \circ \Lambda_{\exp_k} \circ \Phi^{-1} : \delta_j \mapsto \delta_{j+k}$ yields the $*$-homomorphism

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Therefore, $L_\infty(\mathbb{T})$ is isomorphic to the **group von Neumann algebra**

$$\mathcal{L}(\mathbb{Z}) = \overline{\Big\{ \sum_{k \in \Lambda} a_k \lambda(k) : a_k \in \mathbb{C}, \ \Lambda \subset \mathbb{Z} \text{ finite} \Big\}}^{\mathrm{w}^*} \subset \mathcal{B}(\ell_2(\mathbb{Z})).$$

Only $\mathcal{L}(\mathrm{G})$ survives for not abelian groups, but not $L_\infty(\widehat{\mathrm{G}})$ unless G is abelian!!!

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$\mathcal{L}(\mathrm{G})$ = Model of quantum group. Imp in NC geometry and operator algebra.

Noncommutative L_p norm in $\mathcal{L}(\mathrm{G})$

The natural quantum measure in $\mathcal{L}(\mathrm{G})$ is

$$\tau_{\mathrm{G}}\Big(\sum_{g \in \mathrm{G}} \widehat{f}(g)\lambda(g)\Big) = \widehat{f}(e) \quad \text{(extends to } \mathrm{G} \text{ unimodular).}$$

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Given $p>0$ and $f=\sum_g \widehat{f}(g)\lambda(g)\in\mathcal{L}(\mathrm{G})$, we set

$$\|f\|_p^p=\tau_{\mathrm{G}}\big[\,|f|^p\big]=\tau_{\mathrm{G}}\big[(f^*f)^{\frac{p}{2}}\big]$$

by functional calculus in $\mathcal{B}(\ell_2(\mathrm{G}))$. It turns out that $L_p(\mathcal{L}(\mathrm{G}))=L_p(\widehat{\mathbf{G}})$ —defined as the closure of $\mathcal{L}(\mathrm{G})$ wrt the noncommutative L_p norm above— is isometrically isomorphic to the commutative space $L_p(\widehat{\mathrm{G}})$ for any abelian group G.

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G **abelian**	G **not abelian**
χ_g	$\lambda(g)$
$L_\infty(\widehat{\mathrm{G}})$	$\mathcal{L}(\mathrm{G}) = L_\infty(\widehat{\mathbf{G}})$
Haar measure	τ_{G}
Fourier coefficient	$\widehat{f}(g) = \tau_{\mathrm{G}}(f\lambda(g)^*)$
Plancherel theorem	$\langle f, f\rangle_{L_2(\mathcal{L}(\mathrm{G}))} = \sum \lvert\widehat{f}(g)\rvert^2$

Translation invariance – Fourier multipliers

Fourier multipliers over $\mathbb{Z}$

$$\underbrace{\sum\nolimits_k \widehat{f}(k) \exp_k}_{f} \mapsto \underbrace{\sum\nolimits_k m(k) \widehat{f}(k) \exp_k}_{T_m f}$$

are characterized by $T_m f(x - x_0) = T_m f_{x_0}(x)$ for $f_{x_0}(x) = f(x - x_0)$. Consider the comultiplication map $\Delta(\exp_k) = \exp_k \otimes \exp_k$. It can be easily checked that the translation invariance above can be rephrased by

$$\Delta \circ T_m = (T_m \otimes id) \circ \Delta = (id \otimes T_m) \circ \Delta.$$

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THE EXACT SAME IDENTITES
CHARACTERIZE FOURIER MULTIPLIERS OVER G

$$\boxed{T_m f = \sum_{g \in \mathrm{G}} m(g)\widehat{f}(g)\lambda(g) = \int_{\mathrm{G}} m(g)\widehat{f}(g)\lambda(g)\, d\mu(g)}$$

Basic geometric group theory

Affine representations

We look for maps $b : \mathrm{G} \to \mathcal{H}$ such that

$\mathrm{dist}(g,h) = \big\| b(g) - b(h) \big\|_{\mathcal{H}}$ defines a good pseudo-metric over G.

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Substantial information of G is encoded by its **orthogonal group representations** $\pi : \mathrm{G} \to \mathcal{O}(\mathcal{H})$. The map $\Pi(g) \in \mathsf{Aff}(\mathcal{H})$ given by $\Pi(g)[u] = \pi_g(u) + b(g)$ defines an **affine representation** when

$$\Pi(g_1 g_2) = \Pi(g_1) \circ \Pi(g_2) \ \Leftrightarrow \ \pi_{g_1}(b(g_2)) = b(g_1 g_2) - b(g_1).$$

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A G-**cocycle** is any triple $(\mathcal{H}, \pi, b)$ which arises from some affine representation Π as above. Every G-cocycle gives rise naturally to the **length function** $\psi : \mathrm{G} \to \mathbb{R}_+$ defined by $\psi_b(g) = \langle b(g), b(g) \rangle_{\mathcal{H}}$. It satisfies $\psi_b(e) = 0$ and $\psi_b(g) = \psi_b(g^{-1})$ and it is a conditionally negative function

$$\sum\nolimits_g a_g = 0 \ \Rightarrow \ \sum\nolimits_{g,h} \overline{a_g} a_h \psi_b(g^{-1}h) \le 0.$$

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SCHOENBERG THM. Cocycles $b \leftrightarrow$ Lengths $\psi_b \leftrightarrow$ Markov $\star$-processes $e^{-t\psi_b}$

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Substantial information of G is encoded by its **orthogonal group representations** $\pi : \mathrm{G} \to \mathcal{O}(\mathcal{H})$. The map $\Pi(g) \in \mathrm{Aff}(\mathcal{H})$ given by $\Pi(g)[u] = \pi_g(u) + b(g)$ defines an **affine representation** when

$$\Pi(g_1 g_2) = \Pi(g_1) \circ \Pi(g_2) \ \Leftrightarrow \ \pi_{g_1}(b(g_2)) = b(g_1 g_2) - b(g_1).$$

A G-**cocycle** is any triple $(\mathcal{H}, \pi, b)$ which arises from some affine representation Π as above. Every G-cocycle gives rise naturally to the **length function** $\psi : \mathrm{G} \to \mathbb{R}_+$ defined by $\psi_b(g) = \langle b(g), b(g) \rangle_{\mathcal{H}}$. It satisfies $\psi_b(e) = 0$ and $\psi_b(g) = \psi_b(g^{-1})$ and it is a conditionally negative function

$$\sum_g a_g = 0 \ \Rightarrow \ \sum_{g,h} \overline{a_g} a_h \psi_b(g^{-1}h) \leq 0.$$

SCHOENBERG THM. Cocycles $b \leftrightarrow$ Lengths $\psi_b \leftrightarrow$ Markov $\star$-processes $e^{-t\psi_b}$

More flexibility ($\mathrm{SL_n}(\mathbb{R})$): Non-orthogonal π + Non-Hilbert $\mathcal{H}$ + Quasi-cocycles.

