# On sharp constants in local and global Hausdorff-Young inequalities


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# I. Introduction: Global Hausdorff-Young (H-Y) for $\mathbb{R}^d$

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x}\, dx$$

Interpolation between

$$\|\hat{f}\|_\infty \le \|f\|_1, \quad \text{and} \quad \|\hat{f}\|_2 = \|f\|_2 \quad \text{(Plancherel)},$$

leads to

$$\|\hat{f}\|_{p'} \le C\|f\|_p, \qquad 1 \le p \le 2, \tag{1}$$

if $1/p' + 1/p = 1$, with $C \le 1$.

Denote by $H_p(\mathbb{R}^d)$ the best constant $C$ in (1).

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Denote by $H_p(\mathbb{R}^d)$ the best constant $C$ in (1).

**Babenko '61 (for $p'$ even integer), Beckner '75 for general $p$ :**

The best constant is given by (with Gaussians minimizing)

$$H_p(\mathbb{R}^d) = B_p^d, \quad \text{with} \quad B_p := (p^{1/p}/(p'^{1/p'})^{1/2}.$$

# Global H-Y on more general l.c. groups

Let $G$ be a loc. compact, unimodular group of type I, with unitary dual $\hat{G}$, endowed with the usual Mackey Borel structure.

## Abstract Plancherel formula for $G$

There exists a unique Plancherel measure $\mu$ on $\hat{G}$ so that

$$\int_G |\varphi(g)|^2 \, dg = \int_{\hat{G}} \mathrm{tr}\,(\hat{\varphi}(\pi)^* \hat{\varphi}(\pi))\, d\mu(\pi), \tag{2}$$

with

$$\hat{\varphi}(\pi) = \pi(\varphi) := \int_G \varphi(g)\pi(g)\, dg, \qquad \varphi \in L^1(G) \cap L^2(G).$$

Again, interpolation with the trivial estimate

$$\sup_{\pi \in \hat{G}} \|\hat{f}(\pi)\| \leq \|f\|_{L^1(G)}, \qquad f \in L^1(G),$$

leads to

## Kunze/Stein '60, Lipsman '74: H-Y for $G$

$$\|\hat{f}\|_{L^{p'}} \leq C\|f\|_p, \qquad 1 \leq p \leq 2,\ f \in L^1(G) \cap L^p(G), \tag{3}$$

with $C \leq 1$,

where

$$\|\hat{f}\|_{L^{p'}} = \|\hat{f}\|_{L^{p'}(\hat{G})} := \left( \int_{\hat{G}} \|\hat{f}(\pi)\|_{\mathcal{S}^{p'}}^{p'} \, d\mu(\pi) \right)^{1/p'};$$

$$\|T\|_{\mathcal{S}^q} := \left( \mathrm{tr}\, (T^* T)^{q/2} \right)^{1/q} \quad \text{the Schatten-q-norm of } T.$$

We denote by $H_p(G) \leq 1$ the optimal constant $C$ in (3).

## Global H-Y on more general l.c. groups

- Clearly $H_p(G) = 1$, if $G$ is compact (choose $f \equiv 1$).
- For connected, non-compact groups, one expects that $H_p(G) < 1$, if $1 < p < 2$.
- Several authors have indeed proved results in this direction, for various classes of non-compact groups, in particular for connected nilpotent, resp. solvable, Lie groups, including:
  – Eymard/Terp for the $ax + b$-group
  – Russo, Innoue; Russo/Klein;
  – Baklouti/Samoui/Ludwig;
  – Führ

# Local Hausdorff-Young

Assume $G$ is a Lie group, and the support of $f \in L^1(G) \cap L^2(G)$ "shrinks" towards the identity element $e \in G$. Then, in the limit, it appears plausible that the best H-Y-constant tends towards the one for the underlying Lie algebra $\mathfrak{g}$ of $G$, i.e., $B_p^{\dim \mathfrak{g}} = B_p^{\dim G}$.

More precisely, if $U \subset G$ is an open neighborhood of $e$, let

$$\begin{aligned} H_p(U) &:= \sup_{\operatorname{supp} f \subset U, \|f\|_p = 1} \|\hat{f}\|_{L^{p'}}, \\ H_p^{\mathrm{loc}}(G) &:= \inf_U H_p(U). \end{aligned}$$

– Clearly, $H_p(G) \geq H_p^{\mathrm{loc}}(G)$.

