

Free boundary results for real elliptic operators with variable coefficients

José María Martell

joint work with

J. Cavero, S. Hofmann, and T. Toro



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Introduction
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The Laplacian
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Elliptic operators
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Sketch of the proof
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Section 1

Introduction

Introduction: The Laplacian

Theorem (F. & M. Riesz 1916)

$\Omega \subset \mathbb{C}$ **simply connected domain** with **rectifiable boundary**

$$\text{harmonic measure } \omega \ll \sigma = \mathcal{H}^1|_{\partial\Omega}$$

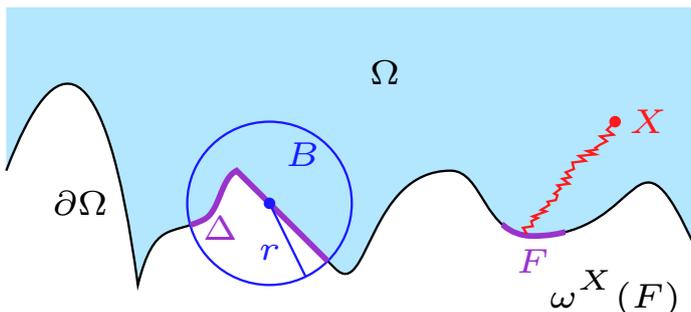
- [Lavrentiev 1936] Quantitative version
- [Bishop-Jones 1990] F. & M. Riesz can fail without some topology
- **Goals** \rightsquigarrow

{	<ul style="list-style-type: none"> • Higher dimensions • Qualitative vs. Quantitative • Converse • Elliptic operators 	}	\rightsquigarrow	Connectivity ???
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Harmonic measure

- $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, **open**
- **Harmonic measure** $\{\omega^X\}_{X \in \Omega}$ family of “probabilities” on $\partial\Omega$

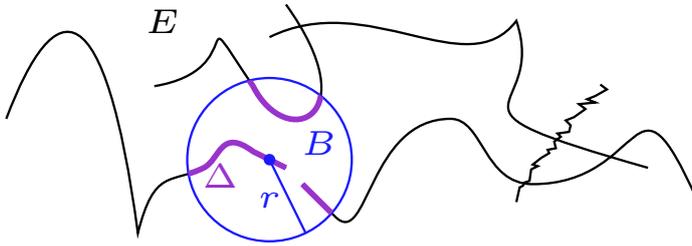
$$u(X) = \int_{\partial\Omega} f(x) d\omega^X(x) \text{ solves } (D) \begin{cases} \mathcal{L}u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = f \in C_c(\partial\Omega) \end{cases}$$



- **Surface ball**
 $\Delta(x, r) = B(x, r) \cap \partial\Omega, x \in \partial\Omega$
- $\sigma = \mathcal{H}^n|_{\partial\Omega}$

- $\partial\Omega$ **ADR** $\rightsquigarrow \sigma(\Delta(x, r)) \approx r^n, x \in \partial\Omega$

Rectifiability and Uniform rectifiability



- Surface ball
 $\Delta(x, r) = B(x, r) \cap E, x \in E$
- $\sigma = \mathcal{H}^n|_E$
- E ADR $\rightsquigarrow \sigma(\Delta(x, r)) \approx r^n$

● Rectifiability Countable union of Lipschitz Images σ -a.e.

● Uniform Rectifiability ADR and Big Pieces of Lipschitz Images

● [David-Semmes 1991]

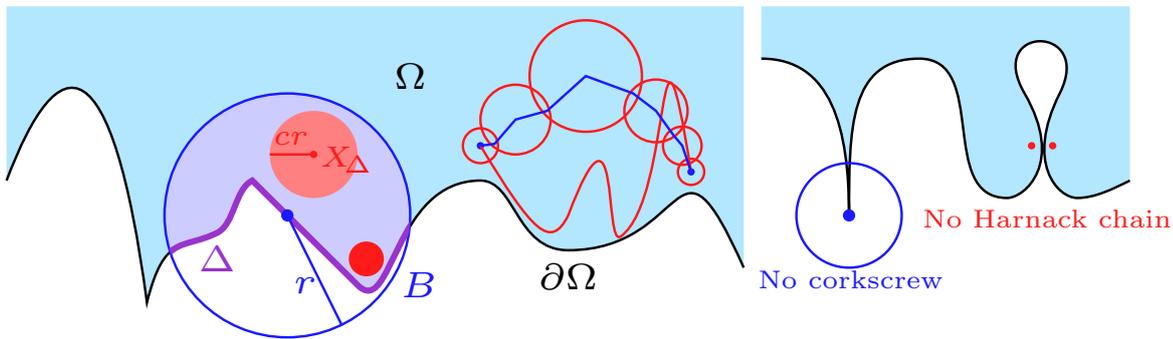
E is UR $\iff E$ is ADR + ALL “nice SIO” are bounded in $L^2(E)$

