# *Multipliers on compact Lie groups*

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# *Plan of the talk*

I. Multiplier theorems on $\mathbb{R}^n$ and $\mathbb{T}^n$

II. Fourier analysis on compact Lie groups $G$

III. Derivatives and Sobolev spaces on $\widehat{G}$.

IV. Fourier multiplier theorems on compact Lie groups.

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*Fourier multipliers on $\mathbb{T}$*

Let $\sigma : \mathbb{Z} \to \mathbb{C}$ be a sequence on the integers. The corresponding Fourier multipliers on the torus is defined via

$$\mathrm{Op}(\sigma)\phi(e^{i\theta}) = \sum_{\ell=-\infty}^{+\infty} e^{i\ell\theta}\sigma(\ell)\widehat{\phi}(\ell),$$

when $\phi(e^{i\theta}) = \sum_{\ell=-\infty}^{+\infty} e^{i\ell\theta}\widehat{\phi}(\ell)$ has only finite terms (e.g.).

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*Sur les multiplicateurs des séries de Fourier, Studia Math.* ***8****, 78-91.*

Assume that the suprema

$$\sup_{\ell} |\sigma(\ell)| < \infty \quad \text{and} \quad \sup_{j\in\mathbb{N}_0} \sum_{2^j \leq \ell \leq 2^{j+1}} |\sigma(\ell) - \sigma(\ell+1)| + |\sigma(-\ell) - \sigma(-\ell-1)|$$

are finite. Then $\mathrm{Op}(\sigma)$ is bounded on $L^2(\mathbb{T})$ and on $L^p(\mathbb{T})$, $1 < p < \infty$.

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If $\sigma : \mathbb{R}^n \to \mathbb{C}$ satisfies $|\partial^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}$ for $|\alpha| \leq [n/2] + 1$, then the Fourier multiplier operator $\mathrm{Op}(\sigma)$ defined via

$$\mathcal{F}\left(\mathrm{Op}(\sigma)\phi\right) := \sigma\widehat{\phi} \quad \phi \in \mathcal{S}(\mathbb{R}^n),$$

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*Multiplier Theorem 2 (Hörmander, 1960)*

If $\|\sigma\|_{l.u.H^s(\mathbb{R}^n),\eta} := \sup_{r>0} \|\sigma(r\,\cdot)\ \eta\|_{H^s(\mathbb{R}^n)}$ is finite for some non-zero function $\eta \in \mathcal{D}(0,\infty)$ and $s > n/2$, then $\mathrm{Op}(\sigma)$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

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*Note:*

1. for $s \in \mathbb{N}$, $\|\sigma\|_{l.u.H^s(\mathbb{R}^n),\eta} \lesssim \max_{|\alpha|\leq s} \sup_{\xi\in\mathbb{R}^n} |\xi|^\alpha |\partial^\alpha \sigma(\xi)|$. So Thm 2 $\Rightarrow$ Thm 1.
2. for $s > n/2$, $\|\sigma\|_{l.u.H^s(\mathbb{R}^n),\eta} \asymp \sup_{\mathbb{R}^n} |\sigma| + \sup_{r>0} r^{s-n/2} \|\sigma\ \eta(r^{-1}\cdot)\|_{\dot{H}^s(\mathbb{R}^n)}$.

*Today's questions/problems:*

- definition of Fourier multiplier operators on a connected compact Lie group $G$ which is usually non-abelian.
- condition on the Fourier multiplier symbol to ensure $L^p(G)$-boundedness?
- where does it stand in the literature on the subject?

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*Notation*

- if $\pi$ is a representation of $G$, then $d_\pi$=its dimension, $\mathcal{H}_\pi$ its space and, $\pi_{i,j}$ its entries once a basis has been chosen.
- $\text{Rep}(G)$:= set of (finite dim.) representations modulo equivalence.
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*The Peter-Weyl Theorem (1927)*

The dual $\widehat{G}$ is discrete and countable.
The functions $\sqrt{d_\pi}\pi_{i,j}$, $1 \leq i, j \leq d_\pi$, $\pi \in \widehat{G}$, form an orthonormal basis of the Hilbert space $L^2(G)$.

