

Sharp weighted norm estimates beyond Calderón-Zygmund theory

Dorothee Frey

Delft University of Technology, Netherlands

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Harmonic analysis - A_2 theorem

Weights: $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\omega \geq 0$, $\omega \in (0, \infty)$ a.e.

A_p condition: $p \in (1, \infty)$. $\omega \in A_p$ if

$$[\omega]_{A_p} = \sup_B \left(\int_B \omega \, dx \right) \left(\int_B \omega^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty$$

Theorem (Hytönen)

T Calderón-Zygmund operator, $\omega \in A_2$, then

$$\|Tf\|_{L^2(\omega)} \leq c_T [\omega]_{A_2} \|f\|_{L^2(\omega)}.$$

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Corollary (Rubio de Francia extrapolation)

$p \in (1, \infty)$. T CZO, $\omega \in A_p$, then

$$\|Tf\|_{L^p(\omega)} \leq c_{p,T} [\omega]_{A_p}^{\max(1, \frac{1}{p-1})} \|f\|_{L^p(\omega)}.$$

Riesz transform

$$\|\nabla L^{-1/2} f\|_2 \leq c \|f\|_2$$

for e.g. $L = -\operatorname{div} A \nabla$ with $A \in L^\infty(\mathbb{R}^n; M^n(\mathbb{C}))$ unif. elliptic

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Assumptions on L

- L injective, θ -accretive operator, $\theta \in [0, \frac{\pi}{2})$, with dense domain in $L^2(\mathbb{R}^n)$
- $\exists 1 \leq p_0 < 2 < q_0 \leq \infty$ s.th.

$$\|e^{-tL}\|_{L^{p_0}(B_1) \rightarrow L^{q_0}(B_2)} \leq C |B_1|^{-1/p_0} |B_2|^{1/q_0} e^{-c \frac{d(B_1, B_2)^2}{t}}$$

for all balls B_1, B_2 of radius \sqrt{t} .

Replacement of Calderón-Zygmund conditions

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- 2 $\exists N \in \mathbb{N}$ s.th. for all balls B_1, B_2 with radius \sqrt{t} ,

$$\|T(tL)^N e^{-tL}\|_{L^{p_0}(B_1) \rightarrow L^{q_0}(B_2)} \leq C |B_1|^{-1/p_0} |B_2|^{1/q_0} \left(1 + \frac{d(B_1, B_2)^2}{t}\right)^{-\frac{n+1}{2}}$$

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- 3 $\exists p_1 \in [p_0, 2)$:

$$\left(\int_{B(x,r)} |Te^{-r^2L}f|^{q_0} dx \right)^{1/q_0} \lesssim \inf_{y \in B(x,r)} \mathcal{M}_{p_1}(Tf)(y) + \inf_{y \in B(x,r)} \mathcal{M}_{p_1}(f)(y).$$

(Weighted) Boundedness of T

- ① L^2 **unweighted**: by assumption / solution of Kato problem [AHLMcT](#)

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$$\|Tf\| \leq c_p \|f\|_p, \quad p \in (p_0, q_0);$$

[Hofmann-Martell](#), [Blunck-Kunstmann](#); [Auscher-Coulhon-Duong-Hofmann](#)

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reverse Hölder class RH_q :

$$[\omega]_{RH_q} = \sup_B \left(\int_B \omega^q dx \right)^{1/q} \left(\int_B \omega dx \right) < \infty$$

$$\omega \in A_p \cap RH_q \iff \omega^q \in A_{q(p-1)+1}$$

Interaction - Weighted sharp bounds

L , T as above, with critical exponents p_0, q_0 .