Section 2

The Laplacian

[Jerison-Kenig 1982] NTA/CAD domains

- Openness \rightsquigarrow Corkscrew condition
- Path-connectedness \rightsquigarrow Harnack chain condition



- Ω is CAD \equiv $\left\{ \begin{array}{l} \bullet$ **Interior** Corkscrew and Harnack Chain
 \bullet **Exterior** Corkscrew
 \bullet $\partial\Omega$ ADR
- Ω is 1-sided CAD \equiv $\left\{ \begin{array}{l} \bullet$ **Interior** Corkscrew and Harnack Chain
 \bullet $\partial\Omega$ ADR

F. & M. Riesz extension: Version I

Theorem (F. & M. Riesz 1916)

$\Omega \subset \mathbb{C}$ simply connected domain, $\partial\Omega$ rectifiable $\implies \omega \ll \sigma$

Theorem (David-Jerison 1990; Semmes 1989)

$$\bullet \Omega \subset \mathbb{R}^{n+1} \text{ CAD} \implies \begin{cases} \omega \in A_\infty(\sigma) \\ \frac{\omega(F)}{\omega(\Delta)} \lesssim \left(\frac{\sigma(F)}{\sigma(\Delta)} \right)^\theta, F \subset \Delta \end{cases}$$

- Where is the rectifiability?

[David-Jerison 1990]: Ω CAD $\implies \partial\Omega$ UR

- CAD \rightsquigarrow Exterior corkscrew
- Why do we need the exterior information? \rightsquigarrow PDE properties!!!

F. & M. Riesz extension: Version II

Theorem

- $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD

$$\textcircled{1} \partial\Omega \text{ UR} \iff \textcircled{2} \Omega \text{ CAD} \iff \textcircled{3} \omega \in A_\infty$$

$$\textcircled{1} \implies \textcircled{2} \text{ [Azzam, Hofmann, Nyström, Toro, M.]}$$

- UR \implies Existence of exterior corkscrews

$$\textcircled{2} \implies \textcircled{3} \text{ [David-Jerison; Semmes]}$$

$$\textcircled{3} \implies \textcircled{1} \text{ [Hofmann, Uriarte, M.]}$$

- $\partial\Omega$ is UR $\iff \iint_{\mathbb{R}^{n+1}} |\nabla^2 \mathcal{S}f|^2 \text{dist}(\cdot, \partial\Omega) dX \lesssim \int_{\partial\Omega} |f|^2 d\sigma$

- Local Tb Theorem: $\{b_Q\}_{Q \in \mathbb{D}} \rightsquigarrow \{\omega^{X_Q}\}_{Q \in \mathbb{D}}$ **Harmonicity!!!**

- PDE properties in 1-sided CAD domains

F. & M. Riesz extension: Version II

Theorem

- $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD

$$\textcircled{1} \partial\Omega \text{ UR} \iff \textcircled{2} \Omega \text{ CAD} \iff \textcircled{3} \omega \in A_\infty$$

- Can we consider other elliptic operators in $\textcircled{3}$?

$$\textcircled{2} \implies \textcircled{3}: \text{[Kenig-Pipher] or [Fefferman-Kenig-Pipher]}$$

$$\textcircled{3} \implies \textcircled{1}: \nabla^2 \mathcal{S} \text{ seems difficult!!!}$$

$$\textcircled{3} \implies \textcircled{2}: \begin{cases} \text{Problem 1: PDE properties} \\ \text{Problem 2: Existence of Exterior Corkscrews} \end{cases}$$