Elementary group cocycles

- **The group $\mathbb{Z}^n$**
  - **Trivial cocycle**
    $\mathcal{H} = \mathbb{R}^n$, π trivial and $b = id$.
    $\psi_b(k) = |k|^2$ and $e^{-t\psi_b}$ = heat semigroup.
    Underlying cocycle in Euclidean Fourier analysis [JMP, GAFA 2014]
  - **Poisson cocycle**
    $\mathcal{H} = L_2(\mathbb{R}^n, \mu)$ infinite-dimensional!!
    $\psi_b(k) = |k|$ and $e^{-t\psi_b}$ = Poisson semigroup.
    Links Euclidean Fourier analysis and NC geometry [JMP, JEMS 2016]
  - **Directional cocycle**
    $\mathcal{H} = \mathbb{R}$ one-dimesional and π trivial.
    $b(k) = \langle k, x \rangle$ injective if $x_1, x_2, \ldots, x_n$ are $\mathbb{Z}$-independent.
    Right endpoint BMO for directional Hilbert transforms [PR, Crelle's J 2016]

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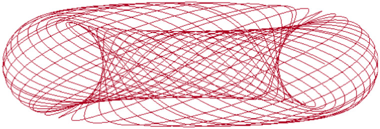
- **Donut cocycle of $\mathbb{R}$**

Pick α/β irrational and $\mathcal{H} = \mathbb{R}^4 \simeq \mathbb{C}^2$
Then the map $b(\xi) = (1,1) - (e^{2\pi i \alpha \xi}, e^{2\pi i \beta \xi})$
defines a **geodesic flow** on $\mathbb{T}^2$ with **dense orbit**



It is an **inner cocycle** associated to $\pi_\xi(z) = (e^{2\pi i \alpha \xi} z_1, e^{2\pi i \beta \xi} z_2)$
New results for idempotent L_p-multipliers in $\mathbb{R}$ [CPPR, Forum Math Σ 2015]

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- **Cayley cocycle of $\mathbb{F}_2$**

  $\mathcal{H} = \mathbb{R}[\mathbb{F}_2]$ with $||$-Gromov inner product.
  $\pi = \lambda$ and $b(w) = \delta_w - \delta_e$ yields the Cayley graph length $\psi_b(w) = |w|$.
  Directional Hilbert transforms in the free group [MR, Preprint 2016]

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- **Other proper cocycles (later)...**
  - **Inf-dim cocycles of $\mathrm{SL}_2(\mathbb{R})$**
  - **Non-orthogonal ones of $\mathrm{SL}_3(\mathbb{R})$**

Smooth Fourier multipliers

Noncommutative Riesz transforms

- **A bit of history**
  - **Dimension-free estimates**

$$\|f\|_p \sim_{c(p)} \Big\| \Big(\sum_{j=1}^n |R_j f|^2 \Big)^{\frac12} \Big\|_p.$$

Gundy/Varopoulos [CR Paris '79] + Stein [Bull AMS '83]
Duoandikoetxea/Rubio de Francia [CR Paris '85] + Pisier [LNM '88]

  - **P.A. Meyer's semigroup approach**

$$\|f\|_p \sim_{c(p)} \Big\| \Gamma\big(A^{-\frac12} f, A^{-\frac12} f\big)^{\frac12} \Big\|_p,$$

where Γ is the so-called Carré du Champ of $S_t = \exp(-tA)$

$$\Gamma(f_1, f_2) = \frac12 \Big(\overline{A(f_1)} f_2 + \overline{f_1} A(f_2) - A(\overline{f_1} f_2) \Big) = \textbf{gradient form.}$$

Meyer [LNM '84] + Bakri [LNM '87]
Lust-Piquard [JFA '98, CMP '99, Adv Math '04].

  - **Riesz-Poisson fails Meyer's conjecture in $\mathbb{R}^n$ for $p < 2$!!**

Noncommutative phenomena for non-diffusion (commutative) semigroups.

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  - If ψ c.n. length over G discrete $\rightsquigarrow (\mathcal{H}_\psi, \pi_\psi, b_\psi)$

$$R_{\psi,u} : \sum_g \widehat{f}(g)\lambda(g) \mapsto 2\pi i \sum_g \frac{\langle b_\psi(g), u\rangle_{\mathcal{H}_\psi}}{\sqrt{\psi(g)}} \widehat{f}(g)\lambda(g),$$

    for any $u \in \mathcal{H}_\psi$. Note $\sqrt{\psi(g)} = \|b_\psi(g)\|_{\mathcal{H}_\psi} \rightsquigarrow$ **Standard symbol**.
  - Shoenberg thm: $A_\psi(\lambda(g)) = \psi(g)\lambda(g)$ generates a Markov process.
  - **Diff form**: $R_{\psi,u} = \partial_{\psi,u} A_\psi^{-\frac{1}{2}}$ with $\partial_{\psi,u}(\lambda(g)) = 2\pi i \langle b_\psi(g), u\rangle_{\mathcal{H}_\psi} \lambda(g)$.

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Theorem A [Junge-Mei-Parcet, JEMS '16]

If $f \in L_p^\circ(\widehat{\mathrm{G}})$ and ψ c.n. length on G:

$$\|f\|_p \sim_{c(p)} \begin{cases} \inf\limits_{R_{\psi,j} f = a_j + b_j} \Big\| \Big(\sum\limits_{j \geq 1} a_j^* a_j \Big)^{\frac{1}{2}} \Big\|_p + \Big\| \Big(\sum\limits_{j \geq 1} \widetilde{b}_j \widetilde{b}_j^* \Big)^{\frac{1}{2}} \Big\|_p & p \leq 2, \\ \max \Big\{ \Big\| \Big(\sum\limits_{j \geq 1} |R_{\psi,j} f|^2 \Big)^{\frac{1}{2}} \Big\|_p, \Big\| \Big(\sum\limits_{j \geq 1} |R_{\psi,j} f^*|^2 \Big)^{\frac{1}{2}} \Big\|_p \Big\} & p \geq 2. \end{cases}$$

The $\widetilde{b}_j$'s are twisted forms of the b_j's, which coincide for trivial action π_ψ.

Remarks. The same result holds for unimodular groups.
Thm A implies classical result in $\mathbb{R}^n$ via the trivial cocycle.
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- **Proof = Pisier + Khintchine**
  - **Pisier's identity**

$$\sqrt{\frac{2}{\pi}}\,\delta(-\Delta)^{-\frac{1}{2}}f = (id_{L_\infty(\mathbb{R}^n)} \otimes Q)\Big(\text{p.v. } \frac{1}{\pi}\int_{\mathbb{R}} \beta_t f \frac{dt}{t}\Big)$$

$\delta\varphi(x,y) = \langle \nabla\varphi(x), y\rangle$, $Q =$ Gaussian proj and $\beta_t f(x,y) = f(x+ty)$.

