

FIRST ORDER APPROACH TO L^T ESTIMATES FOR THE STOKES OPERATOR ON LIPSHITZ DOMAINS

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VERY WEAKLY LIPSCHITZ DOMAINS

$\Omega \subset \mathbb{R}^3$ a bounded open set s.t.

$$\Omega = \bigcup_{j=1}^N \varphi_j(B) \quad \varphi_j : B \rightarrow \varphi_j(B) \text{ unif. locally Lipschitz}$$

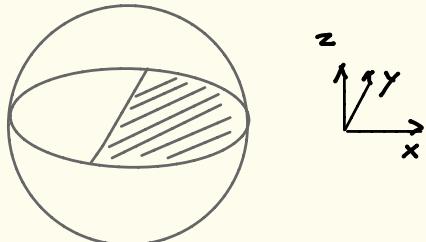
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$$1 = \sum_{j=1}^N \chi_j \quad \text{with} \quad \chi_j \in C^\infty(\Omega), \quad \text{supp } \chi_j \subseteq \varphi_j(B)$$

Examples

- bounded strongly Lipschitz domains
- bounded weakly Lipschitz domains (two bricks)
- a ball cut in the middle

$$\{x^2+y^2+z^2 < 1\} \setminus \{z=0, x \geq 0, x^2+y^2 < 1\}$$



De Rham complex on Ω

$1 < p < \infty$, $\Omega \subseteq \mathbb{R}^3$ bounded open subset

The exterior algebra on \mathbb{R}^3 is

$$\Lambda = \begin{matrix} \wedge^0 \\ \mathbb{C} \end{matrix} \oplus \begin{matrix} \wedge^1 \\ \mathbb{C}^3 \end{matrix} \oplus \begin{matrix} \wedge^2 \\ \mathbb{C}^3 \end{matrix} \oplus \begin{matrix} \wedge^3 \\ \mathbb{C} \end{matrix}$$

The exterior derivative d on Ω is

$$d : 0 \rightarrow L^p(\Omega, \mathbb{C}) \xrightarrow{\nabla} L^p(\Omega, \mathbb{C}^3) \xrightarrow{\text{curl}} L^p(\Omega, \mathbb{C}^3) \xrightarrow{\text{div}} L^p(\Omega, \mathbb{C}) \rightarrow 0.$$

Note that d is an unbounded operator on $L^p(\Omega, \Lambda)$ with domain

$$D^p(d) = \{ u \in L^p(\Omega, \Lambda) : du \in L^p(\Omega, \Lambda) \}$$

Rk: $d^2 = 0$

The dual De Rham complex

The dual of the exterior derivative $d : D^p(\Omega) \rightarrow L^p(\Omega, \Lambda)$ is $\underline{d} : D^{p'}(\underline{\Omega}) \rightarrow L^{p'}(\Omega)$

$$\underline{d} : 0 \leftarrow L^{p'}(\Omega, \mathbb{C}) \xleftarrow{-\operatorname{div}} L^{p'}(\Omega, \mathbb{C}^3) \xleftarrow{\text{and}} L^{p'}(\Omega, \mathbb{C}^3) \xleftarrow{-\nabla} L^{p'}(\Omega, \mathbb{C}) \leftarrow 0$$

$(\frac{1}{p} + \frac{1}{p'} = 1)$ where the — includes boundary conditions : the domain of \underline{d} in $L^{p'}(\Omega, \Lambda)$ is the completion of $C_c^\infty(\Omega, \Lambda)$ in the graph norm $\|u\|_p + \|\underline{d}u\|_{p'}$

$$\text{Again : } \underline{d}^2 = 0$$

Remark If Ω is a bounded strongly Lipschitz domain, then

$$D^{p'}(\underline{\Omega}) = \{ u \in L^{p'}(\Omega, \Lambda) : \underline{d}u \in L^p(\Omega, \Lambda) \text{ & } v \lrcorner u = 0 \text{ on } \partial\Omega \}$$

N.B. Integration by parts formula :

$$\langle v \lrcorner u, v \rangle_{\partial\Omega} = \langle u, \operatorname{div} v \rangle_{\Omega} - \langle \underline{d}u, v \rangle_{\Omega} .$$

The Hodge-Dirac operator

Define $D_H = d + \underline{\Delta}$ on $L^2(\Omega, \wedge)$ with domain $D^2(d) \cap D^2(\underline{\Delta})$:

