Second order parabolic equations with complex coefficients

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Second order parabolic operators

$$\mathcal{H}u = (\partial_t + \mathcal{L})u := \partial_t u - \mathrm{div}_X\, A(X,t)\nabla_X u = 0 \tag{0.1}$$

in $\mathbb{R}^{n+2}_+ = \{(X,t) = (x_0, x_1, .., x_n, t) \in \mathbb{R}^{n+1} \times \mathbb{R} : \ x_0 > 0\}$.

$$\begin{aligned} &(i) \qquad \kappa|\xi|^2 \leq \mathrm{Re}\, A(X,t)\xi\cdot\bar{\xi} = \mathrm{Re}\,\Big(\sum_{i,j=0}^{n} A_{i,j}(X,t)\xi_i\bar{\xi}_j\Big),\\ &(ii) \qquad |A(X,t)\xi\cdot\zeta| \leq C|\xi||\zeta|. \end{aligned} \tag{0.2}$$

Regularity/structural assumptions on A.

$(D2)$, $(N2)$ and $(R2)$

Dirichlet problem with data $f \in L^2(\mathbb{R}^{n+1}, \mathbb{C})$:

$$\begin{aligned} \mathcal{H}u &= 0 \text{ in } \mathbb{R}^{n+2}_+, \\ \lim_{x_0 \to 0} u(x_0, \cdot, \cdot) &= f(\cdot, \cdot), \\ \sup_{x_0 > 0} ||u(x_0, \cdot, \cdot)||_2 + |||x_0 \nabla u||| &< \infty. \end{aligned}$$

Neumann problem with data $g \in L^2(\mathbb{R}^{n+1}, \mathbb{C})$:

$$\begin{aligned} \mathcal{H}u &= 0 \text{ in } \mathbb{R}^{n+2}_+, \\ \lim_{x_0 \to 0} \partial_{\nu_A} u(x_0, \cdot, \cdot) &= g(\cdot, \cdot), \\ \widetilde{N}_*(\nabla u) &\in L^2(\mathbb{R}^{n+1}). \end{aligned}$$

$(D2)$, $(N2)$ and $(R2)$

$$
\begin{aligned}
\partial_t &= D_t^{1/2} H_t D_t^{1/2} (= |\tau|^{1/2} i\mathrm{sign}(\tau)|\tau|^{1/2}),\\
\|f\|_{\mathbb{H}(\mathbb{R}^{n+1},\mathbb{C})} &:= \|\nabla_x f\|_2 + \|H_t D^t_{1/2} f\|_2\\
&\approx ||\mathbb{D}\widehat{f}||_2 := \||(\xi,\tau)\|\widehat{f}\|_2\\
&\approx \|\sqrt{|\xi|^2 + i\tau}\widehat{f}\|_2 = \|\sqrt{\partial_t - \mathrm{div}_x \nabla_x} f\|_2.
\end{aligned}
$$

Regularity problem with data $f \in \mathbb{H}(\mathbb{R}^{n+1},\mathbb{C})$:

$$
\begin{aligned}
\mathcal{H}u &= 0 \text{ in } \mathbb{R}^{n+2}_+,\\
\lim_{x_0\to 0} u(x_0,\cdot,\cdot) &= f(\cdot,\cdot),\\
\widetilde{N}_*(\nabla u) \in L^2(\mathbb{R}^{n+1}),\ \widetilde{N}_*(H_t D^t_{1/2} u) &\in L^2(\mathbb{R}^{n+1}).
\end{aligned}
$$

Elliptic problems: some recent results

1. Alfonseca M. A., Auscher P., Axelsson A., Hofmann S., Kim S. Analyticity of layer potentials and L^2 solvability of boundary value problems for divergence form elliptic equations with complex L^∞ coefficients, Adv. Math., 226 (2011), 4533-4606.
2. Auscher P., Axelsson A., Hofmann S. Functional calculus of Dirac operators and complex perturbations of Neumann and Dirichlet problems. J. Funct. Anal. 255 (2008), no. 2, 374-448.
3. Auscher P., Axelsson A. Weighted maximal regularity estimates and solvability of non-smooth elliptic systems I, Invent. Math. 184 (2011), no. 1, 47-115.
4. Hofmann, S., Kenig, C.E., Mayboroda S., and Pipher, J. Square function/non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators, J. Amer. Math. Soc. 28 (2015), 483-529.

Elliptic problems: the Kato square root estimate

- Auscher, P., Hofmann, S., Lacey, M., McIntosh, A., and Tchamitchian, P. The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$, Ann. of Math. (2) 156, 2 (2002), 633-654.

$$\mathcal{L}_{||} := -\operatorname{div}_x A_{||||}(x)\nabla_x,\ \mathcal{L}_{||}^* = -\operatorname{div}_x A_{||||}^*(x)\nabla_x.$$

$$\sqrt{\mathcal{L}_{||}}f = a\int_0^\infty (I+\lambda^2\mathcal{L}_{||})^{-3}\lambda^3\mathcal{L}_{||}^2 f\,\frac{d\lambda}{\lambda}.$$

$$\begin{aligned}|\langle\sqrt{\mathcal{L}_{||}}f,g\rangle|^2 &\leq a^2\Big(\int_0^\infty\int_{\mathbb{R}^n}|\lambda(I+\lambda^2\mathcal{L}_{||})^{-1}\mathcal{L}_{||}f|^2\,\frac{dxd\lambda}{\lambda}\Big)\\ &\quad\times\Big(\int_0^\infty\int_{\mathbb{R}^n}|\lambda^2\mathcal{L}_{||}^*(I+\lambda^2\mathcal{L}_{||}^*)^{-2}g|^2\,\frac{dxd\lambda}{\lambda}\Big)\\ &\leq c||\nabla_x f||_2||g||_2.\end{aligned}$$

Parabolic problems: recent results

1. Square functions estimates and the Kato problem for second order parabolic operators, Advances in Mathematics 293 (2016), 1-36, submitted Jun 2015.
2. (with Castro, A. and Sande, O.) Boundedness of single layer potentials associated to divergence form parabolic equations with complex coefficients, to appear in Calculus of Variations and Partial Differential Equations, submitted in Oct 2015.
3. L^2 Solvability of boundary value problems for divergence form parabolic equations with complex coefficients, submitted in Dec 2015.
4. (with P. Auscher and M. Egert) Boundary value problems for parabolic systems via a first order approach, submitted in Jul 2016.

Outline of lectures

$$\mathcal{H}u = (\partial_t + \mathcal{L})u := \partial_t u - \mathrm{div}_X\, A(X,t)\nabla_X u = 0$$

in $\mathbb{R}^{n+2}_+ = \{(X,t) = (x_0, x_1, .., x_n, t) \in \mathbb{R}^{n+1} \times \mathbb{R} : \ x_0 > 0\}$.

1. Part I - parabolic problems: ideas and concepts.
2. Part II - the second order approach to BVPs for parabolic equations with complex coefficients.
3. Part III - the first order approach to BVPs for parabolic equations with complex coefficients.

Notation and conventions

$$(X, t) = (x_0, x, t) = (x_0, x_1, .., x_n, t) =: (\lambda, x_1, .., x_n, t) = (\lambda, x, t).$$

$$\nabla_X = \nabla_{\lambda,x} = (\partial_\lambda, \nabla_x) = (\partial_\lambda, \nabla_{||}),$$
$$\mathrm{div}_X = \mathrm{div}_{\lambda,x} = (\partial_\lambda, \mathrm{div}_x) = (\partial_\lambda, \mathrm{div}_{||}).$$

$$\mathcal{H}_{||} := \partial_t + \mathcal{L}_{||} = \partial_t - \mathrm{div}_{||}(A_{||||}(X, t)\nabla_{||}\cdot).$$

$$\mathcal{H}^* = -\partial_t + \mathcal{L}^*,\ \mathcal{H}_{||}^* = -\partial_t + \mathcal{L}_{||}^*.$$

Given $(\lambda, x, t) \in \mathbb{R}^{n+2}$ and $r > 0$: $\Lambda = \Lambda_r(\lambda) := (\lambda - r, \lambda + r)$, $Q = Q_r(x) := B(x, r) \subset \mathbb{R}^n$, $I = I_r(t) := (t - r^2, t + r^2)$, $\Delta = \Delta_r(x, t) = Q_r(x) \times I_r(t)$.

Parabolic cubes: $\Delta \subset \mathbb{R}^{n+1}$, $\Lambda \times \Delta \subset \mathbb{R}^{n+2}$.

Part I - parabolic problems: ideas and concepts

$$\mathcal{H}u = (\partial_t + \mathcal{L})u := \partial_t u - \operatorname{div}_X A(X,t)\nabla_X u = 0$$

in $\mathbb{R}^{n+2}_+ = \{(X,t) = (x_0, x_1, .., x_n, t) \in \mathbb{R}^{n+1} \times \mathbb{R} : \ x_0 > 0\}$.

1. Reinforced weak solutions to $\mathcal{H}u = 0$.
2. Discovering hidden coercivity in $\mathcal{H}$.
3. Existence of reinforced weak solutions.
4. Sectoriality and maximal accretivity of $\mathcal{H}_{\|}$.
5. An associated first order system $\partial_\lambda F + PMF = 0$.
6. Bisectoriality of PM.
7. The core: square function/quadratic estimates.
8. Differences t-independent/-dependent coefficients.
9. Real coefficients: parabolic measure $\omega(X,t,\cdot)$.

Weak solutions

u is a weak solution on $\mathbb{R}^{n+1}_+ \times \mathbb{R}$ if $u \in L^2_{\text{loc}}(\mathbb{R}; \mathrm{W}^{1,2}_{\text{loc}}(\mathbb{R}^{n+1}_+))$ and for all $\phi \in \mathrm{C}^\infty_0(\mathbb{R}^{n+2}_+)$,

$$\int_{\mathbb{R}} \iint_{\mathbb{R}^{n+1}_+} \left(A\nabla_{\lambda,x} u \cdot \overline{\nabla_{\lambda,x}\phi} - u \cdot \overline{\partial_t \phi}\right) dx d\lambda dt = 0. \tag{1.1}$$

(1.1) implies $\partial_t u \in L^2_{loc}(\mathbb{R}; \mathrm{W}^{-1,2}_{\text{loc}}(\mathbb{R}^{n+1}_+))$.