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  - **A crossed product extension of the NC Khintchine inequality.**
  - $A_\psi = \delta_\psi^* \delta_\psi$ **(Sauvageot thm) is the analogue of $-\Delta = \nabla^* \circ \nabla$.**

Noncommutative Riesz transforms

- **A bit of history**
- **Dimension-free estimates**
- **Proof = Pisier + Khintchine**
- **New Riesz transforms, even commutative...**
  - **The Riesz-Poisson transform**

Integrability restriction [Fefferman, Acta Math '70]

Given $A = (-\Delta)^{\frac{1}{2}}$ and $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\Gamma_A(f,f) = \int_{\mathbb{R}_+} P_s |\nabla P_s f|^2 \, ds \notin L_p(\mathbb{R}^n) \quad \text{for} \quad p \leq \frac{2n}{n+1},$$

where $\nabla g(x,s) = (\partial_x g, \partial_s g)$ includes both spatial and time derivatives.

Meyer's approach **fails** for $p < 2$ and n large $\rightsquigarrow$ What is the right form?

By a variation of the Poisson cocycle...
Theorem A solves it for any fractional laplacian in $\mathbb{R}^n$...
Dim-free estimates for Riesz potencials $\rightsquigarrow$ **Noncommutative approach**.

It will be useful below for **smooth Fourier multipliers** in group algebras.

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    Let Γ_0 be an LCA group and $\Gamma = \Gamma_0 \times \Gamma_0 \times \cdots \times \Gamma_0$. Let $\delta \in \Gamma_0$ be torsion free. Introduce $\partial_j f(\gamma) = f(\gamma) - f(\gamma_1, \ldots, \delta\gamma_j, \ldots, \gamma_n)$ and corresponding discrete laplacian $\mathcal{L} = \sum_j \partial_j^* \partial_j$ and Riesz transforms $R_{\delta,j} = \partial_j \mathcal{L}^{-\frac{1}{2}}$.

Discrete laplacians [Lust-Piquard, Adv Math '04]

If $p \geq 2$

$$\|f\|_{L_p(\Gamma)} \sim_{c(p)} \Big\| \Big(\sum_{j=1}^n |R_{\delta,j} f|^2 + |R_{\delta,j}^* f|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p(\Gamma)}.$$

$1 < p < 2$: Other dim-free estimates + Counterexample for this form.

Thm A more flexible: It admits torsion and many other new laplacians.

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  - **Word length laplacians over $\mathbb{Z}_n$ and $\mathbb{F}_n$**

$$\Big\| \sum_{j \in \mathbb{Z}_n} \widehat{f}(j) \chi_j \Big\|_{L_p(\widehat{\mathbb{Z}}_n)} \sim_{c(p)} \Big\| \Big(\sum_{k \in \mathbb{Z}_n} \Big| \sum_{j \in \Lambda_k} \frac{\widehat{f}(j)}{\sqrt{j \wedge (n-j)}} \chi_j \Big|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p(\widehat{\mathbb{Z}}_n)}$$

with $\Lambda_k = \big\{ j \in \mathbb{Z}_{2m} : j - k \equiv s \ (2m) \text{ with } 0 \le s \le m-1 \big\}$ when $n = 2m$.

$$\|f\|_{L_p(\widehat{\mathbf{F}}_n)} \sim_{c(p)} \Big\| \Big(\sum_{h \neq e} \Big| \sum_{g \geq h} \frac{\widehat{f}(g)}{\sqrt{|g|}} \lambda(g) \Big|^2 + \Big| \sum_{g \geq h} \frac{\overline{\widehat{f}(g^{-1})}}{\sqrt{|g|}} \lambda(g) \Big|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p(\widehat{\mathbf{F}}_n)}.$$

Smooth Fourier multipliers over G

- **A Hörmander-Mihlin theorem**

If $\|m\|_{\mathsf{M}_p(\mathbb{R}^n)} = \|T_m\|_{p\to p}$ for $\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$...

Classical HM theorem [Dokl Akad '56 + Acta Math '60]

Let $1 < p < \infty$:

i) [Mihlin, 1956] If $m \in \mathcal{C}^{[\frac{n}{2}]+1}(\mathbb{R}^n \setminus \{0\})$

$$\|m\|_{\mathsf{M}_p(\mathbb{R}^n)} \le c_n \sup_{\xi\neq 0} \sup_{|\beta|\le[\frac{n}{2}]+1} |\xi|^{|\beta|} \big|\partial_\xi^\beta m(\xi)\big|.$$

ii) [Hörmander, 1960] If $m \in \mathcal{C}^{[\frac{n}{2}]+1}(\mathbb{R}^n \setminus \{0\})$

$$\|m\|_{\mathsf{M}_p(\mathbb{R}^n)} \le c_n \sup_{\substack{R>0 \\ |\beta|\le[\frac{n}{2}+1]}} \Big(\frac{1}{R^{n-2|\beta|}} \int_{R<|\xi|<2R} \big|\partial_\xi^\beta m(\xi)\big|^2 d\xi \Big)^{\frac{1}{2}}.$$

iii) [Sobolev space formulation] If φ is a cutoff in $1 < |\xi| < 2$

$$\|m\|_{\mathsf{M}_p(\mathbb{R}^n)} \le c_n \sup_{j\in\mathbb{Z}} \Big\| \big(1 + |\ |^2\big)^{\frac{n}{4}+\varepsilon} \big(\widehat{\varphi\, m(2^j \cdot)}\big) \Big\|_{L_2(\mathbb{R}^n)}.$$

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Theorem B [Junge-Mei-Parcet, GAFA '14 + JEMS '16]

Let G be a discrete group and let $\psi : \mathrm{G} \to \mathbb{R}_+$ be a c.n. length giving rise to a n-dimensional cocycle $(\mathcal{H}_\psi, \pi_\psi, b_\psi)$. Given $1 < p < \infty$, a Littlewood-Paley decomposition $(\varphi_j)_{j \in \mathbb{Z}}$ in $\mathbb{R}^n$ and $\varepsilon > 0$, the following inequality holds

$$\|m\|_{\mathrm{M}_p(\widehat{\mathbf{G}})} \lesssim_{c(p,n)} |m(e)| + \inf_{m = \widetilde{m} \circ b_\psi} \underbrace{\Big\{ \sup_{j \in \mathbb{Z}} \Big\| \mathrm{D}_{\frac{n}{2}+\varepsilon} \Big(\sqrt{\psi_\varepsilon}\, \varphi_j\, \widetilde{m} \Big) \Big\|_{L_2(\mathbb{R}^n)} \Big\}}_{\text{Smaller than classical term!!!}}.$$

Also $L_\infty \to \mathrm{BMO}$ estimates under slightly stronger $\widetilde{m}$-regularity assumptions.

Approach 1. NCCZ + Cocycle BMO.
Mihlin type result + NC Littlewood-Paley.

Approach 2. Quantum Probability methods.
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H-M(ε) multipliers are L-P averages of ψ_ε-Riesz transforms!