D_H is self-adjoint, bisectorial of angle $\omega = 0$

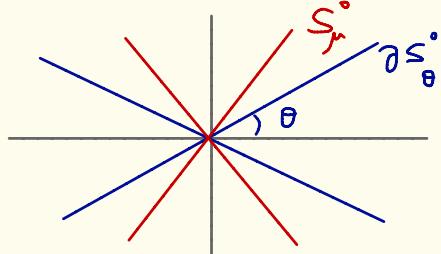
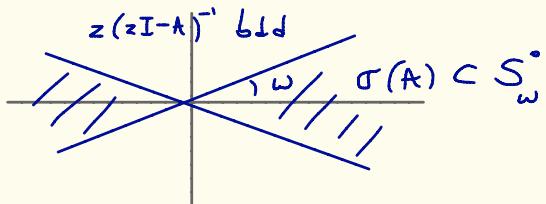
So D_H admits a bounded holomorphic functional calculus, i.e.,

$\forall \mu > 0 \quad \forall f: S_\mu^\circ \rightarrow \mathbb{C}$ holomorphic s.t. $\sup_{z \in S_\mu^\circ} \left(\frac{|z|^s |f(z)|}{1 + |z|^{2s}} \right) < \infty$

$$\|f(D_H)u\| \leq K_\theta \|f\|_{L^\infty(S_\theta^\circ)} \|u\| \quad \forall 0 < \theta < \mu$$

where

$$f(A)u = \frac{1}{2\pi i} \int_{\partial S_\theta^\circ} f(z) (zI - D_H)^{-1} u \, dz$$



The Hodge decomposition (in $L^2(\Omega, \Lambda)$)

$$\begin{aligned}
 L^2(\Omega, \Lambda) &= N^2(d) \xrightarrow{\perp} R^2(\bar{\delta}) \\
 &\quad \cup \qquad \perp \cap \\
 &= R^2(d) \oplus N^2(\bar{\delta}) \\
 &= R^2(d) \oplus \tilde{R}(\bar{\delta}) \oplus \mathcal{W}^2(D_{\bar{\delta}}) \\
 &\qquad \qquad \qquad \text{if } N^2(D_{\bar{\delta}}) = N^2(d) \cap N^2(\bar{\delta}) \\
 &\qquad \qquad \qquad \text{is finite dimensional}
 \end{aligned}$$

In particular, restricting to 1-forms, i.e., on $L^2(\Omega, \Lambda^1) = L^2(\Omega, \mathbb{C}^3)$:

$$\begin{aligned}
 L^2(\Omega, \mathbb{C}^3) &= R^2(\nabla) \oplus N^2(\underline{\operatorname{div}}) \\
 \text{if } \mathcal{H}^2 &= \left\{ u \in L^2(\Omega, \mathbb{C}^3) : \operatorname{div} u = 0 \text{ in } \Omega \right. \\
 &\quad \left. \cdot \cdot \cdot u = 0 \text{ on } \partial\Omega \right\} \\
 &\quad \uparrow \\
 &\quad \text{if } \Omega \text{ is smooth enough}
 \end{aligned}$$

The Hodge Laplacian

$$-\Delta_H = d\underline{\delta} + \underline{\delta}d \quad \text{with domain} \quad D^2(d\underline{\delta}) \cap D^2(\underline{\delta}d) \subset L^2(\Omega, \wedge)$$

is non negative, self adjoint

$$d: 0 \rightarrow L^2(\Omega, \mathbb{C}) \xrightarrow{\nabla} L^2(\Omega, \mathbb{C}^3) \xrightarrow{\text{curl}} L^2(\Omega, \mathbb{C}^3) \xrightarrow{\text{div}} L^2(\Omega, \mathbb{C}) \rightarrow 0$$

$$0 \leftarrow \quad \xleftarrow{-\underline{\text{div}}} \quad \xleftarrow{\underline{\text{curl}}} \quad \xleftarrow{-\underline{\nabla}} \quad \leftarrow \quad : \underline{\delta}$$

$$-\Delta_H = -\underline{\text{div}} \nabla \oplus -\nabla \underline{\text{div}} + \underline{\text{curl}} \text{curl} \oplus \text{curl} \underline{\text{curl}} - \nabla \underline{\text{div}} \oplus -\underline{\text{div}} \nabla$$

$$\begin{array}{lll} \text{on 0-forms} & \text{on 1-forms} & \text{on 2-forms} \\ \text{on 3-forms} & & \end{array}$$