A problem: if we somehow want to control

$$||\nabla_{\lambda,x} u||_2 + ||H_t D^t_{1/2} u||_2$$

we notice a lack of coercivity in the form in (1.1).

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Function spaces

$\dot{H}^{1/2}(\mathbb{R})$ is the homogeneous Sobolev space of order 1/2: it is the completion of $C_0^\infty(\mathbb{R})$ for the semi-norm $\|D_t^{1/2}\cdot\|_2$.

$\dot{\mathrm{E}}$ is the energy space: it is the closure of $C_0^\infty\left(\overline{\mathbb{R}_+^{n+2}}\right)$ w.r.t.

$$\|v\|_{\dot{\mathrm{E}}} := \left(\|\nabla_{\lambda,x} v\|^2_{L^2(\mathbb{R}_+^{n+2})} + \|H_t D_t^{1/2} v\|^2_{L^2(\mathbb{R}_+^{n+2})}\right)^{1/2} < \infty.$$

Modulo constants, $\dot{\mathrm{E}}$ is a Hilbert space.

$\dot{H}^s_{\pm\partial_t-\Delta_x}$: the closure of functions $v \in \mathcal{S}(\mathbb{R}^{n+1})$ with Fourier support away from the origin for the norm $\|\mathcal{F}^{-1}((|\xi|^2 \pm i\tau)^s \widehat{v})\|_2$.

Reinforced weak solutions

u is a *reinforced weak solution* on $\mathbb{R}^{n+1}_+ \times \mathbb{R}$ if

$$u \in \dot{\mathrm{E}}_{\mathrm{loc}} := \dot{H}^{1/2}(\mathbb{R}; L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)) \cap L^2_{loc}(\mathbb{R}; \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+))$$

and if for all $\phi \in \mathrm{C}^\infty_0(\mathbb{R}^{n+2}_+)$,

$$\int_0^\infty \iint_{\mathbb{R}^{n+1}} \left(A\nabla_{\lambda,x} u \cdot \overline{\nabla_{\lambda,x}\phi} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2}\phi}\right) dxdtd\lambda = 0.$$

If $u \in \dot{H}^{1/2}(\mathbb{R})$ and $\phi \in \mathrm{C}^\infty_0(\mathbb{R})$ then

$$\int_{\mathbb{R}} H_t D_t^{1/2} u \cdot \overline{D_t^{1/2}\phi}\, dt = -\int_{\mathbb{R}} u \cdot \overline{\partial_t \phi}\, dt.$$

A reinforced weak solution is a weak solution in the usual sense on $\mathbb{R}^{n+1}_+ \times \mathbb{R}$.

Discovering hidden coercivity

Consider the modified sesquilinear form

$$a_\delta(u,v) = \iiint_{\mathbb{R}^{n+2}_+} A\nabla_{\lambda,x}u \cdot \overline{\nabla_{\lambda,x}(1+\delta H_t)v}\, d\lambda dx dt + \iiint_{\mathbb{R}^{n+2}_+} H_t D_t^{1/2}u \cdot \overline{D_t^{1/2}(1+\delta H_t)v}\, d\lambda dx dt,$$

where $\delta > 0$ is a (real) degree of freedom.

If we fix $\delta > 0$ small enough, then

$$\operatorname{Re} a_\delta(u,u) \geq (\kappa - C\delta)\|\nabla_{\lambda,x}u\|_2^2 + \delta\|H_t D_t^{1/2}u\|_2^2$$

where κ, C are the ellipticity constants for A.

Lemma

$\dot{\mathrm{E}}/\mathbb{C}$ continuously embeds into $\mathrm{C}([0,\infty); \dot{H}^{1/4}_{\partial_t-\Delta_x})$. Any $f \in \dot{H}^{1/4}_{\partial_t-\Delta_x}$ has an extension $v \in \dot{\mathrm{E}}$ such that $v|_{\lambda=0} = f$.

Proof: $\||\tau|^{1/4}\widehat{v}|_{\lambda=\lambda_0}\|_2^2 + \||\xi|^{1/2}\widehat{v}|_{\lambda=\lambda_0}\|_2^2$ equals

$$
\begin{aligned}
&2\,\mathrm{Re}\int_{\lambda_0}^{\infty}\left(\left(|\tau|^{1/2}\widehat{v}, \partial_\lambda\widehat{v}\right) + \left(|\xi|\widehat{v}, \partial_\lambda\widehat{v}\right)\right)d\lambda \\
&\leq \int_0^{\infty}\left(\||\tau|^{1/2}\widehat{v}\|_2^2 + \||\xi|\widehat{v}\|_2^2 + 2\|\partial_\lambda\widehat{v}\|_2^2\right)d\lambda.
\end{aligned}
$$

Conversely, given $f \in \mathrm{C}_0^{\infty}(\mathbb{R}^{n+1})$, we can define

$$
v(\lambda, x, t) = \mathcal{F}^{-1}\left(\mathrm{e}^{-\lambda(|\xi|^2+i\tau)^{1/2}}\widehat{f}\right)(x,t).
$$

Then $\|v\|_{\dot{\mathrm{E}}} \lesssim \|f\|_{\dot{H}^{1/4}_{\partial_t-\Delta_x}}$.

Energy solutions - Dirichlet problem

An energy solution to (0.1) with Dirichlet boundary data $u|_{\lambda=0} = f \in \dot{H}^{1/4}_{\partial_t - \Delta_x}$ is a reinforced weak sol $u \in \dot{\mathrm{E}}$ such that

$$\iiint_{\mathbb{R}^{n+2}_+} \left(A\nabla_{\lambda,x} u \cdot \overline{\nabla_{\lambda,x} v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v}\right) d\lambda dx dt = 0,$$

$\forall\ v \in \dot{\mathrm{E}}_0 =$ the subspace of $\dot{\mathrm{E}}$ with zero boundary trace.

Existence. Take an extension $w \in \dot{\mathrm{E}}$ of the data f and apply the Lax-Milgram lemma to a_δ on $\dot{\mathrm{E}}_0$ to obtain some $u \in \dot{\mathrm{E}}_0$ such that

$$a_\delta(u, v) = -a_\delta(w, v) \qquad (v \in \dot{\mathrm{E}}_0).$$

Uniqueness. $\tilde{u}$ a solution: then $a_\delta(u + w - \tilde{u}, u + w - \tilde{u}) = 0$ and hence $\|u + w - \tilde{u}\|_{\dot{\mathrm{E}}} = 0$ by coercivity.

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Energy solutions - Neumann problem

$$\partial_{\nu_A} u(\lambda, x, t) := [1, 0, \ldots, 0] \cdot (A\nabla_{\lambda,x} u)(\lambda, x, t).$$

An energy solution to (0.1) with Neumann boundary data $\partial_{\nu_A} u|_{\lambda=0} = f \in \dot{H}^{-1/4}_{\partial_t - \Delta_x}$ is a reinforced weak sol $u \in \dot{\mathrm{E}}$,

$$\iiint_{\mathbb{R}^{n+2}_+} \left(A\nabla_{\lambda,x} u \cdot \overline{\nabla_{\lambda,x} v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v}\right) d\lambda dx dt = -\langle f, v|_{\lambda=0}\rangle,$$

$\forall v \in \dot{\mathrm{E}}$ $\langle \cdot, \cdot \rangle$ denotes the pairing of $\dot{H}^{-1/4}_{\partial_t - \Delta_x}$ with $\dot{H}^{1/4}_{-\partial_t - \Delta_x}$.

Solving the Neumann problem: find $u \in \dot{\mathrm{E}}$

$$a_\delta(u, v) = -\langle f, (1 + \delta H_t) v|_{\lambda=0}\rangle \qquad (v \in \dot{\mathrm{E}}).$$

Lax-Milgram applied to a_δ on $\dot{\mathrm{E}}$ yields a unique such u.

Maximal accretivity and sectoriality

$$V := H^{1/2}(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}; \mathrm{W}^{1,2}(\mathbb{R}^n)).$$

Consider $H_{||} := \partial_t - \mathrm{div}_x\, A_{||||}(x,t)\nabla_x : V \to V^*$ defined via

$$\langle H_{||}u, v\rangle := \iint_{\mathbb{R}^{n+1}} \left(A_{||||}\nabla_x u \cdot \overline{\nabla_x v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v} \right) dxdt,$$

$u, v \in V$. $\mathrm{D}(H_{||}) = \{u \in V : H_{||}u \in L^2(\mathbb{R}^{n+1})\}$.

If $\theta \in \mathbb{C}$ with Re $\theta > 0$, then

$$\theta + H_{||} : \mathrm{D}(H_{||}) \to L^2(\mathbb{R}^{n+1}, \mathbb{C})$$

is bijective and the resolvent satisfies the estimate

$$\|(\theta + H_{||})^{-1} f\|_2 \leq \frac{1}{\mathrm{Re}\ \theta}\|f\|_2.$$

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$$\|(\theta + H_{||})^{-1} f\|_2 \leq \frac{1}{\mathrm{Re}\ \theta} \|f\|_2.$$

Maximal accretivity and sectoriality

$H_{||}$ is maximal accretive with domain
$\mathrm{D}(H_{||}) = \{u \in V : H_{||}u \in L^2(\mathbb{R}^{n+1}, \mathbb{C})\}$ in $L^2(\mathbb{R}^{n+1}, \mathbb{C})$.
$H_{||}$ is sectorial.

$H_{||}$ has a bounded H^∞ calculus and there is a square root $\sqrt{H_{||}}$ abstractly defined by functional calculus.

The inequality

$$||\sqrt{H_{||}} f||_2^2 \leq c \int_0^\infty \iint_{\mathbb{R}^{n+1}} |\lambda(I + \lambda^2 H_{||})^{-1} H_{||} f|^2 \frac{dxdtd\lambda}{\lambda},$$

does hold for all $f \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C})$.

Note: no assumptions on $A_{||||} = A_{||||}(x, t)$ besides measurability and uniform ellipticity have been imposed.