# *In other words: group Fourier series*

*On the torus $\mathbb{T}$,*

the Fourier series of $\phi : \mathbb{T} \to \mathbb{C}$ is $\displaystyle \phi(e^{i\theta}) = \sum_{\ell=-\infty}^{+\infty} \widehat{\phi}(\ell) e^{i\ell\theta}$,

where $\widehat{\phi}(\ell) = (\phi, e^{i\ell\theta})_{L^2(\mathbb{T})}$. Recall $\widehat{\mathbb{T}} = \{e^{i\theta} \mapsto e^{i\ell\theta}, \ell \in \mathbb{Z}\} \sim \mathbb{Z}$.

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*On $G$,*

$$\phi(x) = \sum_{\pi \in \widehat{G}} \sum_{1 \leq i,j \leq d_\pi} d_\pi \underbrace{(\phi, \pi_{i,j})_{L^2(G)}}_{=[\widehat{\phi}]_{j,i}} \pi_{i,j}(x) = \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}\left(\widehat{\phi}(\pi)\ \pi(x)\right),$$

where

$$\widehat{\phi}(\pi) := \int_G \phi(x) \pi(x)^* dx =: (\mathcal{F}_G \phi)(\pi).$$

# Plancherel formula and Fourier multipliers

### Plancherel formula

$$\int_G |\phi(x)|^2 dx = \sum_{\pi \in \widehat{G}} d_\pi \sum_{1 \le i,j \le d_\pi} |(\phi, \pi_{i,j})_{L^2(G)}|^2 = \sum_{\pi \in \widehat{G}} d_\pi \|\widehat{\phi}\|^2_{HS(\mathcal{H}_\pi)},$$

where $\|M\|^2_{HS} = \sum_{i,j} |M_{i,j}|^2$. So $\|\phi\|_{L^2(G)} = \|\widehat{\phi}\|_{L^2(\widehat{G})}$.

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$$(\mathrm{Op}(\sigma)\phi)(x) = \sum_{\pi\in\widehat{G}} d_\pi \mathrm{tr}\left(\widehat{\phi}(\pi)\ \sigma(\pi)\ \pi(x)\right), \quad x \in G.$$

Example: $X^\beta = \mathrm{Op}(\mathcal{F}_G(X))$ where $\mathcal{F}_G(X) = \{\pi(X)^\beta, \pi \in \widehat{G}\}$.

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### Fourier multiplier problem

If $\sigma \in L^\infty(\widehat{G})$, i.e. $\|\sigma\|_{L^\infty(\widehat{G})} := \sup_{\pi\in\widehat{G}} \|\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty$, then $\mathrm{Op}(\sigma)$ is bounded on $L^2(G)$. Condition for $L^p$-boundedness $p \in (1,\infty)$?

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*Definition of $\lambda_\pi$*

If $\mathcal{L}$ denotes the (left and right invariant) Laplace operator, then for any $\pi \in \widehat{G}$, $\pi_{i,j}$ is an eigenfunction of $\mathcal{L}$ with eigenvalue $\lambda_\pi \geq 0$.
The spectrum of $\mathcal{L}$ is $\mathrm{Spec}(\mathcal{L}) = \{\lambda_\pi, \pi \in \widehat{G}\}$, a discrete subset of $[0, \infty)$.

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*Spectral and Fourier multipliers, their kernels*

1. If $f$:Spec$(\mathcal{L}) \to \mathbb{C}$ then $f(\mathcal{L}) = \mathrm{Op}(\sigma)$ with $\sigma(\pi) = f(\lambda_\pi)$. Furthermore, if $f(\mathcal{L}) : \mathcal{D}(G) \to \mathcal{D}'(G)$, then $f(\mathcal{L})\delta_{e_G} \in \mathcal{D}'(G)$ with group Fourier transform $\sigma$.