$\text{bdry cond: } v \cdot u = 0$ $\text{bdry cond: } v \times u = 0$ $\text{bdry cond: } \text{div } u = 0$

Neumann Laplacian $\text{Dirichlet Laplacian}$

The Hodge - Stokes operator

In \mathcal{H}^2 , the Stokes operator with Hodge boundary conditions is defined by

$$S_{\mathcal{H}} u = -\Delta_{\mathcal{H}} u = \underline{\operatorname{curl}} \operatorname{curl} u, \quad u \in \mathcal{H}^2 \text{ with } \operatorname{curl} u \in D^2(\underline{\operatorname{curl}})$$

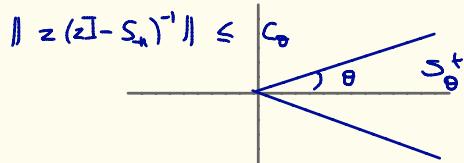
↑
defined on 2-forms

$S_{\mathcal{H}}$ is non-negative self-adjoint in \mathcal{H}^2 ,

$$\mathcal{N}^2(S_{\mathcal{H}}) = \mathcal{D}_{\mathcal{H}} \cap L^2(\Omega, \Lambda^k) = \{u \in L^2(\Omega, \mathbb{C}^3) : \underline{\operatorname{div}} u = 0 \text{ in } \Omega, \operatorname{curl} u = 0 \text{ in } \Omega\}$$

$S_{\mathcal{H}}$ has resolvent bounds in \mathcal{H}^2 : $\forall \theta \in (0, \pi)$

$$\| (zI - S_{\mathcal{H}})^{-1} \|_{\mathcal{L}(\mathcal{H}^2)} \leq C_0 \quad \forall z \in \mathbb{C} \setminus S_{\theta}^+$$



$S_{\mathcal{H}}$ admits also a bounded holomorphic functional calculus in \mathcal{H}^2

L^p questions for D_H , Δ_H , S_H

(H_p) D_H has an L^p Hodge decomposition:

$$L^p(\Omega, \Lambda) = R^0(d) \oplus R^1(d) \oplus N^p(D_H).$$

(R_p) D_H is bisectorial in $L^p(\Omega, \Lambda)$. In particular

$$\| (I + itD_H)^{-1} u \|_p \leq c \| u \|_p \quad \forall t \in \mathbb{R}$$

(F_p) D_H has a bounded holomorphic functional calculus in $L^p(\Omega, \Lambda)$:

$$\| f(D_H) u \|_p \leq C_p \| f \|_\infty \| u \|_p \quad \forall f \in \mathcal{F}_p(S_H)$$

Remark

$$(\mathcal{F}_p) \Rightarrow (H_p)$$

Hodge-Dirac operators on very weakly Lipschitz domains

Theorem 1 (Hodge decomposition)

$\Omega \subseteq \mathbb{R}^n$ a very weakly Lipschitz domain. There exist Hodge exponents $p_H, p^+ = p'$ with $1 \leq p_H < 2 < p^+ \leq \infty$ such that the decomposition

$$L^p(\Omega, \Lambda) = R^p(d) \oplus R^p(\underline{\delta}) \oplus N^p(D_H)$$

holds if, and only if, $p_H < p < p^+$. For p in this range,

- $N^p(d) \cap N^p(\underline{\delta}) = N^p(D_H) = N^2(D_H)$
is finite dimensional
- D_H with domain $D^p(d) \cap D^p(\underline{\delta})$ is a closed operator

Hint : Snüberg's thm + potential maps (see later)

Theorem 2

$\Omega \subseteq \mathbb{R}^n$ a very weakly Lipschitz domain, $1 < p < \infty$.