An associated first order system

$$0 = \partial_t u - \Delta_X u = D_t^{1/2} H_t D_t^{1/2} u - \mathrm{div}_x \nabla_x u - \partial_\lambda \partial_\lambda u.$$

Given a reinforced weak solution u to the heat equation:

$$D_l u(\lambda, x, t) := \begin{bmatrix} \partial_\lambda u(\lambda, x, t) \\ \nabla_x u(\lambda, x, t) \\ H_t D_t^{1/2} u(\lambda, x, t) \end{bmatrix} =: \begin{bmatrix} F_\perp \\ F_\parallel \\ F_\theta \end{bmatrix}.$$

Then

$$\begin{aligned} \partial_\lambda F_\perp &= -\mathrm{div}_x F_\parallel + D_t^{1/2} F_\theta, \\ \partial_\lambda F_\parallel &= \nabla_x F_\perp, \\ \partial_\lambda F_\theta &= H_t D_t^{1/2} F_\perp. \end{aligned}$$

$$\mathbb{L}^2 := L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2}).$$

An associated first order system

$$\partial_\lambda F + PF = 0 \text{ where } P := \begin{bmatrix} 0 & \mathrm{div}_x & -D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -H_t D_t^{1/2} & 0 & 0 \end{bmatrix}.$$

The operator P is independent of λ, defined as an unbounded operator in $\mathbb{L}^2$ with maximal domain. The adjoint of P is

$$P^* = \begin{bmatrix} 0 & \mathrm{div}_x & H_t D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -D_t^{1/2} & 0 & 0 \end{bmatrix}.$$

1. The operator P contains fractional (non-local) time derivatives!
2. P is not self-adjoint: $P \neq P^*$!

An associated first order system

Given a reinforced weak solution u to $\mathcal{H}u = 0$:

$$D_A u(\lambda, x, t) = \begin{bmatrix} \partial_{\nu_A} u(\lambda, x, t) \\ \nabla_x u(\lambda, x, t) \\ H_t D_t^{1/2} u(\lambda, x, t) \end{bmatrix}.$$

Then,

$$|D_A u|^2 \sim |\nabla_{\lambda,x} u|^2 + |H_t D_t^{1/2} u|^2.$$

We split the coefficient matrix A as

$$A(\lambda, x, t) = \begin{bmatrix} A_{\perp\perp}(\lambda, x, t) & A_{\perp\|}(\lambda, x, t) \\ A_{\|\perp}(\lambda, x, t) & A_{\|\|}(\lambda, x, t) \end{bmatrix}.$$

The pointwise transformation

$$A \mapsto \hat{A} := \begin{bmatrix} A_{\perp\perp}^{-1} & -A_{\perp\perp}^{-1} A_{\perp\|} \\ A_{\|\perp} A_{\perp\perp}^{-1} & A_{\|\|} - A_{\|\perp} A_{\perp\perp}^{-1} A_{\perp\|} \end{bmatrix}$$

is a self-inverse bijective transformation of the set of bounded matrices which are strictly accretive.

An associated first order system

$$M := \begin{bmatrix} \hat{A}_{\perp\perp} & \hat{A}_{\perp\|} & 0 \\ \hat{A}_{\|\perp} & \hat{A}_{\|\|} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\mathcal{H}_{\mathrm{loc}}$ is the subspace of $L^2_{loc}(\mathbb{R}_+; \mathbb{L}^2)$ defined by the compatibility conditions

$$\mathrm{curl}_x F_\| = 0, \quad \nabla_x F_\theta = H_t D_t^{1/2} F_\|.$$

Approach: find reinforced weak solutions u with $D_A u \in L^2_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2}))$ by solving

$$\partial_\lambda F + PMF = 0 \tag{1.2}$$

in the weak sense in the space $\mathcal{H}_{\mathrm{loc}}$.

An associated first order system

$$M := \begin{bmatrix} \hat{A}_{\perp\perp} & \hat{A}_{\perp\|} & 0 \\ \hat{A}_{\|\perp} & \hat{A}_{\|\|} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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Approach: find reinforced weak solutions u with $D_A u \in L^2_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2}))$ by solving

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in the weak sense in the space $\mathcal{H}_{\text{loc}}$.

Bisectoriality

The differential equation (1.2) is autonomous and can be solved via semigroup techniques, provided the semigroup is well-defined. This requires that *PM* has a *bounded holomorphic functional calculus*.

T in a Hilbert space is *bisectorial* of *angle* $\omega \in (0, \pi/2)$ if it is closed and its spectrum is contained in the closure of

$$S_\omega := \{z \in \mathbb{C} : |\arg z| < \omega \text{ or } |\arg z - \pi| < \omega\}$$

and if for each $\mu \in (\omega, \pi/2)$ the map $z \mapsto z(z - T)^{-1}$ is uniformly bounded on $\mathbb{C} \setminus S_\mu$.

Bisectoriality of *PM*

Lemma

The operator PM is bisectorial on $\mathbb{L}^2$ *with* $\mathrm{R}(PM) = \mathrm{R}(P)$.

Proof: Consider for $\delta \in \mathbb{R}$,

$$U_\delta := \frac{1}{\sqrt{1+\delta^2}} \begin{bmatrix} 1-\delta H_t & 0 & 0 \\ 0 & 1+\delta H_t & 0 \\ 0 & 0 & \delta - H_t \end{bmatrix}.$$

Write $PM = (PU_\delta)(U_\delta^{-1}M)$.

Claim: PU_δ is self-adjoint and $U_\delta^{-1}M$ is accretive for $\delta > 0$ small enough.

Bisectoriality of PM

$\sqrt{1+\delta^2}PU_\delta$ equals

$$\begin{bmatrix} 0 & \mathrm{div}_x(1+\delta H_t) & -\delta D_t^{1/2} + H_t D_t^{1/2} \\ -\nabla_x(1-\delta H_t) & 0 & 0 \\ -H_t D_t^{1/2} - \delta D_t^{1/2} & 0 & 0 \end{bmatrix}.$$

$U_\delta^{-1}M$ equals

$$\begin{bmatrix} (1+\delta H_t)\hat{A}_{\perp\perp} & (1+\delta H_t)\hat{A}_{\perp\|} & 0 \\ (1-\delta H_t)\hat{A}_{\|\perp} & (1-\delta H_t)\hat{A}_{\|\|} & 0 \\ 0 & 0 & \delta + H_t \end{bmatrix}.$$

Lower block: accretive for all $\delta > 0$ as $\mathrm{Re}(H_t g, g) = 0$.
Upper block: accretive if δ is small enough as $\hat{A}$ accretive.

$$\mathrm{R}(PM) = \mathrm{R}((PU_\delta)(U_\delta^{-1}M)) = \mathrm{R}(PU_\delta) = \mathrm{R}(P).$$

$$(i)\ \int_0^\infty \iint_{\mathbb{R}^{n+1}} |\lambda(I+\lambda^2\mathcal{H}_{||})^{-1}\mathcal{H}_{||}f|^2 \frac{dxdtd\lambda}{\lambda} \sim ||\mathbb{D}f||_2^2 \quad (f \in \dot{\mathsf{E}}_{||}),$$

$$(ii)\ \int_0^\infty \|\lambda PM(1+\lambda^2 PMPM)^{-1}h\|_2^2 \frac{d\lambda}{\lambda} \sim \|h\|_2^2 \quad (h \in \overline{\mathsf{R}(P)}).$$

(i) is a special case of (ii): take

$$M := \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_{||||} & 0 \\ 0 & 0 & 1 \end{bmatrix}, h := \begin{bmatrix} 0 \\ -\nabla_{||} f \\ -H_t D_t^{1/2} f \end{bmatrix} = P \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}.$$

The Kato square root estimate: the case $A_{||||}^* = A_{||||}$ does not follow from abstract functional analysis as $\mathcal{H}_{||}$ not self-adjoint.

The core: square function/quadratic estimates

$$(i)\ \int_0^\infty \iint_{\mathbb{R}^{n+1}} |\lambda(I+\lambda^2\mathcal{H}_{||})^{-1}\mathcal{H}_{||}f|^2\,\frac{dxdtd\lambda}{\lambda} \sim ||\mathbb{D}f||_2^2 \quad (f \in \dot{\mathrm{E}}_{||}),$$

$$(ii)\ \int_0^\infty \|\lambda PM(1+\lambda^2 PMPM)^{-1}h\|_2^2\,\frac{d\lambda}{\lambda} \sim \|h\|_2^2 \quad (h \in \overline{\mathrm{R}(P)}).$$

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The Kato square root estimate: the case $A_{||||}^* = A_{||||}$ does not follow from abstract functional analysis as $\mathcal{H}_{||}$ not self-adjoint.

Differences t-independent/-dependent coefficients

1. Poincare inequalities: local (involving $\nabla_X u$, $\partial_t u$) - non-local (involving $\nabla_X u$, $D_t^\alpha u$, $H_t D_t^\alpha u$).
2. Carleson measure estimate: sufficient to control
$$|(I+\lambda^2\mathcal{H}_{||})^{-1}\operatorname{div}_{||} A_{||||}|^2\lambda\, dxdtd\lambda,$$
-
Carleson measures involving PM (MP).
3. Off-diagonal estimates: strong/classical form (for $\lambda(I+\lambda^2\mathcal{H}_{||})^{-1}\operatorname{div}_{||}$ with constant $e^{-c^{-1}(d_p(E,F)/\lambda)}$) - weaker/novel formulation (for $(1+i\lambda PM)^{-1}$ on cylinders additionally stretched in time).
4. Tb theorem: test functions closer to the elliptic construction - a construction which handles $H_t D_t^{1/2}$.

Part II - the second order approach to BVPs for parabolic equations with complex coefficients

(λ, t)-independent coefficients

$$\mathcal{H}u = (\partial_t + \mathcal{L})u := \partial_t u - \mathrm{div}_X A(X,t)\nabla_X u = 0$$

in $\mathbb{R}_+^{n+2} = \{(X,t) = (x_0, x_1, .., x_n, t) \in \mathbb{R}^{n+1} \times \mathbb{R} : \ x_0 > 0\}$.

$(X,t) = (x_0, x, t) = (x_0, x_1, .., x_n, t) =: (\lambda, x_1, .., x_n, t) = (\lambda, x, t)$.