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2. For any $s \geq 0$, there exists a constant $C > 0$ and $d \in \mathbb{N}_0$ such that for any $f : \mathbb{R} \to \mathbb{C}$ continuous with supp $f \cap [0, \infty) \subset [0, 1]$, we have
$$\forall t \in (0,1] \qquad \| |x|^s f(t\mathcal{L})\delta_{e_G} \|_{L^1(G)} \leq C\sqrt{t}^s \sup_{\ell=0,\dots,d,\ \lambda \geq 0} |f^{(\ell)}(\lambda)|.$$
   Similarly for $L^2(G)$ with $\sqrt{t}^{s-\frac{n}{2}}$. (From heat kernel estimates)

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and if $\sigma : \mathbb{Z} \to \mathbb{C}$ is a symbol, we can consider the discrete derivatives

$$\Delta^+\sigma(\ell) = \sigma(\ell+1) - \sigma(\ell), \quad \Delta^-\sigma(\ell) = \sigma(\ell-1) - \sigma(\ell).$$

These quantities intervened in the Marcinkiewicz Theorem!

If $\sigma(\ell) = \widehat{\phi}(\ell)$ then $\Delta^+\sigma(\ell) = \int_{\mathbb{T}} \phi(e^{i\theta}) \underbrace{(e^{i\theta}-1)}_{|''|\asymp\theta} e^{i\ell\theta} d\theta.$

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*Ruzhansky-Turunen difference operators (2010-14)*

Choose a suitable collection of functions $q_j$, $j = 1, \ldots,$ to replace $(e^{i\theta} - 1)$, and define $\Delta_{q_j}\widehat{\phi} := \widehat{q_j\phi}$.

Remark: implicit definition on $\mathcal{FD}'(G)$, and not on the whole set of symbols $\Sigma = \{\sigma(\pi) \in \mathscr{L}(\mathcal{H}_\pi), \pi\}$. No locality.

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*The symbols extend to fields on* $\mathrm{Rep}(G)$*.*

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$$\Delta_\tau \sigma(\pi) := \sigma(\tau \otimes \pi) - \sigma(\mathrm{Id}_\tau \otimes \pi), \quad \pi \in \mathrm{Rep}(G).$$

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- We will often restrict ourselves to $\tau$ fundamental representation. Recall that any representation occurs in the tensor of a finite number of fundamental representations and that $\mathrm{Fund}(G)$ is finite.
- Each $\Delta_\tau$ preserves locality: $\exists C > 0$ s.t. $\forall \sigma \in \Sigma$, $\forall 0 \le a \le b$: $\sigma(\pi) = 0$ when $\lambda_\pi \notin [a, b] \implies \Delta_\tau \sigma(\pi) = 0$ when $\lambda_\pi \notin [a - C, b + C]$.

# *Link between $\Delta_q$ and $\Delta_\tau$?*

$\Delta_\tau \sigma(\pi) := \sigma(\tau \otimes \pi) - \sigma(\mathrm{Id}_\tau \otimes \pi)$ in the case $\sigma = \widehat{\phi}$ yields:

$$\Delta_\tau \widehat{\phi}(\pi) = \int_G \phi(x)(\tau(x)^* - \mathrm{Id}) \otimes \pi(x)^* dx = \left(\Delta_{q_{i,j}^{(\tau)}} \widehat{\phi}(\pi)\right)_{1 \le i,j \le d_\pi},$$

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where $q_{i,j}^{(\tau)}(x) = [\tau(x)^* - \mathrm{Id}]_{i,j}$.Therefore

$$\sum_{\pi \in \widehat{G}} d_\pi \|\Delta_\tau \phi(\pi)\|^2_{HS(\mathcal{H}_\pi \otimes \mathcal{H}_\tau)} = \sum_{1 \le i,j \le d_\tau} \sum_{\pi \in \widehat{G}} d_\pi \|\mathcal{F}_G(q_{i,j}^{(\tau)} \phi)(\pi)\|^2_{HS(\mathcal{H}_\pi \otimes \mathcal{H}_\tau)}$$

$$= \sum_{1 \le i,j \le d_\tau} \|q_{i,j}^{(\tau)} \phi\|^2_{L^2(G)} = \int_G \|\tau(x)^* - \mathrm{Id}\|^2_{HS(\mathcal{H}_\tau)} |\phi(x)|^2 dx.$$

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*The function $q_2$*

The function given by $q_2(x) := \sum_{\tau\in\mathrm{Fund}(G)}\|\tau(x)^* - \mathrm{Id}\|^2_{HS(\mathcal{H}_\tau)}$ is smooth and non-negative with $q_2(x)\asymp|x|^2$.