**QUESTION:**

If $G$ is Lie group of dimension $d$, is

$$H_p^{\mathrm{loc}}(G) = H_p^{\mathrm{loc}}(\mathfrak{g}) = B_p^d? \tag{4}$$

### Andersson '93, Sjölin '95, Kamaly '96

Yes, if $G = \mathbb{T}^d$.

– **Further motivation for this question:** Articles by Garcia-Cuerva, Marco, and Parcet,'2003/04 on H-Y -estimates for vector valued (more precisely: operator space valued) functions on non-commutative groups.
– An important step in their work consisted in proving the following:

### On local H-Y for central functions on compact Lie groups:

Let $G$ be a compact, semi-simple Lie group, and define the local H-Y-constant constant $H^{\mathrm{loc}}_{p,\mathrm{central}}(G)$ in the same way as before, only restricted to central functions $f$. Then $H^{\mathrm{loc}}_{p,\mathrm{central}}(G) > 0$.

### Theorem 1

If $G$ is a compact Lie group, then

$$H^{\mathrm{loc}}_{p,\mathrm{central}}(G) = B_p^{\dim G}.$$

## II. Global H-Y on nilpotent Lie groups/ the Heisenberg group

**Baklouti, Ludwig, Smaoui '2003**

Let $G$ be a connected, simply-connected nilpotent Lie group, and let $m$ be the dimension of the generic codjoint orbits in $\mathfrak{g}^*$. Then, for $1 \leq p \leq 2$,

$$H_p(G) \leq B_p^{\dim G - \frac{m}{2}}. \tag{5}$$

**The Heisenberg group** $\mathbb{H}_n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, with product

$$(u, v, t) \cdot (u', v', t') := (u + u', v + v', t + t' + \frac{1}{2}(uv' - u'v)).$$

Here, $m = 2n, d = 2n + 1$, so (5) yields

$$H_p(\mathbb{H}_n) \leq B_p^{d-n} = B_p^{n+1}.$$

**Klein/Russo '78**

If $p'$ is an even integer, then $H_p(\mathbb{H}_n) = B_p^d$.

**Remark.** We have an extension of this result (for even integer $p'$) to classes of solvable Lie groups, including all exponential solvable Lie groups.

CONJECTURE: $H_p(\mathbb{H}_n) = B_p^d = B_p^{2n+1}$ for every $p \in [1, 2]$.
The results above on the Heisenberg group can easily be reduced by means of a partial Fourier transform in the center to corresponding results on the Weyl transform $\rho$ : For $f$ on $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$, say a Schwartz function, let

$$\rho(f) := \iint f(u, v) e^{2\pi i(uD+vX)}\, dudv$$

on $L^2(\mathbb{R}^n, dx)$, where $uD := \sum_{j=1}^{n} u_j D_j$ and $vX := \sum_{j=1}^{n} v_j X_j$,

## Weyl transform

with

$$D_j\varphi(x) := \frac{1}{2\pi i}\frac{\partial}{\partial x_j}f(x), \quad X_j\varphi(x) := x_j\varphi(x).$$

Explicitly, $\rho(f)$ is the integral operator

$$\rho(f)\varphi(x) = \int K_f(x,y)\varphi(y)\,dy,$$

with integral kernel

$$K_f(x,y) := \int f(y-x,v)e^{\pi i v(x+y)}\,dv.$$

$f$ on $\mathbb{R}^{2n}$ is polyradial, if

$$f(u,v) = f_0(|z_1|,\cdots,|z_j|),$$

where we have put $z_j := u_j + iv_j \in \mathbb{C}$.

## Theorem 2

If $f \in C_0^\infty(\mathbb{R}^{2n})$ is polyradial, then

$$\|\rho(f)\|_{\mathcal{S}^{p'}} \leq B_p^{2n} \|f\, e^{\frac{\pi}{2}|\cdot|^2}\|_{L^p(\mathbb{R}^{2n})}, \qquad 1 \leq p \leq 2. \tag{6}$$

– So, as $\operatorname{supp} f \to \{0\}$, the right-hand side tends to $B_p^{2n}\|f\|_p$, and we get an analogue to the local result for central functions on compact Lie groups in Theorem 1.

– Regretfully, the result cannot be scaled – one would indeed expect to get the analogue of (15) without the weight $e^{\frac{\pi}{2}|\cdot|^2}$, and even for non-polyradial functions. This would also imply the conjecture above about $H_p(\mathbb{H}_n)$.