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- **A Hörmander-Mihlin theorem**
- **A magic formula for H-M multipliers**

Let $m : \mathbb{R}^n \to \mathbb{C}$ satisfy

$$\|m\|_{\mathsf{W}^2_{\frac{n}{2}+\varepsilon}(\psi_\varepsilon)} = \Big\| \mathsf{D}_{\frac{n}{2}+\varepsilon} \big(\sqrt{\psi_\varepsilon} m \big) \Big\|_{L_2(\mathbb{R}^n)} < \infty.$$

Then, there exists $h \in \mathcal{H}_\varepsilon = L_2(\mathbb{R}^n, \mu_\varepsilon)$ such that

$$m(\xi) = \Big\langle h, \frac{b_\varepsilon(\xi)}{\|b_\varepsilon(\xi)\|_{\mathcal{H}_\psi}} \Big\rangle_{\mu_\varepsilon} = \text{Symbol of } R_{\psi_\varepsilon, h}$$

with $\|m\|_{\mathsf{W}^2_{\frac{n}{2}+\varepsilon}(\psi_\varepsilon)} = \|h\|_{\mathcal{H}_\varepsilon}$. We find one-to-one correspondences

$$\psi_\varepsilon\text{-Riesz transforms} \longleftrightarrow \text{Elements in } \mathsf{W}^2_{\frac{n}{2}+\varepsilon}(\psi_\varepsilon),$$

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Smooth Fourier multipliers over G

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Corollary B1

With the same assumptions

$$\|m\|_{\mathsf{M}_p(\widehat{\mathbf{G}})} \lesssim_{c(p)} |m(e)| + \inf_{m=\widetilde{m}\circ b_\psi} \left\{ \operatorname*{ess\,sup}_{s>0} \left\| \mathsf{D}_{\frac{\dim \mathcal{H}_\psi}{2}+\varepsilon} \Big(\sqrt{\psi_\varepsilon}\, \varphi_s\, \widetilde{m} \Big) \right\|_2 \right\}$$

dimension-free, for any radial Cowling/McIntosh partition of unity $(\varphi_s)_{s>0}$.

Smooth Fourier multipliers over G

- **A Hörmander-Mihlin theorem**
- **A magic formula for H-M multipliers**
- **Dimension free constants via holomorphic calculus**
- **Limiting Besov $B^2_{\frac{n}{2},2}$ conditions with a logarithmic weight**

We had $\psi_\varepsilon(\xi) = \mathrm{k}_n(\varepsilon)|\xi|^{2\varepsilon} = \displaystyle\int_{\mathbb{R}^n} \big(1 - \cos(2\pi\langle \xi, x\rangle)\big) \frac{dx}{|x|^{n+2\varepsilon}}$.

If we pick $\psi_\mu(\xi) = \displaystyle\int_{\mathbb{R}^n} \big(1 - \cos(2\pi\langle \xi, x\rangle)\big) \Big(\chi_{|x|\le 1} + \frac{\chi_{|x|>1}}{1+\log^2|x|}\Big) \frac{dx}{|x|^n} \dots$

Corollary B2

Letting $\mathrm{w}_k = \delta_{k\le 0} + k^2\delta_{k>0}$

$$\|m\|_{\mathrm{M}_p(\widehat{\mathbf{G}})} \lesssim_{c(p,n)} |m(e)| + \sup_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} 2^{nk} \mathrm{w}_k \big\| \widehat{\varphi}_k * (\sqrt{\psi_\mu}\varphi_j \widetilde{m}) \big\|_2^2 \Big)^{\frac{1}{2}}.$$

This is in the line of previous work by **Carbery, Seeger, Baernstein/Sawyer**...

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Maximal estimates and Sobolev dimension

If $\delta(\lambda(g)) = \lambda(g) \otimes \lambda(g)$ and $\sigma(\lambda(g)) = \lambda(g^{-1})$

$$T_m f = \lambda(m) \star f = (\tau \otimes \mathrm{Id})\big(\delta\lambda(m)(\sigma f \otimes \mathbf{1})\big).$$

Theorem C [González-Pérez-Junge-Parcet, Ann Sci ENS '16]

Let G be discrete, $\psi : \mathrm{G} \to \mathbb{R}_+$ an **arbitrary** c.n. length and $\eta(z) = ze^{-z}$. Given $m : \mathrm{G} \to \mathbb{C}$ constant where $\psi = 0$, assume $\lambda(m\eta(t\psi)) = \Sigma_t M_t$ with M_t positive and consider the convolution map $\mathcal{R}f = (M_t^2 \star f)_{t>0}$. If $p > 2$ we find

$$\begin{aligned} \big\| T_m : L_p(\widehat{\mathbf{G}}) \to L_p(\widehat{\mathbf{G}}) \big\| &\lesssim \Big(\sup_{t>0} \|\Sigma_t\|_2 \Big) \Big\| \mathcal{R} : L_{(\frac{p}{2})'} \to L_{(\frac{p}{2})'}(L_\infty) \Big\| \\ &= \Big(\sup_{t>0} \|\Sigma_t\|_2 \Big) \Big(\sup_{\|f\|_{(\frac{p}{2})'} \le 1} \Big\| \sup_{t>0} M_t^2 \star f \Big\|_{(\frac{p}{2})'} \Big). \end{aligned}$$

Remarks. Tradition in classical HA [Bennet, Anal & PDE '14].
By duality, a similar statement also holds for $1 < p < 2$.
Noncommutative square and maximal L_p-norms together (Pisier).

Main application.

Radial multipliers = Spectral multipliers.

Smoothness (Σ_t) wrt Sobolev Dimension (M_t) $\rightsquigarrow$ **Inf-dim cocycles admissible!!**

Maximal estimates and Sobolev dimension

If $\delta(\lambda(g)) = \lambda(g) \otimes \lambda(g)$ and $\sigma(\lambda(g)) = \lambda(g^{-1})$

$$T_m f = \lambda(m) \star f = (\tau \otimes \mathrm{Id})\big(\delta\lambda(m)(\sigma f \otimes \mathbf{1})\big).$$

Theorem C [González-Pérez-Junge-Parcet, Ann Sci ENS '16]

Let G be discrete, $\psi : \mathrm{G} \to \mathbb{R}_+$ an **arbitrary** c.n. length and $\eta(z) = ze^{-z}$. Given $m : \mathrm{G} \to \mathbb{C}$ constant where $\psi = 0$, assume $\lambda(m\eta(t\psi)) = \Sigma_t M_t$ with M_t positive and consider the convolution map $\mathcal{R}f = (M_t^2 \star f)_{t>0}$. If $p > 2$ we find

$$\begin{aligned} \big\| T_m : L_p(\widehat{\mathbf{G}}) \to L_p(\widehat{\mathbf{G}}) \big\| &\lesssim \Big(\sup_{t>0} \|\Sigma_t\|_2 \Big) \Big\| \mathcal{R} : L_{(\frac{p}{2})'} \to L_{(\frac{p}{2})'}(L_\infty) \Big\| \\ &= \Big(\sup_{t>0} \|\Sigma_t\|_2 \Big) \Big(\sup_{\|f\|_{(\frac{p}{2})'} \le 1} \Big\| \sup_{t>0} M_t^2 \star f \Big\|_{(\frac{p}{2})'} \Big). \end{aligned}$$

Remarks. Tradition in classical HA [Bennet, Anal & PDE '14].
By duality, a similar statement also holds for $1 < p < 2$.
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Nonsmooth Fourier multipliers