Section 3

Elliptic operators

F. & M. Riesz extension: Elliptic operators

- $Lu(X) = \operatorname{div}(A\nabla u)(X)$, $X \in \Omega$
- $A(X) = (a_{i,j}(X))_{1 \leq i,j \leq n+1}$ **real, symmetric**,
 $A(X)\xi \cdot \xi \geq \Lambda^{-1}|\xi|^2$ and $|A(X)\xi \cdot \eta| \leq \Lambda|\xi||\eta|$
- ω_L elliptic measure

Theorem (2 \implies 3): Kenig-Pipher; Fefferman-Kenig-Pipher)

- $\Omega \subset \mathbb{R}^{n+1}$ CAD
- $\delta(X) := \operatorname{dist}(X, \partial\Omega)$
- $\left. \begin{array}{l} |\nabla A| \delta \in L^\infty(\Omega) \\ |\nabla A|^2 \delta \text{ is a Carleson measure} \end{array} \right\} \implies \omega_L \in A_\infty$
- $\left. \begin{array}{l} \omega_{L_0} \in A_\infty \\ A \text{ is a Carleson perturbation of } A_0 \end{array} \right\} \implies \omega_L \in A_\infty$

Elliptic operators on 1-sided CAD

- $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD
- $L = \operatorname{div}(A\nabla)$ elliptic operator
- **Goal:** Find classes of matrices A such that

$$\omega_L \in A_\infty \implies \partial\Omega \text{ UR} \quad \text{—or } \Omega \text{ CAD—}$$

Proposition (e.g., Hofmann, Toro, M.)

- $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD
- **The Jerison-Kenig program can be carried out for L**
 - Hölder continuity at $\partial\Omega$
 - Bourgain's estimate
 - Caffarelli-Fabes-Mortola-Salsa's estimate
 - ω_L doubling
 - Comparison principle
 - ...

Furthermore,

- $\omega_L \in RH_p \iff L^{p'}\text{-Dirichlet problem is solvable}$
- $\omega_L \in A_\infty \iff L^q\text{-Dirichlet problem is solvable for large } q$

Free boundary results I

- $A \in \operatorname{Lip}_{\text{loc}}(\Omega)$
- $|\nabla A| \delta \in L^\infty(\Omega)$
- $|\nabla A|$ is a Carleson measure:

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r < \operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(\Delta(x, r))} \iint_{B(x, r) \cap \Omega} |\nabla A(X)| dX < \infty$$

Theorem (Hofmann, Toro, M.)

- $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD
- $\omega_L \in A_\infty \implies \text{Ext. Corkscrews } (\Omega \text{ CAD}) \iff \partial\Omega \text{ UR}$

- Our conditions are stronger than [Kenig-Pipher]
- Converse implication is also true

Free boundary results II: Perturbation

- Good operator: $L_0 u = \operatorname{div}(A_0 \nabla u)$

$$\bullet \omega_{L_0} \in A_\infty = \bigcup_{q>1} RH_q \qquad \bullet \omega_{L_0} \in RH_p$$

- Perturbed operator: $Lu = \operatorname{div}(A \nabla u)$
- Question 1: When $\omega_L \in A_\infty$? \rightsquigarrow Fefferman-Kenig-Pipher
- Question 2: When is $\omega_L \in RH_p$? \rightsquigarrow Dahlberg

Dahlberg's and Fefferman-Kenig-Pipher's perturbation

- Disagreement between L_0 and L

$$a(X) := \sup_{Y \in B(X, \delta(X)/2)} |A_0(Y) - A(Y)| \rightsquigarrow d\mu(X) = \frac{a(X)^2}{\delta(X)} dX$$

Theorem (Cavero, Hofmann, M.)

- Ω 1-sided CAD

$$\textcircled{1} \text{ Carleson measure: } \sup_{\Delta \subset \partial\Omega} \frac{\mu(T_\Delta)}{\sigma(\Delta)} < \infty$$

$$\omega_{L_0} \in A_\infty \implies \omega_L \in A_\infty$$

$$\textcircled{2} \text{ Vanishing Carleson measure: } \lim_{r \rightarrow 0^+} \sup_{\Delta_r \subset \partial\Omega} \frac{\mu(T_{\Delta_r})}{\sigma(\Delta_r)} = 0$$

$$\omega_{L_0} \in RH_p \implies \omega_L \in RH_p$$

Free boundary results II

Corollary (Cavero, Hofmann, M.)