# *Homogeneous Sobolev spaces on $\widehat{G}$ (F. 2016?)*

### *Homogeneous Sobolev space $\dot{H}^1(\widehat{G})$*

- Definition of $\dot{H}^1(\widehat{G})$ as the set of $\sigma \in \Sigma$ s.t.

$$\|\sigma\|^2_{\dot{H}^1(\widehat{G})} := \sum_{\tau \in \mathrm{Fund}(G)} \sum_{\pi \in \widehat{G}} d_\pi \|\Delta_\tau \sigma(\pi)\|^2_{HS(\mathcal{H}_\tau \otimes \mathcal{H}_\pi)} < \infty.$$

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## Definition of $\dot{H}^s(\widehat{G})$, $s \in \mathbb{N}$

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### *Properties of* $\dot{H}^s(\widehat{G})$, $s \in \mathbb{N}$

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- the set of symbol $\sigma \in L^2(\widehat{G})$ such that $\mathcal{F}_G^{-1}\sigma \in L^2(|x|^{2s}dx)$ if $s \leq 0$,
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*Main property*

$\dot{H}^s(\widehat{G})$ quotiented by the kernel of its norm is a Hilbert space isometrically isomorphic to $L^2(q_2^s(x)dx)$. Therefore these sets are growing with $s$, interpolation etc...

# *Leibniz property*

*Leibniz formula*

For any $\sigma_1, \sigma_2 \in \Sigma(G)$ and $\tau, \pi \in \mathrm{Rep}(G)$:

$$\Delta_\tau(\sigma_1\sigma_2)(\pi) = \Delta_\tau(\sigma_1)(\pi)\ \sigma_2(\mathrm{Id}_\tau \otimes \pi) + \sigma_1(\tau \otimes \pi)\ \Delta_\tau(\sigma_2)(\pi).$$

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*Leibniz-type estimates*

For any $s \in \mathbb{N}_0$, $p = 2, \infty$, $\sigma_1, \sigma_2 \in \Sigma$

$$\|\sigma_1\sigma_2\|_{\dot{L}^p_s(\widehat{G})} \leq C_{p,s} \sum_{s_1+s_2=s} \|\sigma_1\|_{\dot{L}^\infty_{s_1}(\widehat{G})} \|\sigma_2\|_{\dot{L}^p_{s_2}(\widehat{G})}.$$

Here $\dot{L}^2_s(\widehat{G}) := \dot{H}^s(\widehat{G})$, and $\dot{L}^\infty_s(\widehat{G})$ is the set of $\sigma \in \Sigma$ s.t.

$$\|\sigma\|_{\dot{L}^\infty_s(\widehat{G})} := \max_{\tau_1,\dots,\tau_s \in \mathrm{Fund}(G)} \sup_{\pi \in \widehat{G}} \|\Delta_{\tau_1} \dots \Delta_{\tau_s} \sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_{\tau_1 \otimes \dots \otimes \tau_s \otimes \mathcal{H}_\pi})} < \infty.$$

Other Leibniz-type estimates with more info on the symbols...

# *Back to Fourier multiplier theorems - Results (F. 2016?)*

*Hörmander-type theorem*

Let $G$ be a compact Lie group of dimension $n$. If $\sigma \in L^\infty(\widehat{G})$ is s.t.

$$\sup_{r>0} r^{s-\frac{n}{2}} \|\sigma \ \eta(r^{-2}\widehat{\mathcal{L}})\|_{\dot{H}^s(\widehat{G})} < \infty \qquad \text{for some } s > \frac{n}{2} \text{ and } \eta \in \mathcal{D}(0,\infty)\backslash\{0\},$$

then $\mathrm{Op}(\sigma)$ is bounded on $L^p(G)$, $p \in (1,\infty)$.