Directional Hilbert transforms

Given a c.n. length $\psi : \mathrm{G} \to \mathbb{R}_+$ with cocycle $(\mathcal{H}_\psi, \pi_\psi, b_\psi)$, let

$$H_{\psi,u} : \sum\nolimits_g \widehat{f}(g)\lambda(g) \mapsto 2\pi i \sum\nolimits_g \mathrm{sgn}\big\langle b_\psi(g), u \big\rangle_{\mathcal{H}_\psi} \widehat{f}(g)\lambda(g).$$

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  Let $\Gamma = \mathcal{H}_\psi \rtimes_\pi \mathrm{G}$ and $\Gamma_{\mathrm{disc}} = \mathcal{H}_{\psi,\mathrm{disc}} \rtimes_\pi \mathrm{G}$

$$\sigma : \mathcal{L}(\mathrm{G}) \ni \lambda(g) \mapsto \exp\big(2\pi i\langle b_\psi(g), \cdot\rangle\big) \rtimes \lambda(g) \in \mathcal{L}(\Gamma_{\mathrm{disc}}).$$

$H_u \rtimes_\pi id_{\mathrm{G}}$ $L_p(\widehat{\mathbf{\Gamma}}_{\mathrm{disc}})$-bounded $\Rightarrow$ $H_{\psi,u}$ $L_p(\widehat{\mathbf{G}})$-bounded

Theorem D [Parcet-Rogers, Crelle's J '16]

If $1 < p \neq 2 < \infty$ and $\dim \mathcal{H}_\psi < \infty$, tfae:

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We also find $L_1 \to L_{1,\infty}$ and $L_\infty \to \mathrm{BMO}$ type estimates for finite orbits.

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$$\Big\| \Big[\sum_{j=1}^\infty |H_{\pi_{g_j}(u)}(f_{g_j})|^2 \Big]^{\frac12} \Big\|_{L_p(\mathbb{R}^n_{\mathrm{bohr}})} \lesssim \Big\| \Big[\sum_{j=1}^\infty |f_{g_j}|^2 \Big]^{\frac12} \Big\|_{L_p(\mathbb{R}^n_{\mathrm{bohr}})} + \text{Row term}$$

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The A_j's admit a HD form of Fefferman's construction in his proof of the ball multiplier theorem

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The orbit $\mathcal{O}_\pi(u)$ is either finite or admits Kakeya shadows

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  Given $\Lambda \subset \mathrm{G}$, set $\mathcal{O}_\pi(\Lambda, u) = \big\{\pi_g(u) : g \in \Lambda\big\}$ and

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Corollary [Parcet-Rogers, AJM '15 + Crelle '16]

$\Omega \subset \mathbb{S}^{n-1}$ HD-lacunar $\Rightarrow M_\Omega$ is $L_q(\mathbb{R}^n)$-bded for $1 < q < \infty$.
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Periodic multipliers [Jodeit, Studia Math '70]

$\mathrm{G} = \mathbb{R}$ and $\psi = 1\mathrm{D}$ donut cocycle $\Rightarrow H_{\psi,u}$ L_p-bded for all u.

Chaotic idempotents [Caspers-Parcet-Perrin-Ricard, Forum Math Σ '15]

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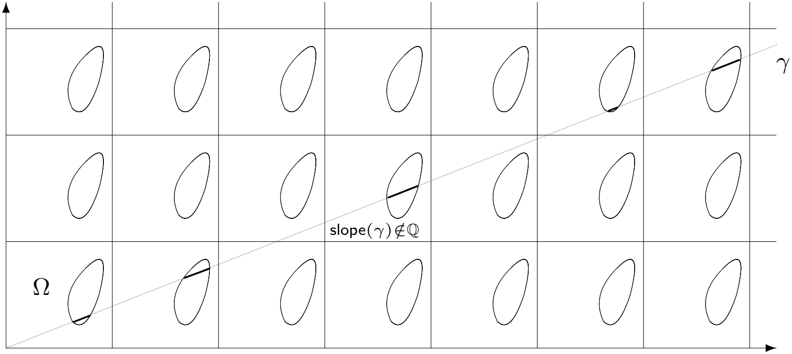

Trick: $\mathbb{Z}_{pq} \simeq \mathbb{Z}_p \times \mathbb{Z}_q$ + Ball multiplier thm

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- **Lacunary subsets of discrete groups**
- **Directional Hilbert transforms over G**

Periodic multipliers [Jodeit, Studia Math '70]

$\mathrm{G} = \mathbb{R}$ and $\psi = 1\mathrm{D}$ donut cocycle $\Rightarrow H_{\psi,u}$ L_p-bded for all u.

Chaotic idempotents [Caspers-Parcet-Perrin-Ricard, Forum Math Σ '15]

$\mathrm{G} = \mathbb{R}$ and $\psi = 2\mathrm{D}$ donut cocycle $\Rightarrow H_{\psi,u}$ L_p-unbded for most u.

Imp. $H_u \rtimes_\pi id_{\mathrm{G}} = H_{\phi,u}$ for certain (simple) cocycle $(\mathcal{K}, \rho, d)$ on Γ_{disc}.

Geometric characterization of L_p-bdness of $H_{\psi,u}$ in terms of $\mathcal{O}_\pi(u)$?

Directional Hilbert transforms

Given a c.n. length $\psi : \mathrm{G} \to \mathbb{R}_+$ with cocycle $(\mathcal{H}_\psi, \pi_\psi, b_\psi)$, let

$$H_{\psi,u} : \sum_g \widehat{f}(g)\lambda(g) \mapsto 2\pi i \sum_g \mathrm{sgn}\langle b_\psi(g), u\rangle_{\mathcal{H}_\psi} \widehat{f}(g)\lambda(g).$$

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H_{ψ,u_w} are L_p-bounded for $1 < p < \infty$. Moreover, if $\mathbb{F}_n = \langle g_j \rangle$

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Open problem for quite some time | Also implications in quantum probability

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Ball multipliers – Results and questions

Given $R > 0$, consider the ball truncations over $\mathbb{F}_n$

$$\mathcal{S}_R : \sum_{w \in \mathbb{F}_n} \widehat{f}(w)\lambda(w) \mapsto \sum_{|w| \le R} \widehat{f}(w)\lambda(w).$$

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A partial result for balls in $\mathbb{F}_n$ [Bożejko-Fendler, Banach Center Pubs '06]

$$\Big| \frac{1}{p} - \frac{1}{2} \Big| > \frac{1}{6} \ \Rightarrow \ \sup_{R>0} \Big\| \mathcal{S}_R : L_p(\widehat{\mathbf{F}}_n) \to L_p(\widehat{\mathbf{F}}_n) \Big\| = \infty.$$

Sketch of proof. The radial subalgebra $\mathcal{R}_n$ of $\mathcal{L}(\mathbb{F}_n)$ is abelian. Their result already holds in $\mathcal{R}_n$. The argument emulates the one which proves that the ball multiplier is not L_p-bded in $\mathbb{R}^n$ when $|1/p - 1/2| > 1/2n$ – see e.g. [Fefferman, Acta Math '70].