$A(\lambda, x, t) = A(x)$.

$\mathcal{E}_\lambda := (I + \lambda^2 \mathcal{H}_{||})^{-1}$, $\mathcal{E}_\lambda^* := (I + \lambda^2 \mathcal{H}_{||}^*)^{-1}$.

$$|||\cdot||| := \left(\iiint_{\mathbb{R}_+^{n+2}} |\cdot|^2 \frac{dxdtd\lambda}{\lambda} \right)^{1/2}.$$

Square function estimates

Theorem

The estimate

$$|||\lambda\mathcal{E}_\lambda\mathcal{H}_{||}f||| + |||\lambda\mathcal{E}_\lambda^*\mathcal{H}_{||}^*f||| \leq c||\mathbb{D}f||_2, \qquad (2.1)$$

hold whenever $f \in \mathbb{H}(\mathbb{R}^{n+1}, \mathbb{C}) = \dot{\mathsf{E}}_{||}$.

(2.1) gives the Kato estimate for t-independent coefficients.

Lemma

Let $\lambda > 0$ *be given. Let* $\mathcal{U}_\lambda := \lambda\mathcal{E}_\lambda \operatorname{div}_{||}$. *Then*

$$\int_0^{l(\Delta)} \iint_\Delta |\mathcal{U}_\lambda A_{|||}|^2 \frac{dxdtd\lambda}{\lambda} \leq c|\Delta| \text{ for all } \Delta \subset \mathbb{R}^{n+1}.$$

Off-diagonal estimates for resolvents

Lemma

Let $\lambda > 0$ *be given. Let*

$$\Theta_\lambda = \{\mathcal{E}_\lambda, \lambda\nabla_{||}\mathcal{E}_\lambda\}, \tilde{\Theta}_\lambda = \{\lambda\mathcal{E}_\lambda \operatorname{div}_{||},\ \lambda^2\nabla_{||}\mathcal{E}_\lambda \operatorname{div}_{||}\}.$$

Let E *and* F *be two closed sets in* $\mathbb{R}^{n+1}$ *,*

$$d = \inf\{||(x-y, t-s)|| :\ (x,t) \in E,\ (y,s) \in F\}.$$

Then

$$(i) \iint_F |\Theta_\lambda f(x,t)|^2\, dxdt \leq ce^{-c^{-1}(d_p(E,F)/\lambda)} \iint_E |f(x,t)|^2\, dxdt,$$

$$(ii) \iint_F |\tilde{\Theta}_\lambda \mathbf{f}(x,t)|^2\, dxdt \leq ce^{-c^{-1}(d_p(E,F)/\lambda)} \iint_E |\mathbf{f}(x,t)|^2\, dxdt,$$

if $f, \mathbf{f} \in L^2$*, and supp* $f \subset E$*, supp* $\mathbf{f} \subset E$*.*

Reduction to the Carleson measure estimate

$$\lambda\mathcal{E}_\lambda\mathcal{H}_{||}f = \lambda\mathcal{E}_\lambda\mathcal{H}_{||}(I - P_\lambda)f + \lambda\mathcal{E}_\lambda\mathcal{H}_{||}P_\lambda f.$$

Using the identity

$$\lambda\mathcal{E}_\lambda\mathcal{H}_{||} = \lambda^{-1}(I - \mathcal{E}_\lambda),$$

$$|||\lambda\mathcal{E}_\lambda\mathcal{H}_{||}(I - P_\lambda)f||| \leq |||\lambda^{-1}(I - P_\lambda)f||| \leq c||\mathbb{D}f||_2.$$

Furthermore,

$$\lambda\mathcal{E}_\lambda\mathcal{H}_{||}P_\lambda f = \lambda\mathcal{E}_\lambda\partial_t P_\lambda f + \lambda\mathcal{E}_\lambda\mathcal{L}_{||}P_\lambda f.$$

We note that

$$\begin{aligned} |||\lambda\mathcal{E}_\lambda\partial_t P_\lambda f||| \leq c|||\lambda\partial_t P_\lambda f||| &\leq |||\lambda(\mathbb{D}P_\lambda)(\mathbb{D}_{n+1}f)||| \\ &\leq c||\mathbb{D}_{n+1}f||_2 \leq c||D_t^{1/2}f||_2. \end{aligned}$$

Term remaining:

$$|||\lambda\mathcal{E}_\lambda\mathcal{L}_{||}P_\lambda f||| = |||\mathcal{U}_\lambda A_{|||}\nabla_{||}P_\lambda f|||, \ \mathcal{U}_\lambda := \lambda\mathcal{E}_\lambda \operatorname{div}_{||}$$

Reduction to the Carleson measure estimate

Let $P_\lambda = \tilde{P}_\lambda^2$ and introduce

$$\mathcal{R}_\lambda = \mathcal{U}_\lambda A_{\|\|} \tilde{P}_\lambda - (\mathcal{U}_\lambda A_{\|\|}) \tilde{P}_\lambda.$$

Then

$$\mathcal{U}_\lambda A_{\|\|} \nabla_{\|} P_\lambda f = \mathcal{U}_\lambda A_{\|\|} P_\lambda \nabla_{\|} f = \mathcal{R}_\lambda \tilde{P}_\lambda \nabla_{\|} f + (\mathcal{U}_\lambda A_{\|\|}) P_\lambda \nabla_{\|} f.$$

$\mathcal{R}_\lambda 1 = 0$ and

$$||\mathcal{R}_\lambda \tilde{P}_\lambda \nabla_{\|} f||_2 \leq c||\lambda \nabla_{\|} \tilde{P}_\lambda \nabla_{\|} f||_2 + ||\lambda^2 \partial_t \tilde{P}_\lambda \nabla_{\|} f||_2,$$

uniformly in λ. Using this

$$|||\mathcal{R}_\lambda \tilde{P}_\lambda \nabla_{\|} f||| \leq c||\nabla_{\|} f||_2.$$

Remaining estimate:

$$|||(\mathcal{U}_\lambda A_{\|\|}) P_\lambda \nabla_{\|} f||| \leq c||\nabla_{\|} f||_2.$$

Test functions for local Tb-theorem

Lemma

Let w be a unit vector in $\mathbb{C}^n$ and let $0 < \epsilon \ll 1$ be a degree of freedom. Given a parabolic cube $\Delta \subset \mathbb{R}^{n+1}$, with center (x_Δ, t_Δ), we let

$$f_{\Delta,w}^{\epsilon} = (I + (\epsilon l(\Delta))^2 \mathcal{H}_{\|})^{-1}(\chi_\Delta(\Phi_\Delta \cdot \bar{w}))$$

where $\Phi_\Delta = x - x_\Delta$ and where $\chi_\Delta = \chi_\Delta(x, t)$ is a smooth cut off for Δ. Then

$$(i) \qquad \iint_{\mathbb{R}^{n+1}} |f_{\Delta,w}^{\epsilon} - \chi_\Delta(\Phi_\Delta \cdot \bar{w})|^2 \, dxdt \leq c(\epsilon l(\Delta))^2 |\Delta|,$$

$$(ii) \qquad \iint_{\mathbb{R}^{n+1}} |\nabla_{\|}(f_{\Delta,w}^{\epsilon} - \chi_\Delta(\Phi_\Delta \cdot \bar{w}))|^2 \, dxdt \leq c|\Delta|.$$

The local Tb-theorem

Lemma

Then there exists $\epsilon \in (0,1)$, *depending only on* n, κ, C, *and a finite set* W *of unit vectors in* $\mathbb{C}^n$, *whose cardinality depends on* ϵ *and* n, *such that*

$$(i) \qquad \iint_{\mathbb{R}^{n+1}} |\mathbb{D} f^{\epsilon}_{\Delta,w}|^2 \, dxdt \leq c_1 |\Delta|,$$

$$(ii) \qquad \iint_{\mathbb{R}^{n+1}} \left(|\partial_t f^{\epsilon}_{\Delta,w}|^2 + |\mathcal{L}_{||} f^{\epsilon}_{\Delta,w}|^2\right) dxdt \leq c_2 |\Delta| / l(\Delta)^2,$$

$$(iii) \qquad \frac{1}{|\Delta|} \int_0^{l(\Delta)} \iint_{\Delta} |\mathcal{U}_\lambda A_{||||}|^2 \frac{dxdtd\lambda}{\lambda}$$

$$\leq c_3 \sum_{w \in W} \frac{1}{|\Delta|} \int_0^{l(\Delta)} \iint_{\Delta} |(\mathcal{U}_\lambda A_{||||}) \cdot \mathcal{S}^{\Delta}_{\lambda} \nabla_{||} f^{\epsilon}_{\Delta,w}| \frac{dxdtd\lambda}{\lambda}.$$

Here $\mathcal{S}^{\Delta}_{\lambda}$ *is a dyadic averaging operator induced by* Δ.

Applications to BVPs: ingredients

To develop a parabolic version of [AAAHK] you need a number of ingredients:

1. Existence theory for resolvents $\mathcal{E}_\lambda := (I + \lambda^2 \mathcal{H}_{||})^{-1}$.
2. Estimates for resolvents: L^2-boundedness, off-diagonal estimates,...
3. De Giorgi–Moser-Nash estimates.
4. Estimates for single layer potentials: kernel estimates, uniform (in λ) L^2-estimates, off-diagonal estimates,...
5. .....
6. Square function estimates (for composed operators).
7. Invertibility by analytic perturbation theory.
8. Real symmetric coefficients: a reverse Hölder inequality for the parabolic Poisson kernel associated to $\mathcal{H}$.

De Giorgi–Moser-Nash estimates

Let $\Lambda \times \Delta \subset \mathbb{R}^{n+2}$, $r = l(\Lambda \times \Delta)$, $\mathcal{H}u = 0$ in $2(\Lambda \times \Delta)$. Then

$$\sup_{\Lambda\times\Delta} |u| \leq c \fint\!\!\fint\!\!\fint_{2(\Lambda\times\Delta)} |u|,$$

and

$$|u(X,t) - u(\tilde{X},\tilde{t})| \leq c\left(\frac{||(X-\tilde{X}, t-\tilde{t})||}{r}\right)^{\alpha} \fint\!\!\fint\!\!\fint_{2(\Lambda\times\Delta)} |u|,$$

whenever (X,t), $(\tilde{X},\tilde{t}) \in \Lambda \times \Delta$.