- Ω 1-sided CAD
- \mathcal{L} Laplacian (or an operator as in [Hofmann, Toro, M.])
- L “Carleson perturbation” of \mathcal{L}

$$\omega_L \in A_\infty \iff \omega_{\mathcal{L}} \in A_\infty \iff \Omega \text{ CAD} \iff \partial\Omega \text{ UR}$$

Section 4

Sketch of the proof

Sketch of the proof: The Laplacian

- $A = I \rightsquigarrow L = \mathcal{L}$ Laplacian
- Goal: $\omega \in A_\infty \implies$ Exterior Corkscrews
- $\partial\Omega$ ADR $\rightsquigarrow \mathbb{D} = \{Q\}_{Q \in \mathbb{D}}$ dyadic grid [David-Semmes; Christ]
- **Bad cubes** $\mathcal{B} = \mathcal{B}(c_0) := \{Q \in \mathbb{D} : c_0\text{-ext. corkscrew fails for } Q\}$
- Goal: Packing condition $\sup_{Q \in \mathbb{D}} \frac{1}{\sigma(Q)} \sum_{Q' \in \mathcal{B} \cap \mathbb{D}_Q} \sigma(Q') =: M < \infty$
 - $Q \in \mathbb{D} \rightsquigarrow \exists Q' \notin \mathcal{B} \cap \mathbb{D}_Q$ with $2^{-[M]} \ell(Q) \leq \ell(Q') \leq \ell(Q)$
 - Q' has a c_0 -ext. cks. $\implies Q$ has a $c_0 2^{-[M]}$ -ext. cks.

Sketch of the proof: Packing condition

- Goal: Packing condition $\sup_{Q \in \mathbb{D}} \frac{1}{\sigma(Q)} \sum_{Q' \in \mathcal{B} \cap \mathbb{D}_Q} \sigma(Q') =: M < \infty$
- **John-Nirenberg reduction:** The following suffices

For every $Q_0 \in \mathcal{D}$, $\exists \mathcal{F} = \{Q_j\}_j \subset \mathbb{D}_{Q_0}$ pairwise disjoint such that

- $\sigma\left(Q_0 \setminus \bigcup_j Q_j\right) \geq \frac{1}{100} \sigma(Q_0)$

- $\sum_{Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}, Q_0}} \sigma(Q') = \sum_{Q \in \mathcal{B} \cap \mathbb{D}_{Q_0} : Q \not\subset Q_j \in \mathcal{F}} \sigma(Q) \leq \frac{M}{100} \sigma(Q_0)$

Sketch of the proof: Stopping time

- $Q_0 \in \mathbb{D} \rightsquigarrow X_0$ (interior) corkscrew relative to $100 Q_0$
- [Bourgain]: $\omega^{X_0}(Q_0) \geq C_0^{-1}$
- Normalization: $\omega := C_0 \sigma(Q_0) \omega^{X_0} \quad \mathcal{G} := C_0 \sigma(Q_0) G(\cdot, X_0)$
- $1 \leq \frac{\omega(Q_0)}{\sigma(Q_0)} \leq C_0$
- Subdivide (dyadically) and stop when $\frac{\omega(Q)}{\sigma(Q)}$ is too big or too small
 - $\mathcal{F} \subset \mathbb{D}_{Q_0}$ pairwise disjoint
 - $\frac{\omega(Q)}{\sigma(Q)} \approx 1$ for $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$ (i.e., $Q \not\subset Q_j$)
 - $\omega \in A_\infty \rightsquigarrow \sigma\left(Q_0 \setminus \bigcup_j Q_j\right) \geq \frac{1}{100} \sigma(Q_0)$

Sketch of the proof: Estimate for $Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}, Q_0}$