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Open problem. Prove that $\sup_{R>0} \Big\| \mathcal{S}_R : L_p(\widehat{\mathbf{F}}_n) \to L_p(\widehat{\mathbf{F}}_n) \Big\| = \infty$ for all $p \neq 2$.

Noncommutative analog of Fefferman's ball multiplier theorem [Ann Math '71] in $\mathbb{F}_n$.

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Noncommutative analog of Fefferman's ball multiplier theorem [Ann Math '71] in $\mathbb{F}_n$.

Other pairs (G, ψ) do not witness curvature... The length $\psi(k_1, k_2) = |k_1| + |k_2|$ on $\mathbb{Z}^2$ admits L_p-summability along dilations of the ψ-balls in its infinite-dimensional cocycle since they become squares with the trivial cocycle $\mathbb{Z}^2 \to \mathbb{R}^2$. An even more challenging problem is to characterize the pairs (G, ψ) witnessing curvature.

Incidence of Kazhdan property (T)

Haagerup property

Definition

A locally compact group G has the **Haagerup property** when it admits a proper cocycle. In other words, when $b_\psi^{-1}(\mathsf{K})$ is compact in G for any compact K in $\mathcal{H}_\psi$.

G **discrete group.** No compact set in $\mathcal{H}_\psi$ admits infinitely many points from $b_\psi(\mathrm{G})$.

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  - Smooth th well-understood
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- **Infinite-dimensional cocycles**
  - Open: Smooth (radial) multipliers
  - Open: Directional Hilbert transforms and balls.
  - Much richer (Riesz transforms)... Interesting cases – $\mathbb{F}_n$ and $\mathrm{SL}_2(\mathbb{R})$.

Kazhdan property (T)

Definition

A locally compact group G has **Kazhdan property (T)** when all of its cocycles are inner. In other words, cocycles of the form $g \mapsto \pi_g(u) - u \Leftrightarrow b_\psi(\mathrm{G})$ bounded in $\mathcal{H}_\psi$.

Kazhdan property (T) – Strong negation of Haagerup property for noncompact G.

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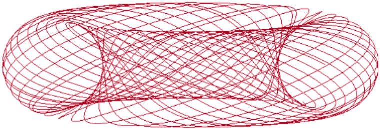
- **A clear geometric obstruction**



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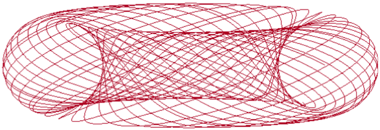

In particular, **no L_p-summability results are possible from inner cocycles**.

- **Important class of groups in HA + OA**

Noncompact semisimple Lie groups with high $\mathbb{R}$-rank (≥ 2) and sublattices

This lead us to "nonorthogonal proper cocycles" of $\mathrm{SL_n}(\mathbb{R})$ and other groups...

Fourier L_p summability over $\mathrm{SL_n}(\mathbb{R})$

Connes' rigidity conjecture

A group G is called ICC when $\left|\left\{g^{-1}hg : g \in \mathrm{G}\right\}\right| = \infty$ for all $h \neq e$.

Connes' rigidity conjecture – 1982

G_1, G_2 ICC with Kazhdan property (T): Does $\mathcal{L}(\mathrm{G}_1) \simeq \mathcal{L}(\mathrm{G}_2)$ imply $\mathrm{G}_1 \simeq \mathrm{G}_2$?

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$\mathsf{PSL}_\mathsf{n}(\mathbb{Z})$ is the quotient of $\mathsf{SL}_\mathsf{n}(\mathbb{Z})$ by its center (trivial for n odd and ± 1 for n even).

Connes' $\mathsf{PSL}_\mathsf{n}(\mathbb{Z})$ conjecture – 1982

The family of group vN algebras $\{\mathcal{L}(\mathsf{PSL}_\mathsf{n}(\mathbb{Z})) : n \geq 3\}$ are pairwise nonisomorphic

If $A_n = \mathsf{SL}_\mathsf{n}(\mathbb{Z})$ and $B_n = \mathbb{Z}^n \rtimes \mathsf{SL}_\mathsf{n}(\mathbb{Z})$, we have $A_n \subset B_n \subset A_{n+1} \ldots$
It is also an open problem to decide whether $\mathcal{L}(B_n) \simeq \mathcal{L}(B_m)$ implies $n = m$.

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G noncompact connected semisimple Lie group...
Λ **lattice in** $\mathrm{G} \rightsquigarrow$ **Is** $\mathbb{R}$**-rank** (G) **an invariant of** $\mathcal{L}(\Lambda)$**?**

CBAP – A tool for classification

Definition

An operator space = quantum Banach sp X is said to have the **CBAP** when there exists a net of finite-rank linear maps $\varphi_\alpha : \mathrm{X} \to \mathrm{X}$ satisfying the properties below:

i) $\lim_\alpha \big\| \varphi_\alpha(x) - x \big\|_{\mathrm{X}} = 0$,

ii) $\sup_\alpha \big\| \varphi_\alpha : \mathrm{X} \to \mathrm{X} \big\|_{\mathrm{cb}} < \infty$.

CBAP = Completely bounded approximation property
Other important approximation properties from Grothendieck, Haagerup...

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CBAP for discrete groups = Fourier L_p-summability

Given a discrete group G and $p < \infty$, it turns out that $\mathrm{X} = \mathrm{C}^*_\lambda(\mathrm{G})$ or $\mathrm{X} = L_p(\widehat{\mathbf{G}})$ have the CBAP when there exists a sequence $m_j : \mathrm{G} \to \mathbb{C}$ of compactly supported functions which converge pointwise to 1 such that

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An invariant of $\mathcal{L}(\mathrm{G})$...

$$\mathcal{L}(\mathrm{G}_1) \simeq \mathcal{L}(\mathrm{G}_2) \Rightarrow \Big[L_p(\widehat{\mathbf{G}}_1) \in \text{CBAP} \Leftrightarrow L_p(\widehat{\mathbf{G}}_2) \in \text{CBAP for all } p > 2 \Big].$$

Group algebras without the CBAP

A key negative result [Haagerup, Unpublished (so far) '88]

The group

$$\mathrm{G} = \mathbb{R}^2 \rtimes \mathsf{SL}_2(\mathbb{R}) \text{ **is not weakly amenable**.}$$

In other words, $\mathrm{C}^*_\lambda(\mathrm{G})$ does not have the CBAP. This immediately implies the same result for $\mathbb{K}^n \rtimes \mathsf{SL}_\mathsf{n}(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{Z}$ and $n \geq 2$. Also for $\mathsf{SL}_\mathsf{n}(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{Z}$ and $n \geq 3$. More generally, the same holds for all connected simple Lie groups with $\mathbb{R}$-rank ≥ 2 and all of their lattices.

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All these group algebras fail CBAP $\rightsquigarrow$ More subtle properties to distinguish...