Layer potentials

$K_t(X, Y)$ kernel of $e^{-t\mathcal{L}}$,

$$\begin{aligned}\Gamma(\lambda, x, t, \sigma, y, s) &= \Gamma(X, t, Y, s)\\ &:= K_{t-s}(X, Y) = K_{t-s}(\lambda, x, \sigma, y).\end{aligned}$$

$$\begin{aligned}\Gamma_\lambda(x, t, y, s) &:= \Gamma(\lambda, x, t, 0, y, s),\\ \Gamma^*_\lambda(y, s, x, t) &:= \Gamma^*(0, y, s, \lambda, x, t),\end{aligned}$$

Associated single layer potentials:

$$\begin{aligned}\mathcal{S}^{\mathcal{H}}_\lambda f(x, t) &:= \iint_{\mathbb{R}^{n+1}} \Gamma_\lambda(x, t, y, s) f(y, s)\, dyds,\\ \mathcal{S}^{\mathcal{H}^*}_\lambda f(x, t) &:= \iint_{\mathbb{R}^{n+1}} \Gamma^*_\lambda(y, s, x, t) f(y, s)\, dyds.\end{aligned}$$

Double layer potentials: $\mathcal{D}^{\mathcal{H}}_\lambda f(x, t)$, $\mathcal{D}^{\mathcal{H}^*}_\lambda f(x, t)$.

Bounded, invertible and good layer potentials

1. The core estimates.
2. Estimates of non-tangential maxs and Sobolev norms. Example: $||\tilde{N}_*^{\pm}(\nabla_x \mathcal{S}_\lambda^{\mathcal{H}} f)||_2 + ||\tilde{N}_*^{\pm}(\nabla_x \mathcal{S}_\lambda^{\mathcal{H}^*} f)||_2 \leq \Gamma ||f||_2$.
3. Existence of boundary operators. Example: $(\pm\frac{1}{2}I + \mathcal{K}^{\mathcal{H}})$, $(\pm\frac{1}{2}I + \tilde{\mathcal{K}}^{\mathcal{H}})$, $\mathbb{D}\mathcal{S}_\lambda^{\mathcal{H}}|_{\lambda=0}$, exist.
4. Estimates for boundary operators. Example: $||(\pm\frac{1}{2}I + \mathcal{K}^{\mathcal{H}})f||_2 \approx ||f||_2 \approx ||(\pm\frac{1}{2}I + \tilde{\mathcal{K}}^{\mathcal{H}})f||_2$.
5. Invertibility of boundary operators. Example: $(\pm\frac{1}{2}I + \mathcal{K}^{\mathcal{H}})$, $(\pm\frac{1}{2}I + \tilde{\mathcal{K}}^{\mathcal{H}})$, are invertible on $L^2(\mathbb{R}^{n+1}, \mathbb{C})$.

$\mathcal{H}$, $\mathcal{H}^*$ have bounded, invertible and good layer potentials with constant $\Gamma \geq 1$.

Theorem

Assume that $\mathcal{H}_0$, $\mathcal{H}_0^*$, $\mathcal{H}_1$, $\mathcal{H}_1^*$ are as above and satisfy De Giorgi-Moser-Nash estimates. Assume that

> $\mathcal{H}_0$, $\mathcal{H}_0^*$, *have bounded, invertible and good layer potentials for some constant* Γ_0.

Then there exists a constant ε_0, depending at most on n, Λ, the De Giorgi-Moser-Nash constants and Γ_0, such that if

$$||A^1 - A^0||_\infty \leq \varepsilon_0,$$

then there exists a constant Γ_1, depending at most on n, Λ, the De Giorgi-Moser-Nash constants and Γ_0, such that

> $\mathcal{H}_1$, $\mathcal{H}_1^*$, *have bounded, invertible and good layer potentials with constant* Γ_1.

Corollary

Assume that $\mathcal{H}_0$, $\mathcal{H}_0^*$, $\mathcal{H}_1$, $\mathcal{H}_1^*$ are as above and satisfy De Giorgi-Moser-Nash estimates. Assume that

> $(D2)$, $(N2)$ *and* $(R2)$ *are uniquely solvable, for* $\mathcal{H}_0$, $\mathcal{H}_0^*$ *by way of layer potentials and for a constant* Γ_0.

Then there exists a constant ε_0, depending at most on n, Λ, the De Giorgi-Moser-Nash constants and Γ_0, such that if

$$||A^1 - A^0||_\infty \leq \varepsilon_0,$$

then then there exists a constant Γ_1, depending at most on n, Λ, the De Giorgi-Moser-Nash constants and Γ_0, such that

> $(D2)$, $(N2)$ *and* $(R2)$ *are uniquely solvable, for* $\mathcal{H}_1$, $\mathcal{H}_1^*$, *by way of layer potentials and with constant* Γ_1.

Solvability for $(D2)$, $(N2)$ and $(R2)$

We establish the solvability for $(D2)$, $(N2)$ and $(R2)$, by way of layer potentials, when the coefficient matrix is either

(i)	a small complex perturbation of a constant (complex) matrix, or,
(ii)	a real and symmetric matrix, or,
(iii)	a small complex perturbation of a real and symmetric matrix.

In cases (i) – (iii) the De Giorgi-Moser-Nash estimates hold.

Good layer potentials: the core estimates

$$|||\cdot|||_{\pm} = \left(\iiint_{\mathbb{R}^{n+2}_{\pm}} |\cdot|^2 \frac{dxdtd\lambda}{|\lambda|}\right)^{1/2}, \ |||\cdot||| := |||\cdot|||_{+}.$$

The core estimates,

$$(i) \qquad \sup_{\lambda\neq 0} ||\partial_\lambda \mathcal{S}_\lambda^{\mathcal{H}} f||_2 + \sup_{\lambda\neq 0} ||\partial_\lambda \mathcal{S}_\lambda^{\mathcal{H}^*} f||_2 \leq \Gamma ||f||_2,$$

$$(ii) \qquad |||\lambda\partial_\lambda^2 \mathcal{S}_\lambda^{\mathcal{H}} f|||_{\pm} + |||\lambda\partial_\lambda^2 \mathcal{S}_\lambda^{\mathcal{H}^*} f|||_{\pm} \leq \Gamma ||f||_2,$$

whenever $f \in L^2(\mathbb{R}^{n+1}, \mathbb{C})$ and where $\mathcal{S}_\lambda^{\mathcal{H}} f$ and $\mathcal{S}_\lambda^{\mathcal{H}^*} f$.

Technical challenge: prove that the core estimates are stable under small complex perturbations of the coefficient matrix.

Invertibility by analytic perturbation theory

Lemma

Assume that $\mathcal{H}_0$, $\mathcal{H}_0^*$, $\mathcal{H}_1$, $\mathcal{H}_1^*$ are as above and satisfy De Giorgi-Moser-Nash estimates. Assume that

> *$\mathcal{H}_0$, $\mathcal{H}_0^*$, have bounded, invertible and good layer potentials for some constant Γ_0.*

If

$$||A^1 - A^0||_\infty \leq \varepsilon_0, \textit{ then}$$

$$\begin{aligned}
||\mathcal{K}^{\mathcal{H}_0} - \mathcal{K}^{\mathcal{H}_1}||_{2\to 2} + ||\tilde{\mathcal{K}}^{\mathcal{H}_0} - \tilde{\mathcal{K}}^{\mathcal{H}_1}||_{2\to 2} &\leq c\varepsilon_0, \\
||\nabla_{||}\mathcal{S}_\lambda^{\mathcal{H}_0}|_{\lambda=0} - \nabla_{||}\mathcal{S}_\lambda^{\mathcal{H}_1}|_{\lambda=0}||_{2\to 2} &\leq c\varepsilon_0, \\
||H_t D^t_{1/2}\mathcal{S}_\lambda^{\mathcal{H}_0}|_{\lambda=0} - H_t D^t_{1/2}\mathcal{S}_\lambda^{\mathcal{H}_1}|_{\lambda=0}||_{2\to 2} &\leq c\varepsilon_0.
\end{aligned}$$

Real coefficients: parabolic measure

Given $f \in C(\mathbb{R}^{n+1}) \cap L^\infty(\mathbb{R}^{n+1})$,

$$u(X,t) := \iint_{\mathbb{R}^{n+1}} f(y,s)\, d\omega(X,t,y,s),$$

gives the solution to the continuous Dirichlet problem

$$\begin{aligned} &\mathcal{H}u = 0 \text{ in } \mathbb{R}^{n+2}_+,\\ &u \in C([0,\infty) \times \mathbb{R}^{n+1}),\\ &u(0,x,t) = f(x,t) \text{ on } \mathbb{R}^{n+1}. \end{aligned}$$

$\{\omega(X,t,\cdot) : (X,t) \in \mathbb{R}^{n+2}_+\}$ is a family of regular Borel measures on $\mathbb{R}^{n+1}$: the $\mathcal{H}$-caloric, or $\mathcal{H}$-parabolic measure.

Parabolic measure is a doubling measure

Given $(x, t) \in \mathbb{R}^{n+1}$ and $r > 0$,

$$A_r^+(x, t) := (4r, x, t + 16r^2).$$

Theorem

Assume that A is real and satisfies (0.2)*. If* $(x_0, t_0) \in \mathbb{R}^{n+1}$*,* $0 < r_0 < \infty$*,* $\Delta := \Delta_{r_0}(x_0, t_0)$*, then*

$$\omega\big(A^+_{4r_0}(x_0, t_0), 2\tilde{\Delta}\big) \leq c\omega\big(A^+_{4r_0}(x_0, t_0), \tilde{\Delta}\big)$$

whenever $\tilde{\Delta} \subset 4\Delta$*.*

The theorem holds more generally in Lip(1,1/2) domains and in parabolic NTA-domains.