Remark: Theorem 1 and 2 can be seen as particular instances of a more general setting, namely that of Gelfand pairs $(G, K)$.

**Alternative approach to $L^{p'}(\hat{G})$** : Let $G$ be any unimodular loc. compact group. Following Segal '50, Kunze '58, let $\mathcal{M}$ be the von Neumann algebra $\mathcal{M} = \mathrm{Cv}^2(G)$ of $L^2$-bounded left convolution operators on $G$. Define a "generalised trace" $\tau$ on its positive part $\mathcal{M}_+$ by

$$\tau(L_{f^* * f}) := \|f\|_2^2 = f^* * f(e) \tag{7}$$

(where $L_h$ denotes the convolution operator $\phi \mapsto h * \phi$) and put

$$\|\hat{f}\|_q := (\tau((L_{f^* * f})^{q/2}))^{1/q}.$$

According to Klein/Russo, if $G$ is separable and of type I, this norm agrees with the previous q-norm, i.e.,

$$\|\hat{f}\|_q = \Big(\int_{\hat{G}} \|\hat{f}(\pi)\|_{\mathcal{S}^q}^q \, d\mu(\pi)\Big)^{1/q} = \|\hat{f}\|_{L^q(\hat{G})} \tag{8}$$

# Alternative approach to $L^{p'}(\hat{G})$

Note: If $h$ is a positive definite function on $G$, then

$$h(e) = \|h\|_\infty = \|L_h\|_{1\to\infty},$$

and in conclusion

$$\|\hat{f}\|_q^q = \|(L_{f^* * f})^{q/2}\|_{1\to\infty}. \tag{9}$$

**Theorem 3**

Let $G$ be a unimodular Lie group, with Lie algebra $\mathfrak{g}$, which we regard as a commutative group under addition. Then

$$H_p^{\mathrm{loc}}(G) \geq H_p(\mathfrak{g}) = B_p^{\dim G}.$$

# Sketch of Proof

Given any open neighborhood $U$ of $e$ in $G$, we need to prove that

$$H_p(\mathfrak{g}) \leq H_p(U). \tag{10}$$

By (9), we have

$$\|\hat{f}\|_{p'}^{p'} = \sup_{\|g\|_1, \|h\|_1 \leq 1} \langle (L_{f^* * f})^{p'/2} g, h \rangle. \tag{11}$$

**Scaling:** Let $\Omega \subseteq \mathfrak{g}$ be an open neighbourhood of the origin such that $\Omega = -\Omega$ and $\exp|_\Omega : \Omega \to \exp(\Omega)$ is a diffeomorphism. For all $f \in C_0(\mathfrak{g}),\ 1 \leq p \leq \infty,\ \lambda > 0$, define $f^{\lambda,p} : G \to \mathbb{C}$ by

$$f^{\lambda,p}(x) = \begin{cases} \lambda^{-n/p} f(\lambda^{-1} \exp|_\Omega^{-1}(x)) & \text{if } x \in \exp(\Omega), \\ 0 & \text{otherwise.} \end{cases}$$

# Sketch of Proof

One easily shows:

$$\begin{aligned} \|f^{\lambda,p}\|_{L^p(G)} &\leq C\|f\|_{L^p(\mathfrak{g})} \\ \lim_{\lambda\to 0}\|f^{\lambda,p}\|_{L^p(G)} &= \|f\|_{L^p(\mathfrak{g})} \end{aligned}$$

Also, using Baker-Campbell-Hausdorff,

$$\exp(X_1)\cdots\exp(X_k) = \exp(X_1+\cdots+X_k+B(X_1,\dots,X_k)),$$

where

$$\lambda^{-1}B(\lambda X_1,\dots,\lambda X_k)\to 0 \quad \text{as } \lambda\to 0,$$

one finds that for $f_1,\dots,f_k,g\in C_0(\mathfrak{g})$

$$\lim_{\lambda\to 0}\langle f_1^{\lambda,1}*\cdots*f_k^{\lambda,1}, g^{\lambda,\infty}\rangle_{L^2(G)} = \langle f_1*\cdots*f_k, g\rangle_{L^2(\mathfrak{g})}. \tag{12}$$