The Lafforgue – de la Salle theorem

Theorem F [Lafforgue - de la Salle, Duke Math J '11]

The groups $\mathrm{G}_n = \mathsf{SL}_{\mathsf{n+1}}(\mathbb{Z})$ with $n \ge 2$ satisfy

$$\Big|\frac{1}{2} - \frac{1}{p}\Big| > \frac{1}{2(\lfloor \frac{n}{2} \rfloor + 1)} \quad \Rightarrow \quad L_p(\widehat{\mathbf{G}}_n) \text{ fails the } \mathrm{CBAP}.$$

Moreover, the same result holds for all lattices in $\mathsf{SL}_{\mathsf{n+1}}(\mathbb{R})$ and all lattices in every connected simple Lie group of $\mathbb{R}$-rank ≥ 9. Also nonarchimidean local fields like $\mathbb{Q}_q$.

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In this final form, the statement above incorporates a refinement by T. de Laat and M. de la Salle [Crelle's J '15]. Together with [Haagerup - de Laat, Duke Math J '13] this implies that all connected simple Lie groups with finite center do not even have Grothendieck's AP.

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- Main ingredient: Gelfand pairs and HA on the n-sphere.

Fourier L_p multipliers over $\mathsf{SL_n}(\mathbb{R})$?

Challenge. Positive results for L_p multipliers over $\mathsf{SL_n}(\mathbb{R})$ and $\mathsf{SL_n}(\mathbb{Z})$!
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Parallel results for lattices in G_n would yield...

- **A complete solution of Connes'** $\mathsf{PSL_n}(\mathbb{Z})$ **conjecture.**
- $\mathbb{R}$-**rank**(G) **is an invariant of** $\mathcal{L}(\Lambda)$ **for all lattices** $\Lambda \subset \mathrm{G}$.

OBSTRUCTION. NC de Leeuw restriction $\mathrm{G} \to \Lambda$ fails $\rightsquigarrow$ ad hoc argument...

Noncommutative de Leeuw restriction

Restriction theorem [de Leeuw, Ann Math '65]

If m is continuous and T_m is $L_p(\mathbb{R}^n)$-bounded

$$T_{m_{|_{\mathrm{H}}}} : \int_{\mathrm{H}} \widehat{f}(h)\chi_h \, d\mu(h) \ \mapsto \ \int_{\mathrm{H}} m(h)\widehat{f}(h)\chi_h \, d\mu(h)$$

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Theorem G [Caspers-Parcet-Perrin-Ricard, Forum Math Σ '15]

If $m : \mathrm{G} \to \mathbb{C}$ is continuous and $\mathrm{H} \subset \mathrm{G}$

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provided $\mathrm{H} \in \mathrm{ADS}$ (ok for H discrete), $\Delta_{\mathrm{G}|_{\mathrm{H}}} = 1$ (standard) and $\mathrm{G} \in [\mathrm{SAIN}]_{\mathrm{H}}$.

If $\mathrm{H} \subset \mathrm{G}$, we say that $\mathrm{G} \in [\mathrm{SAIN}]_{\mathrm{H}}$ (small almost-invariant neighborhoods) when for every $\mathrm{F} \subset \mathrm{H}$ finite, there is a basis $(V_j)_{j \ge 1}$ of symmetric neighborhoods of 1 with

$$\lim_{j \to \infty} \frac{\mu\big((h^{-1}V_j h) \bigtriangleup V_j\big)}{\mu(V_j)} = 0 \quad \text{for all} \quad h \in \mathrm{F}.$$

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Limitations of NC restriction [González-Pérez - de la Salle, Preprint '16]

The SAIN condition is essentially optimal in Theorem G. It fails for $\mathrm{SL_n}(\mathbb{Z}) \subset \mathrm{SL_n}(\mathbb{R})$.

A rough geometric intuition in $\mathsf{SL}_n(\mathbb{R})$

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$$b(g) = g \cdot u - u \quad \text{with} \quad u = (1,1,1)$$

$$\mathsf{SL}_n(\mathbb{R}) \ni g \mapsto b(g) \rtimes g \in \Gamma_n \subset \mathbb{R}^n \rtimes \mathsf{SL}_n(\mathbb{R})$$

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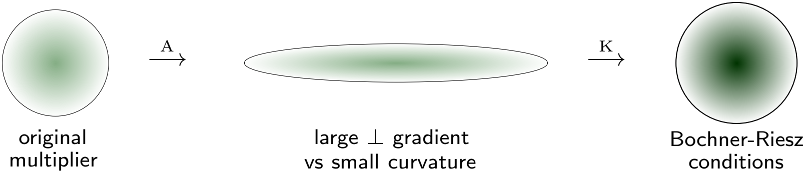

Very **far from rigorous** but hopefully illustrating
—Recall the behavior of $H_u \rtimes id_{\mathrm{G}}$—

Local Hörmander-Mihlin symbols in $\mathsf{SL_n}(\mathbb{R})$

Natural **nonisometric** "proper cocycles"

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Given $n \geq 2$, there exists

$$\Omega_n = \text{ neighborhood of the identity in } \mathsf{SL_n}(\mathbb{R})$$

such that $T_m \colon L_p(\widehat{\mathbf{SL_n}(\mathbb{R})}) \to L_p(\widehat{\mathbf{SL_n}(\mathbb{R})})$ for all $1 < p < \infty$ and all Ω_n-supported $\mathsf{SO_n}$-biinvariant symbols $m : \mathsf{SL_n}(\mathbb{R}) \to \mathbb{C}$ satisfying the β-lifted Hörmander-Mihlin smoothness condition below

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- Dilations of $\mathrm{SL_n}(\mathbb{R})$-multipliers.
- Fourier L_4-summability over $\mathrm{SL_3}(\mathbb{R})$.
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Only qualitative results known so far — Positive definite functions + $\mathcal{C}^\infty$-bumps.

Local HM over $\mathsf{SL}_n(\mathbb{R})$ – Sketch of the proof

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STEP 1. It is easy to check that $\mathrm{SL_n}(\mathbb{R})$ is $(\Omega_n, 1/2)$-amenable for some Ω_n.

- **Local δ-amenability**
- **Matrix amplification**

  Define
  $$\Phi_\alpha = \int_{\mathrm{G}} \varphi_\alpha(g) e_{gg} \, d\mu(g) \in \mathcal{B}(L_2(\mathrm{G})).$$
  Given $1 \leq p \leq \infty$, set $j_{p\alpha} : f \mapsto \Phi_\alpha^{\frac{2}{p}} j(f)$ with
  $$j : \mathcal{M} \rtimes \mathrm{G} \to \mathcal{M} \bar{\otimes} \mathcal{B}(L_2(\mathrm{G})),$$
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STEP 2. The following properties hold for $p \ge 2$:

i) $\big\| j_{p\alpha} : L_p(\mathcal{M} \rtimes_\gamma \mathrm{G}) \to L_p(\mathcal{M} \bar{\otimes} \mathcal{B}(L_2(\mathrm{G}))) \big\|_{\mathrm{cb}} \le 1.$

ii) If in addition G is (Ω, δ)-amenable, we also find that
$$\|f\|_p \le_{\mathrm{cb}} \frac{1}{1-\delta} \lim_\alpha \|j_{p\alpha}(f)\|_p \quad \text{whenever} \quad f_g = 0 \text{ for all } g \notin \Omega.$$