A reverse Hölder inequality for the Poisson kernel

Theorem

$\mathcal{H} = \partial_t - \text{div}_X(A(x)\nabla_X)$. Suppose in addition that A is real and symmetric. Then there exists $c \geq 1$, depending only on n and κ, C, such that

$$\iint_{\Delta} |k^{A_\Delta}(y,s)|^2 \, dyds \leq c|\Delta|^{-1},$$

where $\Delta \subset \mathbb{R}^{n+1}$ is a parabolic cube and $k^{A_\Delta}(y,s)$ is the to $\mathcal{H}$ associated Poisson kernel at $A_\Delta := (l(\Delta), x_\Delta, t_\Delta)$.

This, and other Rellich type estimates, use, in a crucial way, symmetry of A and that A is independent of (λ, t).

Theorem

Assume $\exists$ *system* $\{b_\Delta\}$ *of functions,*

$$\begin{array}{ll} (i) & \displaystyle\iint_{\mathbb{R}^{n+1}} |b_\Delta(x,t)|^2\, dxdt \leq c|\Delta|, \\ (ii) & \displaystyle\int_0^{l(\Delta)} \iint_\Delta |\theta_\lambda b_\Delta(x,t)|^2 \frac{dxdtd\lambda}{\lambda} \leq c|\Delta|, \\ (iii) & \displaystyle c^{-1}|\Delta| \leq Re \iint_\Delta b_\Delta(x,t)\, dxdt. \end{array}$$

Then there exists a constant c such that

$$|||\theta_\lambda f||| = \left(\int_0^\infty \iint_{\mathbb{R}^{n+1}} |\theta_\lambda f(x,t)|^2 \frac{dxdtd\lambda}{\lambda}\right)^{1/2} \leq c||f||_2,$$

whenever $f \in L^2(\mathbb{R}^{n+1}, \mathbb{C})$.

Applying the local Tb-theorem for square functions

$$\theta_\lambda f(x,t) := \iint_{\mathbb{R}^{n+1}} \lambda \partial_\lambda^2 \Gamma_\lambda(x,t,y,s) f(y,s)\, dyds.$$

$$b_\Delta(y,s) := |\Delta| 1_\Delta \tilde{k}_-^{A_\Delta^-}(y,s).$$

$$\begin{aligned}(ii)\ \theta_\lambda b_\Delta(x,t) &= \iint_{\mathbb{R}^{n+1}} \lambda \partial_\lambda^2 \Gamma_\lambda(x,t,y,s) b_\Delta(y,s)\, dyds \\ &= \lambda|\Delta| \iint_\Delta \partial_\lambda^2 \Gamma_\lambda(x,t,y,s) \tilde{k}_-^{A_\Delta^-}(y,s)\, dyds \\ &= \lambda|\Delta| \big(\partial_\lambda^2 \Gamma(\lambda, x, t, -l(\Delta), x_\Delta, t_\Delta)\big).\end{aligned}$$

$$(iii)\ \iint_{\mathbb{R}^{n+1}} b_\Delta(y,s)\, dyds = |\Delta| \tilde{\omega}_-^{A_\Delta^-}(\Delta) \geq c^{-1}|\Delta|.$$

Part III - the first order approach to BVPs for parabolic equations with complex coefficients

The associated first order system

$$A(\lambda, x, t) = A(x, t).$$

Given a reinforced weak solution u to $\mathcal{H}u = 0$:

$$D_A u(\lambda, x, t) := \begin{bmatrix} \partial_{\nu_A} u(\lambda, x, t) \\ \nabla_x u(\lambda, x, t) \\ H_t D_t^{1/2} u(\lambda, x, t) \end{bmatrix}.$$

$$P := \begin{bmatrix} 0 & \mathrm{div}_x & -D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -H_t D_t^{1/2} & 0 & 0 \end{bmatrix}, \quad P^* = \begin{bmatrix} 0 & \mathrm{div}_x & H_t D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -D_t^{1/2} & 0 & 0 \end{bmatrix},$$

$$M := \begin{bmatrix} \hat{A}_{\perp\perp} & \hat{A}_{\perp\|} & 0 \\ \hat{A}_{\|\perp} & \hat{A}_{\|\|} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Correspondence to the first order system

Theorem

If u is a reinforced weak solution, $F := D_A u \in \mathcal{H}_{\text{loc}}$, *then*

$$\iiint_{\mathbb{R}^{n+2}_+} F \cdot \overline{\partial_\lambda \phi}\, d\lambda dx dt = \iiint_{\mathbb{R}^{n+2}_+} MF \cdot \overline{P^* \phi}\, d\lambda dx dt \tag{3.1}$$

for all $\phi \in C_0^\infty(\mathbb{R}^{n+2}_+; \mathbb{C}^{n+2})$. *Conversely, if* $F \in \mathcal{H}_{\text{loc}}$ *satisfies* (3.1) *for all* ϕ, *then there exists a reinforced weak solution u, unique up to a constant, such that* $F = D_A u$.

Conclusion: we can construct all reinforced weak solutions u with $D_A u \in L^2_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2}))$ by solving

$$\partial_\lambda F + PMF = 0 \tag{3.2}$$

in the weak sense in the space $\mathcal{H}_{\text{loc}}$.

Quadratic estimates

PM is a bisectorial operator on $L^2(\mathbb{R}^{n+1};\mathbb{C}^{n+2})$ with angle $\omega = \omega(n,\kappa,C)$ and $\mathrm{R}(PM) = \mathrm{R}(P)$.

By bisectoriality there is a topological splitting

$$L^2(\mathbb{R}^{n+1};\mathbb{C}^{n+2}) = \overline{\mathrm{R}(P)} \oplus \mathrm{N}(PM).$$

Theorem

The following estimate holds for all $h \in \overline{\mathrm{R}(PM)}$

$$\int_0^\infty \|\lambda PM(1+\lambda^2 PMPM)^{-1}h\|_2^2 \,\frac{\mathrm{d}\lambda}{\lambda} \sim \|h\|_2^2 \tag{3.3}$$

Holds with PM, $\mathrm{R}(PM)$, replaced by MP, $\mathrm{R}(MP) = M\mathrm{R}(P)$.

A key observation: hidden coercivity and Šneĭberg

Lemma

There exists $\delta_0 > 0$ such that if p, q with $|\frac{1}{p} - \frac{1}{2}| < \delta_0$ and $|\frac{1}{q} - \frac{1}{2}| < \delta_0$, $\lambda \in \mathbb{R}$, then the resolvent $(1 + i\lambda PM)^{-1}$ is bounded on $L^p(\mathbb{R}; L^q(\mathbb{R}^n; \mathbb{C}^{n+2}))$ with uniform bounds with respect to λ. The same result holds with MP, P^*M or MP^* in place of PM.

Proof: For $1 < p, q < \infty$, we define

$$H_{p,q}(\mathbb{R}^{n+1}) := L^p(\mathbb{R}; W^{1,q}(\mathbb{R}^n)) \cap H^{1/2,p}(\mathbb{R}; L^q(\mathbb{R}^n))$$

equipped with

$$\|u\|_{H_{p,q}} := \Big\| \big\| |u| + |\nabla_x u| + |D_t^{1/2} u| \big\|_{L^q(\mathbb{R}^n)} \Big\|_{L^p(\mathbb{R})}.$$

We let $H^*_{p,q}$ denote the space dual to $H_{p,q}$.

A key observation: hidden coercivity and Šneĭberg

$\lambda = 1$. Given $g \in L^2(\mathbb{R}^{n+1};\mathbb{C}^{n+2})$, $\exists$ unique $\widetilde{g} \in L^2(\mathbb{R}^{n+1};\mathbb{C}^{n+2})$,

$$g = \left[(A\widetilde{g})_\perp \quad \widetilde{g}_\parallel \quad \widetilde{g}_\theta\right]^* .$$

$(1+iPM)^{-1}f = g$ is equivalent to a system for $\widetilde{g}_\perp$

$$\begin{cases} (\partial_t + L_A)\widetilde{g}_\perp = f_\perp - A_{\perp\parallel}f_\parallel - i\,\mathrm{div}_x(A_{\parallel\parallel}f_\parallel) + iD_t^{1/2}f_\theta, \\ \widetilde{g}_\parallel - i\nabla_x\widetilde{g}_\perp = f_\parallel, \\ \widetilde{g}_\theta - iH_tD_t^{1/2}\widetilde{g}_\perp = f_\theta, \end{cases}$$

where

$$L_A := \left[1 \quad i\,\mathrm{div}_x\right] A \begin{bmatrix} 1 \\ i\nabla_x \end{bmatrix} = A_{\perp\perp} + \mathcal{H}_\parallel + \text{first order terms.}$$

$\partial_t + L_A$ admits hidden coercivity, invertible $H_{2,2} \to H^*_{2',2'}$, bounded $H_{p,q} \to H^*_{p',q'}$: we can apply Šneĭberg's lemma.

Off-diagonal type estimates for resolvents

$$C_k(Q \times J) := (2^{k+1}Q \times N^{k+1}J) \setminus (2^k Q \times N^k J)$$

Proposition

There exists $\varepsilon_0 > 0$ *and* $N_0 > 1$ *such that if* $|\frac{1}{q} - \frac{1}{2}| < \varepsilon_0$*, then one can find* $\varepsilon = \varepsilon(n, q, \varepsilon_0) > 0$ *with the following property: given* $N \geq N_0$*, there exists* $C = C(\varepsilon, N_0, q) < \infty$ *such that*

$$\iint_{Q \times 4^j I} |(1 + i\lambda PM)^{-1} h|^q \, dyds \leq CN^{-q\varepsilon k} \iint_{C_k(Q \times 4^j I)} |h|^q \, dyds$$

whenever $Q = B(x, r) \subset \mathbb{R}^n$*,* $I = (t - r^2, t + r^2)$*,* $\lambda \sim r$*,* $j \in \mathbb{N}$*,* $k \in \mathbb{N}^*$ *and provided* $h \in (L^2 \cap L^q)(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$ *has support in* $C_k(Q \times 4^j I)$*.*

Analogous estimates with PM replaced by MP, $P^* M$ or MP^*.