- (i) If $p_H < p < p^+$, then D_H is bisectorial of angle Ω in $L^p(\Omega, \Lambda)$ and for all $\mu \in (0, \frac{\pi}{2})$, D_H admits a bounded S_μ° holomorphic functional calculus in $L^p(\Omega, \Lambda)$.
- (ii) Conversely, if D_H is bisectorial with a bounded holomorphic functional calculus in $L^p(\Omega, \Lambda)$, then $p_H < p < p^+$.
- (iii) For all $r \in (\max\{1, (p_H)_s\}, p^+)$ $\left[q_s = \frac{nq}{n+q} \right]$ and all $\theta \in (0, \frac{\pi}{2})$ there exists $C_{r,\theta} > 0$ s.t. $(I + zD_H)^{-1} : R^r(d) + R^r(\underline{d}) + N^r(D_H) \rightarrow L^r(\Omega, \Lambda)$ with the estimates $\forall z \in \mathbb{C} \setminus S_\theta$
- $$\sup_{z \in \mathbb{C} \setminus S_\theta} \| (I + zD_H)^{-1} u \|_r \leq C_{r,\theta} \| u \|_r \quad \forall u \in R^r(d) + R^r(\underline{d}) + N^r(D_H)$$
- and $\exists K_{r,\theta}$ s.t. $\forall f \in H^\infty(S_\theta^\circ)$,
- $$\| f(D_H) u \|_r \leq K_{r,\theta} \| f \|_\infty \| u \|_r \quad \forall u \in R^r(d) + R^r(\underline{d}) + N^r(D_H).$$

Off diagonal bounds

Proposition

On a very weakly Lipschitz domain Ω , let $q \in (p_+, p^*)$ such that D_H is $L^q(\Omega)$. Let $\mu \in (0, \pi/2)$

Then the families $\{ (I+zD_H)^{-1}, z \in \mathbb{C} \setminus S_\mu \}$, $\{ zd(I+zD_H)^{-1}, z \in \mathbb{C} \setminus S_\mu \}$ and $\{ z\bar{d}(I+zD_H)^{-1}, z \in \mathbb{C} \setminus S_\mu \}$ admit the following off-diagonal bounds

$$\| \mathbf{1}_{E \cap R_z} \mathbf{1}_F u \|_{L^q(\Omega, \Lambda)} \leq C e^{-c \frac{\text{dist}(E, F)}{|z|}} \| u \|_{L^q(\Omega, \Lambda)} \quad \forall z \in \mathbb{C} \setminus S_\mu$$

where $R_z = (I+zD_H)^{-1}$ or $zd(I+zD_H)^{-1}$ or $z\bar{d}(I+zD_H)^{-1}$.

Idea of the proof

$$\begin{aligned} \eta &= e^{2\bar{z}} \\ u &\in L^p(\Omega, \Lambda) \\ v &= (I+zD_H)^{-1}(\mathbf{1}_F u) \end{aligned} \quad \begin{aligned} \bar{z} &\in \text{Lip}(\partial\Omega) \cap L^\infty(\partial\Omega), \quad \bar{z}=1 \text{ on } E, \quad \bar{z}=0 \text{ on } F, \quad \|\nabla \bar{z}\|_\infty \leq \frac{1}{\text{dist}(E, F)} \\ \eta v &= v + [\eta, (I+zD_H)^{-1}] (\mathbf{1}_F u) \\ &\quad \leftarrow \text{optimal choice of } \alpha \end{aligned}$$

Potential operators on the unit ball

$B = B(0,1) \subseteq \mathbb{R}^n$, $\varphi \in C_c^\infty(\frac{1}{2}B)$, $\int \varphi = 1$

$$R_B u^k(x) := \int_B \varphi(y) (x-y) \cdot \int_0^1 t^{k-1} u^k(y+t(x-y)) dt dy, \quad 1 \leq k \leq n$$

$$L^p(\Omega, \lambda) \ni u = \sum_{k=0}^n u^k, \quad u^k \in L^p(\Omega, \lambda^k)$$

Then: $R_B : L^p(\Omega, \lambda) \rightarrow W^{1,p}(\Omega, \lambda) \hookrightarrow L^{\tilde{p}}(\Omega, \lambda)$ $\left[\tilde{p} = \begin{cases} \infty & \text{if } p > n \\ \frac{np}{n-p} & \text{if } p < n \end{cases} \right]$

and $dR_B u + R_B du + K_B u = u$ where $K_B u = (\int \varphi u) \mathbf{1}$ $K_B : L^p \rightarrow L^\infty$ compact

On bilipschitz transformations of the unit ball

$\rho : B \rightarrow \rho B$ uniformly locally Lipschitz, $(\rho^* u)(x) = \text{Jac } \rho(x) \cdot u(\rho(x))$ pull-back

$d_{\rho B} = (\rho^*)^{-1} d_B \rho^* : \text{define } R_{\rho B} = (\rho^*)^{-1} R_B \rho^* \text{ and } K_{\rho B} = (\rho^*)^{-1} K_B \rho^*$