Theorem (Bernicot, F., Petermichl 2016)

$p \in (p_0, q_0)$. There exists $c_p > 0$ such that for all $\omega \in A_{p/p_0} \cap RH_{(q_0/p)'}$

$$\|T\|_{L_\omega^p \rightarrow L_\omega^p} \leq c_p ([\omega]_{A_{p/p_0}} [w]_{RH_{(q_0/p)'}})^\alpha$$

with

$$\alpha := \max \left\{ \frac{1}{p - p_0}, \frac{q_0 - 1}{q_0 - p} \right\}.$$

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Note: With

$$p := 1 + \frac{p_0}{q_0'} \in (p_0, q_0),$$

we have

$$\alpha = \frac{1}{p - p_0} \quad \text{if } p \in (p_0, p],$$

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If $q_0 = p_0'$, then $p = 2$, and one has a sharp L_ω^2 inequality with $\alpha = \frac{1}{2 - p_0}$.

- If $p_0 = 1$ and $q_0 = \infty$, then

$$\alpha = \max \left\{ 1, \frac{1}{p-1} \right\}.$$

\leadsto reproving A_2 conjecture for e.g. $R = \nabla(-\Delta)^{-1/2}$ (not for all CZO)

Remarks

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- no need for pointwise regularity estimates on full kernel
- result holds true on doubling metric measure space

① Holomorphic functional calculus of L

- bounded H^∞ functional calculus on L^2 by assumption;
- extension to L^p for $p \in (p_0, q_0)$ by extrapolation

optimal weighted:

$$\|\varphi(L)\|_{L_\omega^p \rightarrow L_\omega^p} \leq c_{p,\sigma} ([\omega]_{A_{p/p_0}} [\omega]_{RH_{(q_0/p')}})^\alpha \|\varphi\|_\infty$$

for all $\varphi \in H^\infty(S_\sigma^o)$.

② **Riesz transforms** $R = \nabla L^{-1/2}$

- in Euclidean space/ doubling Riemannian manifold with bounded geometry and nonnegative Ricci curvature;

- in a convex doubling subset of \mathbb{R}^n with the Laplace operator associated with Neumann boundary conditions.

Then for all $p \in (1, \infty)$, $\omega \in A_p$,

$$\|R\|_{L_\omega^p \rightarrow L_\omega^p} \lesssim [\omega]_{A_p}^\alpha, \quad \alpha = \max \left\{ 1, \frac{1}{p-1} \right\}$$

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3 Paraproducts associated with L

For $g \in BMO_L(\mathbb{R}^n)$,

$$\Pi_g(f) = \int_0^\infty Q_t^{(k)}(Q_t^{(k)} f \cdot P_t^{(k)} g) \frac{dt}{t}$$

\leadsto sharp weighted algebra properties for fractional Sobolev spaces

4 Fourier multiplier $q_0 = \infty$

linear symbol m on \mathbb{R}^n satisfying Hörmander condition

$$\sup_{R>0} \left(R^{s|\alpha|-n} \int_{R \leq |\xi| \leq 2R} |\partial_\xi^\alpha m(\xi)|^s d\xi \right)^{1/s} < \infty$$

for all $|\alpha| \leq l$, some $s \in (1, 2]$ and $l \in (n/s, n)$.

$$T(f) = T_m(f) : x \mapsto \int e^{ix \cdot \xi} m(\xi) \hat{f}(\xi) d\xi.$$

[Bui, Conde-Alonso, Duong, Hormozi '15]: kernel of T satisfies L^r - L^∞ regularity off-diagonal estimates, $r \in (n/l, \infty)$.

Then assumptions satisfied with $p_0 = r$ and $q_0 = \infty$.

Reproves main result of [Bui, Conde-Alonso, Duong, Hormozi '15].

Structure of proof

Step 1: Sparse domination

Step 2: Weighted estimate for “sparse” operator

For any cube $Q \subseteq \mathbb{R}^n$ there exists a shifted dyadic cube

$$R \in \mathcal{D}^\alpha = \{2^{-k}([0, 1]^n + m + (-1)^k \alpha) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}$$

for some $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, such that

$$Q \subseteq R, \ell(R) \leq 6\ell(Q).$$

Denote $\mathcal{D} := \bigcup_\alpha \mathcal{D}^\alpha$, and call *dyadic set* any element of \mathcal{D} .