Proof of the off-diagonal type estimates ($q = 2$)

We set $J = 4^j I$ and

$$C_k^1 := (2^{k+1}Q \setminus 2^k Q) \times N^{k+1}J, \; C_k^2 := 2^k Q \times (N^{k+1}J \setminus N^k J).$$

We write $h = h_1 + h_2$ with h_i supported in C_k^i.

$$\iint_{Q\times J} |(1+i\lambda PM)^{-1}h_1|^2 dyds \leq \frac{1}{\ell(J)|Q|}\int_{\mathbb{R}}\int_Q |(1+i\lambda PM)^{-1}h_1|^2 dyds.$$

Using spatial off-diagonal estimates we obtain for any $m \in \mathbb{N}$,

$$\begin{aligned}\iint_{Q\times J} |(1+i\lambda PM)^{-1}h_1|^2\, dyds &\lesssim \frac{2^{-km}}{\ell(J)|Q|}\int_{\mathbb{R}}\int_{\mathbb{R}^n} |h_1|^2\, dyds \\ &= 2^{-km}N^{k+1}2^{kn}\iint_{C_k(Q\times J)} |h|^2\, dyds.\end{aligned}$$

Proof of the off-diagonal type estimates ($q = 2$)

Smooth cut-off function $\eta \in C_0^\infty(N^{k-1}J)$, equal to 1 on $N^{k-2}J$ and satisfies $(N^k\ell(J))\|\partial_t\eta\|_\infty \lesssim 1$. With $p > 2$

$$\iint_{Q\times J} |(1+i\lambda PM)^{-1}h_2|^2\, dyds$$

is bounded by

$$\begin{aligned}
&\frac{1}{|Q|}\fint_J\int_{\mathbb{R}^n} |(1+i\lambda PM)^{-1}h_2|^2\, dyds \\
&\leq \frac{1}{|Q|}\left(\fint_J\left(\int_{\mathbb{R}^n} |(1+i\lambda PM)^{-1}h_2|^2 dy\right)^{p/2} ds\right)^{2/p} \\
&= \frac{1}{|Q|\ell(J)^{2/p}}\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n} |\eta(1+i\lambda PM)^{-1}h_2|^2 dy\right)^{p/2} ds\right)^{2/p}.
\end{aligned}$$

As $\eta(t)h_2(x,t) = 0$, we can re-express $\eta(1+i\lambda PM)^{-1}h_2$ using a commutator

$$\begin{aligned}\eta(1+i\lambda PM)^{-1}h_2 &= [\eta,(1+i\lambda PM)^{-1}]h_2\\ &= (1+i\lambda PM)^{-1}[\eta,i\lambda PM](1+i\lambda PM)^{-1}h_2\\ &= (1+i\lambda PM)^{-1}i\lambda[\eta,P]M(1+i\lambda PM)^{-1}h_2,\end{aligned}$$

where

$$[\eta,P] = \begin{bmatrix} 0 & 0 & -[\eta,D_t^{1/2}]\\ 0 & 0 & 0\\ -[\eta,H_tD_t^{1/2}] & 0 & 0\end{bmatrix}.$$

$$\|[\eta,D_t^{1/2}]\|_{L^2(\mathbb{R})\to L^p(\mathbb{R})} + \|[\eta,H_tD_t^{1/2}]\|_{L^2(\mathbb{R})\to L^p(\mathbb{R})} \lesssim (N^k\ell(J))^{-(1-(1/p))}.$$

It follows that $[\eta,P] : L^2(L^2) \to L^p(L^2)$ with norm as above.

Proof of the off-diagonal type estimates ($q = 2$)

$$
\begin{aligned}
&\frac{1}{|Q|\ell(J)^{2/p}}\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n}|\eta(1+i\lambda PM)^{-1}h_2|^2\,dy\right)^{p/2}ds\right)^{2/p}\\
&\lesssim \frac{1}{|Q|\ell(J)^{2/p}}\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n}|i\lambda[\eta,P]M(1+i\lambda PM)^{-1}h_2|^2\,dy\right)^{p/2}ds\right)^{2/p}\\
&\lesssim \frac{|\lambda|^2}{|Q|\ell(J)^{2/p}}\cdot\frac{1}{(N^k\ell(J))^{(2-2/p)}}\int_{\mathbb{R}}\int_{\mathbb{R}^n}|M(1+i\lambda PM)^{-1}h_2|^2\,dyds\\
&\lesssim \frac{|\lambda|^2}{|Q|\ell(J)^2(N^k)^{2-2/p}}\int_{\mathbb{R}}\int_{\mathbb{R}^n}|h_2|^2\,dyds\\
&\lesssim \frac{2^{kn}}{(N^k)^{1-2/p}}\frac{|\lambda|^2}{\ell(J)}\iint_{C_k(Q\times J)}|h|^2\,dyds\\
&\lesssim \frac{2^{kn}}{(N^k)^{1-2/p}}\iint_{C_k(Q\times J)}|h|^2\,dyds.
\end{aligned}
$$

Proof of the off-diagonal type estimates ($q = 2$)

We have proved that

$$\iint_{Q\times J} |(1 + i\lambda PM)^{-1} h|^2 \, dyds \lesssim 2^{-km} N^{k+1} 2^{kn} \iint_{C_k(Q\times J)} |h|^2 \, dyds + \frac{2^{kn}}{(N^k)^{1-2/p}} \iint_{C_k(Q\times J)} |h|^2 \, dyds.$$

First we pick $0 < \varepsilon < \varepsilon_0$ and then p with $\varepsilon < \frac{1}{2} - \frac{1}{p} < \varepsilon_0$. For N large enough $2^{kn} \lesssim N^{-2k\varepsilon}(N^k)^{1-2/p}$ and given any such choice of N, there is a choice of m large verifying

$$2^{-km} N^{k+1} 2^{kn} \lesssim N^{-2k\varepsilon}.$$

Proof of the quadratic estimate

Suffices to prove

$$\int_0^\infty \|\lambda MP(1+\lambda^2 MPMP)^{-1}h\|_2^2 \frac{\mathrm{d}\lambda}{\lambda} \lesssim \|h\|_2^2 \qquad (h \in \mathbb{L}^2) \quad (3.4)$$

and the analogous estimate with MP replaced by M^*P^*.

We set $R_\lambda = (1+i\lambda MP)^{-1}$ for $\lambda \in \mathbb{R}$. Then

$$Q_\lambda = \frac{1}{2i}(R_{-\lambda} - R_\lambda) = \lambda MP(1+\lambda^2 MPMP)^{-1}.$$

As Q_λ vanishes on $\mathrm{N}(MP) = \mathrm{N}(P)$: it is enough to prove (3.4) for $h \in \mathrm{R}(MP)$. $\Theta_\lambda := Q_\lambda M$. Suffices to prove

$$\int_0^\infty ||\Theta_\lambda Pv||_2^2 \frac{d\lambda}{\lambda} \lesssim \|Pv\|_2^2 \qquad (v \in \mathrm{D}(P)). \qquad (3.5)$$

Reduction to a Carleson measure estimate

$$\begin{aligned}\Theta_\lambda P v =& \Theta_\lambda(1-P_\lambda)Pv + (\Theta_\lambda - \gamma_\lambda S_\lambda)P_\lambda P v\\ &+ \gamma_\lambda S_\lambda(P_\lambda - S_\lambda)Pv + \gamma_\lambda S_\lambda P v.\end{aligned}$$

$$\begin{aligned}\int_0^\infty \|\Theta_\lambda(1-P_\lambda)Pv\|_2^2\,\frac{d\lambda}{\lambda} &\lesssim \|Pv\|_2^2,\\ \int_0^\infty \|(\Theta_\lambda - \gamma_\lambda S_\lambda)P_\lambda Pv\|_2^2\,\frac{d\lambda}{\lambda} &\lesssim \int_0^\infty \|\lambda\nabla_x P_\lambda Pv\|_2^2\,\frac{d\lambda}{\lambda}\\ &\quad+\int_0^\infty \|\lambda^{2\alpha}D_t^\alpha P_\lambda Pv\|_2^2\,\frac{d\lambda}{\lambda},\\ \int_0^\infty \|\gamma_\lambda S_\lambda(P_\lambda - S_\lambda)Pv\|_2^2\,\frac{d\lambda}{\lambda} &\lesssim \int_0^\infty \|(P_\lambda - S_\lambda)Pv\|_2^2\,\frac{d\lambda}{\lambda}.\end{aligned}$$

Reduction to a Carleson measure estimate

$$\Theta_\lambda Pv = \Theta_\lambda(1-P_\lambda)Pv + (\Theta_\lambda - \gamma_\lambda S_\lambda)P_\lambda Pv + \gamma_\lambda S_\lambda(P_\lambda - S_\lambda)Pv + \gamma_\lambda S_\lambda Pv.$$

$$\begin{aligned}
\int_0^\infty \|\Theta_\lambda(1-P_\lambda)Pv\|_2^2 \frac{d\lambda}{\lambda} &\lesssim \|Pv\|_2^2, \\
\int_0^\infty \|(\Theta_\lambda - \gamma_\lambda S_\lambda)P_\lambda Pv\|_2^2 \frac{d\lambda}{\lambda} &\lesssim \int_0^\infty \|\lambda \nabla_x P_\lambda Pv\|_2^2 \frac{d\lambda}{\lambda} \\
&\quad + \int_0^\infty \|\lambda^{2\alpha} D_t^\alpha P_\lambda Pv\|_2^2 \frac{d\lambda}{\lambda}, \\
\int_0^\infty \|\gamma_\lambda S_\lambda(P_\lambda - S_\lambda)Pv\|_2^2 \frac{d\lambda}{\lambda} &\lesssim \int_0^\infty \|(P_\lambda - S_\lambda)Pv\|_2^2 \frac{d\lambda}{\lambda}.
\end{aligned}$$

Reduction to a Carleson measure estimate

$$\int_0^\infty \|\gamma_\lambda S_\lambda P v\|_2^2 \frac{d\lambda}{\lambda} \lesssim \|\gamma_\lambda\|_C^2 \|Pv\|_2^2 \qquad (v \in \mathrm{D}(P)),$$

$$\|\gamma_\lambda\|_C^2 := \sup_{\Delta \in \Box} \frac{1}{|\Delta|} \int_0^{\ell(\Delta)} \iint_\Delta |\gamma_\lambda(x,t)|^2 \frac{dxdtd\lambda}{\lambda}.$$

Test functions for local Tb-theorem: one can construct test functions which can handle the non-local terms appearing in P.