## Sketch of Proof

Also $(f^{\lambda,1})^* = (f^*)^{\lambda,1}$, and thus an earlier estimate implies

$$\|L_{(f^{\lambda,1})^* * f^{\lambda,1}}\|_{L^2(G)\to L^2(G)} \leq \|(f^{\lambda,1})^* * f^{\lambda,1}\|_{L^1(G)} \leq C^2\|f\|^2_{L^1(\mathfrak{g})}.$$

Hence the positive operators $L_{(f^{\lambda,1})^* * f^{\lambda,1}}$ are bounded on $L^2(G)$ uniformly in $\lambda$. Therefore, for all $\Phi \in C([0,\infty))$, the operator $\Phi(L_{(f^{\lambda,1})^* * f^{\lambda,1}})$ is bounded and depends only on $\Phi|_K$, where $K = [0, C^2\|f\|^2_{L^1(\mathfrak{g})}]$. Let

$$\mathcal{F} := \Big\{\Phi \in C(K) : \forall g, h \in C_0(\mathfrak{g}) : \\ \lim_{\lambda\to 0}\langle\Phi(L_{(f^{\lambda,1})^* * f^{\lambda,1}})g^{\lambda,2}, h^{\lambda,2}\rangle_{L^2(G)} = \langle\Phi(L_{f^* * f})g, h\rangle_{L^2(\mathfrak{g})}\Big\}. \quad (13)$$

$F$ is a linear subspace of $C(K)$, closed under conjugation. Moreover, $\mathcal{F}$ contains all polynomials:

## Sketch of Proof

Indeed,if $\Phi(u) = u^N$, then by (12), as $\lambda \to 0$,

$$\langle \Phi(L_{(f^{\lambda,1})^* * f^{\lambda,1}}) g^{\lambda,2}, h^{\lambda,2} \rangle_{L^2(G)} = \langle ((f^*)^{\lambda,1} * f^{\lambda,1})^{(*N)} * g^{\lambda,1}, h^{\lambda,\infty} \rangle$$
$$\longrightarrow \langle (f^* * f)^{(*N)} * g, h \rangle = \langle \langle \Phi(L_{f^* * f}) g, h \rangle_{L^2(\mathfrak{g})}.$$

– Finally, one easily sees that $\mathcal{F}$ is closed in $C(K)$. Thus, by Stone-Weierstraß,

$$\mathcal{F} = C(K).$$

– Choosing $\Phi(u) := u^{p'/2}$ in $\mathcal{F}$, by (13) we obtain:

# Sketch of Proof

For all $\forall f, g, h \in C_0(G)$,

$$\lambda^{-d(p'-1)} \langle (L_{(f^{\lambda,\infty})^* * f^{\lambda,\infty}})^{p'/2} g^{\lambda,1}, h^{\lambda,1} \rangle_{L^2(G)} \to \langle (L_{f^* * f})^{p'/2} g, h \rangle_{L^2(\mathfrak{g})} \quad (14)$$

as $\lambda \to 0$.

– Assuming that $\lambda$ is so small that $f^{\lambda,\infty}, g^{\lambda,1}, h^{\lambda,1}$ are supported in $U$, and recalling (11), i.e.,

$$\sup_{\|g\|_1, \|h\|_1 \leq 1} \langle (L_{f^* * f})^{p'/2} g, h \rangle = \|\hat{f}\|_{p'}^{p'},$$

we see that the left-hand side of (14) is bounded by

$$\begin{aligned} &\lambda^{-d(p'-1)} \big( H_p(U) \| f^{\lambda,\infty} \|_p \big)^{p'} \| g^{\lambda,1} \|_1 \| h^{\lambda,1} \|_1 \\ &= H_p(U)^{p'} \| f^{\lambda,p} \|_p^{p'} \| g^{\lambda,1} \|_1 \| h^{\lambda,1} \|_1. \end{aligned}$$

Since $\lim_{\lambda \to 0} \| \varphi^{\lambda,p} \|_{L^p(G)} = \| \varphi \|_{L^p(\mathfrak{g})}$, in the limit as $\lambda \to 0$, we find that

$$\langle (L_{f^* * f})^{p'/2} g, h \rangle_{L^2(\mathfrak{g})} \leq H_p(U)^{p'} \| f \|_p^{p'} \| g \|_1 \| h \|_1.$$

## Sketch of Proof

Hence

$$\|\hat{f}\|_{L^{p'}(\mathfrak{g})} \leq H_p(U)\|f\|_{L^p(\mathfrak{g})}.$$

This implies $H_p(\mathfrak{g}) \leq H_p(U)$, and since $U$ was arbitrary, we arrive at the desired estimate

$$H_p(\mathfrak{g}) \leq H_p^{\mathrm{loc}}(G).$$

□

### Thanks

for your attention!