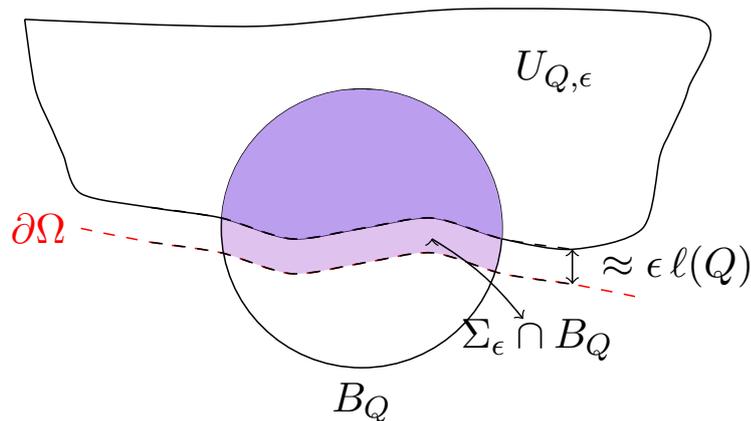
- $Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}, Q_0} \rightsquigarrow \Phi$ smooth cut-off for B_Q

▶ Skip

$$\begin{aligned}
 \sigma(Q) &\stackrel{Q \in \mathbb{D}_{\mathcal{F}, Q_0}}{\approx} \omega(Q) \approx \int_{\partial\Omega} \Phi d\omega = \iint_{\Omega} \nabla \mathcal{G}(X) \cdot \nabla \Phi(X) dX \\
 &= \iint_{\Omega} (\nabla \mathcal{G}(X) - \vec{\beta}) \cdot \nabla \Phi(X) dX + \vec{\beta} \cdot \iint_{\Omega} \nabla \Phi(X) dX \\
 &= \iint_{\Omega} (\nabla \mathcal{G}(X) - \vec{\beta}) \cdot \nabla \Phi(X) dX - \vec{\beta} \cdot \iint_{\Omega_{\text{ext}}} \nabla \Phi(X) dX \\
 &\lesssim \ell(Q)^{-1} \left(\iint_{\Omega \cap B_Q} |\nabla \mathcal{G}(X) - \vec{\beta}| dX + |\vec{\beta}| |\Omega_{\text{ext}} \cap B_Q| \right) \\
 &=: \ell(Q)^{-1} (I + II)
 \end{aligned}$$

Sketch of the proof: Estimate for $Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}, Q_0}$

- $Q \in \mathcal{B} \implies II = |\vec{\beta}| |\Omega_{\text{ext}} \cap B_Q| \lesssim c_0 \ell(Q)^{n+1} \approx c_0 \ell(Q) \sigma(Q)$
- $I = \iint_{\Omega \cap B_Q} |\nabla \mathcal{G}(X) - \vec{\beta}| dX \leq \iint_{U_{Q,\epsilon}} \dots + \iint_{\Sigma_\epsilon \cap B_Q} = I_1 + I_2$



$$I_2 \lesssim \epsilon \ell(Q) \sigma(Q)$$

Sketch of the proof: Estimate for $Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}, Q_0}$

- Ω 1-sided CAD $\rightsquigarrow U_{Q,\epsilon}$ connected union of Whitney boxes
- $\delta(X) \approx_\epsilon \ell(Q)$ for $X \in U_{Q,\epsilon}$
- Poincaré inequality in $U_{Q,\epsilon}$ Constant depends on $\epsilon!!!$
- Caffarelli-Fabes-Mortola-Salsa's $\rightsquigarrow \frac{\mathcal{G}(X)}{\delta(X)} \approx_\epsilon \frac{\omega(Q)}{\sigma(Q)}$ for $X \in U_{Q,\epsilon}$
- $Q \in \mathbb{D}_{\mathcal{F}, Q_0} \rightsquigarrow \frac{\omega(Q)}{\sigma(Q)} \approx 1$

$$I_1 = \iint_{U_{Q,\epsilon}} |\nabla \mathcal{G}(X) - \vec{\beta}| dX \lesssim_\epsilon \ell(Q) \iint_{U_{Q,\epsilon}} |\nabla^2 \mathcal{G}(X)| dX$$

$$\lesssim_\epsilon \ell(Q) \sigma(Q)^{\frac{1}{2}} \left(\iint_{U_{Q,\epsilon}} |\nabla^2 \mathcal{G}(X)|^2 \mathcal{G}(X) dX \right)^{\frac{1}{2}}$$