Consequences of the quadratic estimate

For any bounded holomorphic function $b : S_\mu \to \mathbb{C}$ the functional calculus operator $b(PM)$ on $\overline{\mathrm{R}(P)}$ is bounded by $\|b(PM)\|_{\overline{\mathrm{R}(P)}\to\overline{\mathrm{R}(P)}} \lesssim \|b\|_{L^\infty(S_\mu)}$.

If b is unambiguously defined at the origin, then $b(PM)$ extends to $L^2(\mathbb{R}^{n+1};\mathbb{C}^{n+2})$ by $b(0)$ on $\mathrm{N}(PM)$.

$\mathrm{H}^\pm(PM) := \chi^\pm(PM)\overline{\mathrm{R}(P)}$ yields the *generalized Hardy space* decomposition,

$$\overline{\mathrm{R}(P)} = \mathrm{H}^+(PM) \oplus \mathrm{H}^-(PM).$$

Solving the first order system

Generalized *Cauchy extension* in the upper half-space: for $h \in \overline{\mathsf{R}(P)}$ and $\lambda > 0$, $(C_0^+ h)(\lambda, \cdot) := e^{-\lambda PM}\chi^+(PM)h$.

Proposition

$F := C_0^+ h$ of $h \in \overline{\mathsf{R}(P)}$. Then $\partial_\lambda F + PMF = 0$ in the strong sense $F \in \mathsf{C}([0,\infty); \overline{\mathsf{R}(P)}) \cap \mathsf{C}^\infty((0,\infty); \mathsf{D}(PM))$, and

$$\sup_{\lambda>0} \|F_\lambda\|_2 \sim \|\chi^+(PM)h\|_2 \sim \sup_{\lambda>0} \fint_\lambda^{2\lambda} \|F_\mu\|_2^2 \, d\mu,$$

$$\lim_{\lambda\to 0} F_\lambda = \chi^+(PM)h, \qquad \lim_{\lambda\to\infty} F_\lambda = 0,$$

$$\int_0^\infty ||\lambda\partial_\lambda F||_2^2 \frac{d\lambda}{\lambda} \sim \|\chi^+(PM)h\|_2^2.$$

$\partial_\lambda F + PMF = 0$ also in the weak sense (3.1).

Solving the first order system

Theorem

Let $F \in L^2_{\text{loc}}(\mathbb{R}_+; \overline{\mathrm{R}(P)})$ be a solution of (1.2) in the weak sense such that

$$\sup_{\lambda>0} \fint_{\lambda}^{2\lambda} \|F_\mu\|_2^2 \, d\mu < \infty. \tag{3.6}$$

Then F has an L^2 limit $h \in \mathrm{H}^+(PM)$ at $\lambda = 0$ and F is given by the Cauchy extension of h.

$$\sup_{\lambda>0} \fint_{\lambda}^{2\lambda} \iint_{\mathbb{R}^{n+1}} (|\nabla_{\lambda,x} u|^2 + |H_t D_t^{1/2} u|^2) \, dxdtd\mu$$

$$\sim \|h\|_2^2 \sim \|\partial_{\nu_A} u|_{\lambda=0}\|_2^2 + \|\nabla_x u|_{\lambda=0}\|_2^2 + \|H_t D_t^{1/2} u|_{\lambda=0}\|_2^2.$$

Kato square root estimate for t-dependent coefficients

$$V := H^{1/2}(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}; \mathrm{W}^{1,2}(\mathbb{R}^n)).$$

Theorem

The operator $\mathcal{H}_{||} = \partial_t - \mathrm{div}_x A_{||||}(x,t)\nabla_x$ arises from an accretive form, it is maximal-accretive in $L^2(\mathbb{R}^{n+1})$ and

$$\|\sqrt{\mathcal{H}_{||}}\, u\|_2 \sim \|\nabla_x u\|_2 + \|D_t^{1/2} u\|_2 \qquad (u \in V).$$

Proof.

$$M := \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_{||||} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

PM and $[PM] = \mathrm{sgn}(PM)PM$ share the same domain.

Proof of the Kato estimate

$\|PMh\|_2 \sim \|[PM]h\|_2$.

But $[PM]$ is the square root of $(PM)^2$ and

$$(PM)^2 = \begin{bmatrix} \partial_t - \operatorname{div}_x A_{\|\|}\nabla_x & 0 & 0 \\ 0 & -\nabla_x \operatorname{div}_x A_{\|\|} & -H_t D_t^{1/2} \operatorname{div}_x A_{\|\|} \\ 0 & \nabla_x D_t^{1/2} & \partial_t \end{bmatrix}.$$

Let $h = [f, 0, 0]^*$.

$$\begin{aligned} \|\nabla_x f\|_2 + \|H_t D_t^{1/2} f\|_2 \sim & \|PMh\|_2 \\ \sim & \|[PM]h\|_2 \sim \|(\partial_t - \operatorname{div}_x A_{\|\|}\nabla_x)^{1/2} f\|_2. \end{aligned}$$

Controlling the non-tangential maximal function

For $(x,t) \in \mathbb{R}^{n+1}$ we define the non-tangential maximal function

$$\widetilde{N}_* F(x,t) = \sup_{\lambda>0} \left(\iiint_{\Lambda \times Q \times I} |F(\mu,y,s)|^2 d\mu dy ds \right)^{1/2},$$

$\Lambda = (c_0\lambda, c_1\lambda)$, $Q = B(x, c_2\lambda)$ and $I = (t - c_3\lambda^2, t + c_3\lambda^2)$.

Theorem

Let $h \in \overline{\mathrm{R}(PM)}$ and let $F = (C_0^+ h)(\lambda,\cdot) := e^{-\lambda PM}\chi^+(PM)h$. Then

$$\|\widetilde{N}_* F\|_2 \sim \|h\|_2,$$

and

$$\lim_{\lambda\to 0} \iiint_{\Lambda \times Q \times I} |F(\mu,y,s) - h(x,t)|^2 d\mu dy ds = 0$$

for almost every $(x,t) \in \mathbb{R}^{n+1}$.

Proof of the non-tangential maximal function estimate

1. New reverse Hölder estimates for reinforced weak solutions:

$$\left(\iiint_{\Lambda\times Q\times I}|\nabla_{\lambda,x}u|^2+|H_tD_t^{1/2}u|^2 d\mu dyds\right)^{1/2}$$
$$\lesssim\sum_{m\geq 0}2^{-m}\iiint_{8\Lambda\times 8Q\times 4^m I}|\nabla_{\lambda,x}u|+|H_tD_t^{1/2}u|+|D_t^{1/2}u|d\mu dyds.$$

2. Quadratic estimates.
3. Off-diagonal estimates.

Applications to BVPs

For $-1 \leq s \leq 0$ we let

$$\|F\|_{\mathcal{E}_s} := \begin{cases} \|\widetilde{N}_*(F)\|_2 & \text{(if } s = 0\text{)}, \\ \left(\int_0^\infty \|\lambda^{-s} F\|_2^2 \, \frac{d\lambda}{\lambda}\right)^{1/2} & \text{(otherwise)}, \end{cases}$$

and define the solution classes

$$\mathcal{E}_s := \{F \in L^2_{loc}(\mathbb{R}^{n+2}_+; \mathbb{C}^{n+2}); \|F\|_{\mathcal{E}_s} < \infty\}.$$

Given $s \in [-1, 0]$, the regularity of the data, we consider BVPs

$$(R)^{\mathcal{H}}_{\mathcal{E}_s} : \mathcal{H}u = 0,\ D_A u \in \mathcal{E}_s,\ u|_{\lambda=0} = f \in \dot{H}^{s/2+1/2}_{\partial_t - \Delta_x}$$
$$(N)^{\mathcal{H}}_{\mathcal{E}_s} : \mathcal{H}u = 0,\ D_A u \in \mathcal{E}_s,\ \partial_{\nu_A} u|_{\lambda=0} = f \in \dot{H}^{s/2}_{\partial_t - \Delta_x}.$$

Well-posedness results: λ-independent coefficients

Theorem

1. *$(R)^{\mathcal{H}}_{\mathcal{E}_s}$ and $(N)^{\mathcal{H}}_{\mathcal{E}_s}$ are compatibly well-posed when $-1 \leq s \leq 0$ and $A(x,t)$ has block structure.*
2. *$(R)^{\mathcal{H}}_{\mathcal{E}_s}$ and $(N)^{\mathcal{H}}_{\mathcal{E}_s}$ are compatibly well-posed when $-1 \leq s \leq 0$ and $A(x,t) = A$ with A constant.*
3. *$(R)^{\mathcal{H}}_{\mathcal{E}_s}$ and $(N)^{\mathcal{H}}_{\mathcal{E}_s}$ are compatibly well-posed when $-1 \leq s \leq 0$ and $A(x,t) = A(x)$ is hermitian.*
4. *Results for $(R)^{\mathcal{H}}_{\mathcal{E}_s}$ and $(N)^{\mathcal{H}}_{\mathcal{E}_s}$ when $A(x,t)$ has a triangular structure.*

In 3: by Rellich type estimates

$$||h||^2 \lesssim \int_0^\infty \|D_t^{1/4} F_\theta\|_2^2 d\lambda = -(h_\perp, h_\theta) - \int_0^\infty (\nabla u, [A, H_t D_t^{1/2}] \nabla u) d\lambda.$$

Further results and open areas for generalization

Further results:

1. Layer potentials.
2. Well-posedness results for equations with λ-dependent coefficients.
3. All our results apply to parabolic systems without any changes but in the accretivity condition.

Areas for further explorations:

1. Degenerate parabolic equations and systems.
2. Boundary value problems with other spaces of data.
3. Higher order equations and systems.
4. Time/space fractional equations.

References

1. Square functions estimates and the Kato problem for second order parabolic operators, Advances in Mathematics 293 (2016), 1-36.
2. (with Castro, A. and Sande, O.) Boundedness of single layer potentials associated to divergence form parabolic equations with complex coefficients, to appear in Calculus of Variations and Partial Differential Equations.
3. L^2 Solvability of boundary value problems for divergence form parabolic equations with complex coefficients, submitted.
4. (with P. Auscher and M. Egert) Boundary value problems for parabolic systems via a first order approach, submitted.