# Proof of Theorem 2

## Theorem 2

If $f \in C_0^\infty(\mathbb{R}^{2n})$ is polyradial, then

$$\|\rho(f)\|_{\mathcal{S}^{p'}} \leq B_p^{2n} \|f\, e^{\frac{\pi}{2}|\cdot|^2}\|_{L^p(\mathbb{R}^{2n})}, \qquad 1 \leq p \leq 2. \tag{15}$$

**Proof.** Key identity relating Laguerre polynomials to Bessel functions:

$$L_k^\alpha(x) = \frac{e^x x^{-\alpha/2}}{k!} \int_0^\infty t^{k+\alpha/2} J_\alpha(2\sqrt{xt}) e^{-t}\, dt, \qquad x > 0. \tag{16}$$

– In order to avoid technicalities, let us assume $n = 1$.
– If $f(z) = f_0(|z|)$ is a radial $L^1$-function on $\mathbb{R}^2$, one may represent the operator $\rho(f)$ with respect to the orthonormal basis of Hermite functions $h_k, k \in \mathbb{N}$, of $L^2(\mathbb{R})$ as an infinite diagonal matrix, with diagonal elements

$$\tilde{f}(k) := \langle \rho(f) h_k, h_k \rangle = \int_{\mathbb{R}^2} f(z) e^{-(\pi/2)|z|^2} L_k^0(\pi|z|^2)\, dz, \qquad k \in \mathbb{N}. \tag{17}$$

# Proof of Theorem 2

In particular,

$$\|\rho(f)\|_{\mathcal{S}^q} = \|\tilde{f}\|_{\ell^q}.$$

Recall also that the Euclidean Fourier transform of any radial $L^1$- function $g(z) = g_0(|z|)$ on $\mathbb{R}^2$ can written in polar coordinates as

$$\hat{g}(\zeta) = 2\pi \int_0^\infty g_0(r) J_0(2\pi|\zeta|r) r\, dr. \tag{18}$$

Put $F(z) = e^{(\pi/2)|z|^2} f(z)$. Since also $\hat{F}$ is radial, we may write $\hat{F}(\zeta) = \hat{F}_0(|\zeta|)$. Combining (16) – (18), we obtain

$$\tilde{f}(k) = \int_0^\infty \hat{F}_0\left(\sqrt{t/\pi}\right) \frac{t^k}{k!} e^{-t}\, dt. \tag{19}$$

# Proof of Theorem 2

For suitable functions $\varphi$ on the positive real line, let us write

$$\breve{\varphi}(k) := \int_0^\infty \varphi(t) \frac{t^k}{k!} e^{-t}\, dt, \qquad k \in \mathbb{N}.$$

We claim that

$$\|\breve{\varphi}\|_{\ell^q} \le \|\varphi\|_{L^q(\mathbb{R}^+, dt)}, \quad 1 \le q \le \infty. \tag{20}$$

Indeed, this estimate is trivial for $q = \infty$, since the $\frac{t^k}{k!} e^{-t}\, dt$ are probability measures, and for $p = 1$ we may estimate

$$\sum_{k=0}^\infty |\breve{\varphi}(k)| \le \int_0^\infty |\varphi(t)| \sum_{k=0}^\infty \frac{t^k}{k!} e^{-t}\, dt = \|\varphi\|_1.$$

Thus, (20) follows by the complex interpolation theorem of Riesz and Thorin.

# Proof of Theorem 2

From (20) and (19) we get

$$\|\tilde{f}\|_{\ell^q} \leq \Big( \int_0^\infty |\hat{F}_0(\sqrt{t/\pi})|^q \, dt \Big)^{1/q} = \|\hat{F}\|_{L^q(\mathbb{R}^2)},$$

and thus by the **sharp Hausdorff-Young inequality** on $\mathbb{R}^2$ we obtain

$$\|\rho(f)\|_{\mathcal{S}^q} = \|\tilde{f}\|_{\ell^q} \leq B_p^2 \|F\|_p,$$

hence (15). $\square$

**Thanks**

for your attention!