\rightarrow same mapping properties & $d_{\rho B} R_{\rho B} + R_{\rho B} d_{\rho B} + K_{\rho B} = I$

Potential maps on very weakly Lipschitz domains (vwL)

Proposition

On a vwL domain Ω , there are potential maps for all $p \in (1, \infty)$

$$R: L^p(\Omega, \Lambda) \rightarrow \overset{\circ}{L}{}^p(\Omega, \Lambda) \cap D^p(\underline{J})$$

$$K: L^p(\Omega, \Lambda) \rightarrow L^\infty(\Omega, \Lambda) \text{ compact}$$

$$S: L^p(\Omega, \Lambda) \rightarrow \overset{\circ}{L}{}^p(\Omega, \Lambda) \cap D^p(\underline{\Sigma})$$

$$L: L^p(\Omega, \Lambda) \rightarrow L^\infty(\Omega, \Lambda) \text{ compact}$$

s.t.,

$$dR + Rd + K = I, \quad \underline{\delta} S + S \underline{\delta} + L = I$$

$$dK = 0$$

$$K = 0 \text{ on } R^p(\underline{J})$$

$$\underline{\delta} L = 0$$

$$L = 0 \text{ on } R^p(\underline{\Sigma})$$

$$dR u = u \text{ if } u \in R^p(\underline{J})$$

$$\rightsquigarrow$$

$$\underline{\delta} Su = u \text{ if } u \in R^p(\underline{\Sigma})$$

Corollary

On a vwL domain Ω

- d and $\underline{\delta}$ are closed (unbdd) operators in $L^p(\Omega, \Lambda)$, $1 < p < \infty$

- $\{R^p(\underline{J}), 1 < p < \infty\}$ and $\{R^p(\underline{\Sigma}), 1 < p < \infty\}$ are complex interpolation scales

Idea of the proof of bisectionality (dim 3)

Start from $p=2$ and extrapolate

$$p = 2_S = \frac{6}{5} \quad u \in R^{C_4}(d) : u = dR u$$

$|z| \geq \text{diam } \Omega$:

$$\begin{aligned} \| (I + z D_H)^{-1} u \| &= \| (I + z D_H)^{-1} dR u \|_{C_4} \lesssim \| (I + z D_H)^{-1} dR u \|_2 \\ &\stackrel{\text{def}}{\lesssim} \frac{1}{|z|} \| R u \|_2 \lesssim \| u \|_{C_5} \end{aligned}$$

$0 < |z| = t < \text{diam } \Omega$: C_k^t ($k \in J$) cubes in \mathbb{R}^3 with side-length t and corners at points in $t\mathbb{Z}^3$ which intersect Ω

$$Q_k^t = 4C_k^t \cap \Omega \quad : \quad \Omega = \bigcup_{k \in J} Q_k^t \quad \gamma_k \in C_c(4C_k^t, [\bar{a}_1]) : \quad \|\nabla \gamma_k\|_\infty \leq \frac{1}{t}$$

$$\sum \gamma_k^2 = 1 \text{ on } \Omega$$

$$\gamma_k^2 u = d w_k + \frac{1}{t} v_k$$

$$w_k = \gamma_k R(\gamma_k u) : \|w_k\|_2, \|v_k\|_2 \lesssim \|\mathbf{1}_{Q_k^t} u\|_{C_5}$$

+ off diagonal bounds

Estimates on the Hodge exponents on strongly Lipschitz domains

Theorem 3

Let $\Omega \subset \mathbb{R}^n$ be a bounded strongly Lipschitz domain. Then the Hodge exponents associated with the Hodge decomposition

$$L^2(\Omega, \Lambda) = \mathcal{R}^T(d) \oplus \mathcal{R}^T(\delta) \oplus \mathcal{N}(D_H)$$

are estimated as follows :

$$p_H \leq \frac{(2+\varepsilon)n}{n(1+\varepsilon)+1} < \frac{2n}{n+1} < \frac{2n}{n-1} < \frac{(2+\varepsilon)n}{n-1} \leq p^H.$$

In particular, if $n=2$ $p_H < \frac{4}{3} < 4 < p^H$

and if $n=3$ $p_H < \frac{3}{2} < 3 < p^H$



THANK YOU FOR YOUR ATTENTION