Definition (Sparse collection)

A collection of dyadic sets $\mathcal{S} := (P)_{P \in \mathcal{S}} \subset \mathcal{D}$ is said to be *sparse* if for each $P \in \mathcal{S}$ one has

$$(1) \quad \sum_{Q \in ch_{\mathcal{S}}(P)} |Q| \leq \frac{1}{2}|P|,$$

where $ch_{\mathcal{S}}(P)$ is the collection of \mathcal{S} -children of P : the maximal elements of \mathcal{S} that are strictly contained in P .

Step 1: Sparse domination

Theorem

$p \in (p_0, q_0)$. There exists $C > 0$ such that for all $f \in L^p$ and $g \in L^{p'}$ both supported in $5Q_0$ for some $Q_0 \in \mathcal{D}$, there exists a sparse collection $S \subset \mathcal{D}$ (depending on f, g) with

$$\left| \int_{Q_0} Tf \cdot g \, dx \right| \leq C \sum_{P \in S} |P| \left(\int_{5P} |f|^{p_0} \, dx \right)^{1/p_0} \left(\int_{5P} |g|^{q'_0} \, dx \right)^{1/q'_0}.$$

Stopping time argument for maximal operator

Elements in the proof

Maximal operator $T^\#$ of T

$$T^\# f(x) = \sup_{\substack{B \text{ ball} \\ B \ni x}} \left(\int_B |Te^{-r(B)^2 L} f|^{q_0} dy \right)^{1/q_0}, \quad x \in M,$$

for $f \in L_{\text{loc}}^{q_0}$.

Proposition

$T^\#$ is of weak type (p_0, p_0) and bounded in L^p for $p \in (p_0, 2]$.

Via Cotlar-type inequality, extrapolation.

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Run stopping time argument with sets

$$E = \left\{ x \in Q_0 \mid \max\{M_{Q_0, p_0}^* f(x), T_{Q_0}^\# f(x)\} > \eta \left(\int_{5Q_0} |f|^{p_0} dx \right)^{1/p_0} \right\}.$$

Step 2: Weighted estimates for sparse operators

Proposition

$p_0, q_0 \in [1, \infty]$ with $p_0 < q_0$, and $p \in (p_0, q_0)$. Suppose S is a bounded operator on L^p , and suppose there exists $c > 0$ such that for all $f \in L^p$ and $g \in L^{p'}$ there exists a sparse collection S with

$$|\langle S(f), g \rangle| \leq c \sum_{P \in S} \left(\int_{5P} |f|^{p_0} dx \right)^{1/p_0} \left(\int_{5P} |g|^{q_0'} dx \right)^{1/q_0'} |P|.$$

Then there exists a constant $C = C(S, p, p_0, q_0)$ such that for every weight $\omega \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'}$, the operator S is bounded on L^p_ω with

$$\|S\|_{L^p_\omega \rightarrow L^p_\omega} \leq C \left([\omega]_{A_{\frac{p}{p_0}}} [\omega]_{RH_{\left(\frac{q_0}{p}\right)'}} \right)^\alpha,$$

with

$$\alpha := \max \left\{ \frac{1}{p - p_0}, \frac{q_0 - 1}{q_0 - p} \right\}.$$

Optimality

Only for $n = 1$.

For $p > 1$, $\omega_\alpha : x \mapsto |x|^\alpha$ is in A_p if and only if $-1 < \alpha < p - 1$,

$$[\omega_{-1+\varepsilon}]_{A_p} \simeq \varepsilon^{-1}, \quad [\omega_{p-1-\varepsilon}]_{A_p} \simeq \varepsilon^{-(p-1)}$$

as $\varepsilon \rightarrow 0$.

If $s > 1$, then $\omega_{-1/s+\varepsilon}$ is critical for RH_s ,

$$[\omega_{-1/s+\varepsilon}]_{RH_s} \simeq \varepsilon^{-1/s}.$$

sparse collection $\mathcal{S} = \{I_n := [0, 2^{-n}] : n \in \mathbb{N}\}$

Proposition

$p \in (p_0, q_0)$. There exist functions f, g such that $\phi(r) = r^\alpha$ is the best possible choice, asymptotically as $r \rightarrow \infty$.

Thank you for your attention.