Sketch of the proof: Estimate for $Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}, Q_0}$

- $Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}, Q_0}$

$$\sigma(Q) \lesssim c_0 \sigma(Q) + \epsilon \sigma(Q) + C_\epsilon \sigma(Q)^{\frac{1}{2}} \left(\iint_{U_{Q, \epsilon}} |\nabla^2 \mathcal{G}(X)|^2 \delta(X) dX \right)^{\frac{1}{2}}$$

- Picking c_0 and ϵ small enough

$$\sigma(Q) \lesssim_\epsilon \iint_{U_{Q, \epsilon}} |\nabla^2 \mathcal{G}(X)|^2 \mathcal{G}(X) dX \quad \forall Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}, Q_0}$$

- $\Omega_\star = \text{int} \left(\bigcup_{Q \in \mathbb{D}_{\mathcal{F}, Q_0}} U_{Q, \epsilon}^\star \right)$ is 1-sided CAD

$$\sum_{Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}, Q_0}} \sigma(Q) \lesssim \iint_{\Omega_\star} |\nabla^2 \mathcal{G}(X)|^2 \mathcal{G}(X) dX \stackrel{???}{\lesssim} \sigma(Q_0)$$

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Sketch of the proof: Integrations by parts

- Goal: $\iint_{\Omega_\star} |\nabla^2 \mathcal{G}|^2 \mathcal{G} dX \lesssim \sigma(Q_0)$

- \mathcal{G} harmonic $\implies \partial \mathcal{G}$ harmonic

$$2 |\nabla(\partial \mathcal{G})|^2 \mathcal{G} = \mathcal{L}((\partial \mathcal{G})^2) \mathcal{G} = \text{div} [\nabla((\partial \mathcal{G})^2) \mathcal{G} - (\partial \mathcal{G})^2 \nabla \mathcal{G}]$$

- “Divergence Theorem”

$$\begin{aligned} 2 \iint_{\Omega_\star} |\nabla(\partial \mathcal{G})|^2 \mathcal{G} dX &= \iint_{\Omega_\star} \text{div} [\nabla((\partial \mathcal{G})^2) \mathcal{G} - (\partial \mathcal{G})^2 \nabla \mathcal{G}] dX \\ &= \int_{\partial \Omega_\star} [\nabla((\partial \mathcal{G})^2) \mathcal{G} - (\partial \mathcal{G})^2 \nabla \mathcal{G}] \cdot \vec{\nu} d\mathcal{H}^n \\ &\lesssim \int_{\partial \Omega_\star} [|\nabla^2 \mathcal{G}| |\nabla \mathcal{G}| \mathcal{G} + |\nabla \mathcal{G}|^3] d\mathcal{H}^n \end{aligned}$$

Sketch of the proof: Integrations by parts

- We know

$$\frac{\mathcal{G}(X)}{\delta(X)} \approx_\epsilon \frac{\omega(Q)}{\sigma(Q)} \approx 1, \quad X \in U_{Q,\epsilon}, \quad Q \in \mathbb{D}_{\mathcal{F},Q_0}$$

- Mean Value property + Caccioppoli's inequality

$$|\nabla \mathcal{G}(X)| \lesssim \frac{\mathcal{G}(X)}{\delta(X)} \approx 1 \quad \delta(X) |\nabla^2 \mathcal{G}(X)| \lesssim \frac{\mathcal{G}(X)}{\delta(X)} \approx 1$$

- Finally,

$$\begin{aligned} \iint_{\Omega_\star} |\nabla(\partial \mathcal{G})|^2 \mathcal{G} \, dX &\lesssim \int_{\partial \Omega_\star} [|\nabla^2 \mathcal{G}| |\nabla \mathcal{G}| \mathcal{G} + |\nabla \mathcal{G}|^3] \, d\mathcal{H}^n \\ &\lesssim \mathcal{H}^n(\partial \Omega_\star) \approx (\text{diam}(\partial \Omega_\star))^n \approx \ell(Q_0)^n \approx \sigma(Q_0) \quad \square \end{aligned}$$

Thank you for your attention!!!