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Classical proof with Calderòn-Zygmund theory. Only $r = 2^j$, $j \geq j_0$, needed. The sharpness in $s$ is obtained by considering the spectral case:

*Case of spectral multipliers in $\mathcal{L}$*

For $s' > s > \frac{n}{2}$ with $s \geq 1$ and $\eta \in \mathcal{D}(0,\infty)\backslash\{0\}$,

$$\|f(\widehat{\mathcal{L}})\|_{L^\infty(\widehat{G})} + \sup_{r>0} r^{s-\frac{n}{2}} \|\sigma \ \eta(r^{-2}\widehat{\mathcal{L}})\|_{\dot{H}^s(\widehat{G})} \leq C_{s,s',\eta} \|f\|_{l.u.H^{s'}(\mathbb{R}),\eta}.$$

Cq: $f(\mathcal{L}) = \mathrm{Op}(f(\widehat{\mathcal{L}}))$ is bounded on $L^p(G)$, $p \in (1,\infty)$. Known on compact manifold (Sogge 1993).

# *Consequences if the Hörmander-type theorem*

*Mihlin-type theorem*

Let $G$ be a compact Lie group of dimension $n$. If $\sigma \in L^\infty(\widehat{G})$ is s.t. for $s' = 1, \ldots, [n/2] + 1$ and $\tau_1, \ldots, \tau_{s'} \in \text{Fund}(G)$

$$\forall \pi \in \widehat{G} \qquad \|\Delta_{\tau_1} \ldots \Delta_{\tau_{s'}} \sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_{\tau_1} \otimes \ldots \mathcal{H}_{\tau_{s'}} \otimes \mathcal{H}_\pi)} \leq C_{s'} (1 + \lambda_\pi)^{-\frac{s'}{2}},$$

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*Marcinkiewicz-type theorem*

Let $G$ be a compact Lie group of dimension $n$. For any $\sigma \in \Sigma$ satisfying $\sup_{\pi \in \widehat{G}} \|\sigma(\pi)\|_{HS(\mathcal{H}_\pi)} < \infty$ and

$$\sup_{j \in \mathbb{N}} \sum_{2^j \leq \lambda_\pi \leq 2^{j+1}} \|\Delta_{\tau_1} \ldots \Delta_{\tau_{[n/2]+1}} \sigma(\pi)\|_{HS(\mathcal{H}_{\tau_1} \otimes \ldots \mathcal{H}_{\tau_{[n/2]+1}} \otimes \mathcal{H}_\pi)} < \infty,$$

then $\mathrm{Op}(\sigma)$ is bounded on $L^p(G)$, $p \in (1, \infty)$.

# *Why was it interesting to me?*

*Interesting proofs of e.g.*

- estimate: $\forall s \in \mathbb{N} \quad \forall t \in (0,1) \quad \forall k \in \mathbb{N} \quad \|e^{ike^{-t\widehat{\mathcal{L}}}}\|_{\dot{H}^s(\widehat{G})} \leq C_s \sqrt{t}^{s-\frac{n}{p}} k^s$,
  - for $k = 1$, heat kernel estimates
  - for $k > 1$, $\Delta_\tau \sigma^k(\pi) = \Delta_\tau \sigma(\pi) \sum_{j_1+j_2=k-1} \sigma(\tau \otimes \pi)^{j_1} \sigma(\mathrm{Id}_\tau \otimes \pi)^{j_2}$, and Leibniz-type estimates.

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- characterisation: $\|\sigma\|_{\dot{H}^s(\widehat{G})}=0 \iff \sigma\in\mathcal{F}_G\mathfrak{U}_{s-1}(\mathfrak{g})$.
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$\exists$ *large literature of Fourier multipliers on compact groups*

- very often spectral in $\mathcal{L}$ (estimates for Bochner-Riesz in B. Dreseler 1986)
- or central (L. Vretare 1974, N. Weiss 1972), thus $\sigma(\pi)$ is scalar and $\pi \sim$ its highest weight $\in \mathfrak{g}^*$,
- only exceptions known to me: R. Coifman + G. Weiss 1974 on $SU(2)$, M. Ruzhansky + J. Wirth 2014.

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6. other settings: notion of intrinsic difference operators on e.g. the Heisenberg group (contraction?)

THANK YOU FOR YOUR ATTENTION.