Local HM over $\mathsf{SL}_n(\mathbb{R})$ – Sketch of the proof

- **Local δ-amenability**
- **Matrix amplification**

  Define
  $$\Phi_\alpha = \int_{\mathrm{G}} \varphi_\alpha(g) e_{gg} \, d\mu(g) \in \mathcal{B}(L_2(\mathrm{G})).$$
  Given $1 \le p \le \infty$, set $j_{p\alpha} : f \mapsto \Phi_\alpha^{\frac{2}{p}} j(f)$ with
  $$j : \mathcal{M} \rtimes \mathrm{G} \to \mathcal{M} \bar{\otimes} \mathcal{B}(L_2(\mathrm{G})),$$
  $$j\Big(\int_{\mathrm{G}} f_g \rtimes \lambda(g) \, d\mu(g) \Big) = \Big(\gamma_{g^{-1}}(f_{gh^{-1}}) \Big)_{\mathrm{G} \times \mathrm{G}}.$$

STEP 2. The following properties hold for $p \ge 2$:

i) $\big\| j_{p\alpha} : L_p(\mathcal{M} \rtimes_\gamma \mathrm{G}) \to L_p(\mathcal{M} \bar{\otimes} \mathcal{B}(L_2(\mathrm{G}))) \big\|_{\mathrm{cb}} \le 1.$

ii) If in addition G is (Ω, δ)-amenable, we also find that

$$\|f\|_p \le_{\mathrm{cb}} \frac{1}{1-\delta} \lim_\alpha \|j_{p\alpha}(f)\|_p \quad \text{whenever} \quad f_g = 0 \text{ for all } g \notin \Omega.$$

Basic idea. $\mathsf{SL}_n(\mathbb{R}) \to L_\infty(\mathbb{R}^{n^2}) \rtimes \mathsf{SL}_n(\mathbb{R}) \to L_\infty(\mathbb{R}^{n^2}) \bar{\otimes} \mathcal{B}(L_2(\mathsf{SL}_n(\mathbb{R}))).$

Local HM over $\mathsf{SL_n}(\mathbb{R})$ – Sketch of the proof

- **Local δ-amenability**
- **Matrix amplification**
- **An operator to bound**

If $\sigma : \mathcal{L}(\mathsf{SL_n}(\mathbb{R})) \to L_\infty(\mathbb{R}^{n^2}_{\mathrm{bohr}}) \rtimes \mathsf{SL_n}(\mathbb{R})$ is the β-embedding

$$\|T_m f\|_p = \|\sigma T_m f\|_p = \|(T_{\widetilde{m}} \rtimes id)\sigma f\|_p \leq_{\mathrm{cb}} \frac{1}{1-\delta} \lim_\alpha \Big\| j_{p\alpha}\big((T_{\widetilde{m}} \rtimes id)\sigma f\big)\Big\|_p.$$

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Moreover, in the algebra $\mathcal{R}_{\mathrm{bohr}} = L_\infty(\mathbb{R}^{n^2}_{\mathrm{bohr}}) \bar{\otimes} \mathcal{B}(L_2(\mathsf{SL_n}(\mathbb{R})))$ we have

$$j_{p\alpha}\big((T_{\widetilde{m}} \rtimes id)\sigma f\big) = \underbrace{\Big(g^{-1} T_{\widetilde{m}}\ g\Big)}_{\Lambda} \bullet \Big(\varphi_\alpha(g)^{\frac{2}{p}} g^{-1} \cdot (\sigma(f)_{gh^{-1}})\Big) = \Lambda \bullet j_{p\alpha}\sigma f.$$

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The L_2-bdness of Λ is trivial. The goal is to prove $\Lambda : \mathcal{R}_{\mathrm{bohr}} \to \mathrm{BMO}(\mathcal{R}_{\mathrm{bohr}})$.
Using de Leeuw decompactification: $\mathcal{R}_{\mathrm{bohr}} \rightsquigarrow \mathcal{R} = L_\infty(\mathbb{R}^{n^2}) \bar{\otimes} \mathcal{B}(L_2(\mathsf{SL_n}(\mathbb{R})))$.
$\mathrm{BMO}(\mathcal{R}) = \mathrm{BMO}_r(\mathcal{R}) \cap \mathrm{BMO}_c(\mathcal{R}) \Rightarrow$ **AIM** $= \Lambda : \mathcal{R} \to \mathrm{BMO}_\dagger(\mathcal{R})$ for $\dagger = r, c$.

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STEP 3. Our strong HM smoothness condition implies $\Lambda : \mathcal{R} \to \mathrm{BMO}_c(\mathcal{R})$.

This follows adapting techniques in [JMP, GAFA '14] for nonequivariant actions.

Local HM over $\mathrm{SL}_n(\mathbb{R})$ – Sketch of the proof

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- **Matrix amplification**
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- **The box diagonalization**

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  Adapting [JMP, GAFA '14] once more, we see that

$$\Lambda : \mathcal{R} \to \mathrm{BMO}_r(\mathcal{R}) \Leftrightarrow \inf_{\substack{\widetilde{m}(g\xi) = \langle A_\xi, B_g \rangle_{\mathcal{K}} \\ \mathcal{K} \text{ Hilbert}}} \Big(\sup_{\xi \in \mathbb{R}^{n^2}} \|A_\xi\|_{\mathcal{K}} \sup_{g \in \mathsf{SL_n}(\mathbb{R})} \|B_g\|_{\mathcal{K}} \Big) < \infty.$$

  Equivalent to $(\widetilde{m}(g\xi))_{g\xi}$ being a **Schur multiplier** $\mathcal{B}(L_2(\mathbb{R}^{n^2})) \to \mathcal{B}(L_2(\mathsf{SL_n}(\mathbb{R})))$.

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- Λ acts on the matrix $A_m f = j_{p\alpha} \sigma T_m f$ as a Schur multiplier.
- $\operatorname{supp} m \subset \Omega_n \Rightarrow j_{p\alpha}\sigma T_m f$ is a strip-diagonal matrix $g^{-1}h \in \Omega_n$.
- A box diagonalization exploiting the geometry of $\mathsf{SL}_n(\mathbb{R})$ is possible.
- $\widetilde{m} \mapsto \widetilde{m}_g$ preserves HM constants $\Rightarrow$ Select the central box $(g,h) \in \Omega_n^2$.

Local HM over $\mathrm{SL_n}(\mathbb{R})$ – Sketch of the proof

- **Local δ-amenability**
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Since $g = k'\sigma k \in K\Sigma K = \Omega_n$, we find for $\xi \in \mathbb{S}^{n^2-1}$ that $|g\xi| = |\sigma k\xi|$ and

$$\Big\| \left(|\sigma k \xi|^{-\varepsilon} \right)_{\sigma k, \xi} \Big\|_{\mathrm{schur}} = \sup_{\sigma \in \Sigma} \Big\| \left(|\sigma k \xi|^{-\varepsilon} \right)_{k, \xi} \Big\|_{\mathrm{schur}} \sim \Big\| \left(|k \xi|^{-\varepsilon} \right)_{k, \xi} \Big\|_{\mathrm{schur}}$$

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  Similar ideas than for the Riesz transform...

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Thank you!