The Boundary value problems for second order elliptic operators satisfying Carleson condition

Martin Dindoš

Interactions of Harmonic Analysis and Operator Theory,
Birmingham, 13-16 September 2016

Table of contents

Dirichlet, Neumann and Regularity boundary value problems

Let $L = \text{div}\, A\nabla u$ be a second order elliptic operator with bounded **real** measurable coefficients $A = (a_{ij})$ on a Lipschitz domain Ω. That is there is $\Lambda > 0$ such that

$$\Lambda^{-1}|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2.$$

(Matrix A does not have to be symmetric).

Dirichlet, Neumann and Regularity boundary value problems

Let $L = \text{div } A\nabla u$ be a second order elliptic operator with bounded **real** measurable coefficients $A = (a_{ij})$ on a Lipschitz domain Ω. That is there is $\Lambda > 0$ such that

$$\Lambda^{-1}|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2.$$

(Matrix A does not have to be symmetric).

Let $\Gamma(.)$ be a collection of nontangential cones with vertices a boundary points $Q \in \partial\Omega$. We define the non-tangential maximal function at Q relative to Γ by

$$N(u)(Q) = \sup_{X \in \Gamma(Q)} |u(X)|.$$

We also consider a weaker version of this object

$$\widetilde{N}(u)(Q) = \sup_{X \in \Gamma(Q)} \left(\delta(X)^{-n} \int_{B_{\delta(X)/2}(X)} |u(Y)|^2 \, dY \right)^{\frac{1}{2}}.$$

Let $\Gamma(.)$ be a collection of nontangential cones with vertices a boundary points $Q \in \partial\Omega$. We define the non-tangential maximal function at Q relative to Γ by

$$N(u)(Q) = \sup_{X \in \Gamma(Q)} |u(X)|.$$

We also consider a weaker version of this object

$$\widetilde{N}(u)(Q) = \sup_{X \in \Gamma(Q)} \left(\delta(X)^{-n} \int_{B_{\delta(X)/2}(X)} |u(Y)|^2 \, dY \right)^{\frac{1}{2}}.$$

L^p Dirichlet problem

Definition

Let $1 < p \leq \infty$. The Dirichlet problem with data in $L^p(\partial\Omega, d\sigma)$ is solvable (abbreviated $(D)_p$) if for every $f \in C(\partial\Omega)$ the weak solution u to the problem $Lu = 0$ with continuous boundary data f satisfies the estimate

$$\|N(u)\|_{L^p(\partial\Omega, d\sigma)} \lesssim \|f\|_{L^p(\partial\Omega, d\sigma)}.$$

The implied constant depends only the operator L, p, and the Lipschitz constant of the domain.

L^p Neumann problem

Definition

Let $1 < p < \infty$. The Neumann problem with boundary data in $L^p(\partial\Omega)$ is solvable (abbreviated $(N)_p$), if for every $f \in L^p(\partial\Omega) \cap C(\partial\Omega)$ such that $\int_{\partial\Omega} f d\sigma = 0$ the weak solution u to the problem

$$\begin{cases} Lu & = 0 \quad \text{in } \Omega \\ A\nabla u \cdot \nu & = f \quad \text{on } \partial\Omega \end{cases}$$

satisfies

$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}.$$

Again, the implied constant depends only the operator L, p, and the Lipschitz constant of the domain. Here ν is the outer normal to the boundary $\partial\Omega$.

Regularity problem

Definition

Let $1 < p < \infty$. The regularity problem with boundary data in $H^{1,p}(\partial\Omega)$ is solvable (abbreviated $(R)_p$), if for every $f \in H^{1,p}(\partial\Omega) \cap C(\partial\Omega)$ the weak solution u to the problem

$$\begin{cases} Lu & = 0 \quad \text{in } \Omega \\ u|_{\partial B} & = f \quad \text{on } \partial\Omega \end{cases}$$

satisfies

$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_T f\|_{L^p(\partial\Omega)}.$$

Again, the implied constant depends only the operator L, p, and the Lipschitz constant of the domain.

Negative result

Theorem

There exists a bounded measurable matrix A on a unit disk D satisfying the ellipticity condition such that the Dirichlet problem $(D)_p$, the Regularity problem $(R)_p$ and the Neumann problem $(N)_p$ are not solvable for any $p \in (1, \infty)$.

Hence solvability requires extra assumption on the regularity of coefficients of the matrix A.

Negative result

Theorem

There exists a bounded measurable matrix A on a unit disk D satisfying the ellipticity condition such that the Dirichlet problem $(D)_p$, the Regularity problem $(R)_p$ and the Neumann problem $(N)_p$ are not solvable for any $p \in (1, \infty)$.

Hence solvability requires extra assumption on the regularity of coefficients of the matrix A.

The Carleson condition - motivation

Consider the boundary value problems associated with a smooth elliptic operator in the region above a graph $t = \varphi(x)$, for φ Lipschitz.

Consider a mapping $\Phi : \mathbb{R}^n_+ \to \{X = (x,t); t > \varphi(x)\}$ defined by

$$\Phi(X) = (x, c_0 t + (\theta_t * \varphi)(x))$$

where $(\theta_t)_{t>0}$ is smooth compactly supported approximate identity and c_0 is large enough so that Φ is one to one.

The Carleson condition - motivation

Consider the boundary value problems associated with a smooth elliptic operator in the region above a graph $t = \varphi(x)$, for φ Lipschitz.

Consider a mapping $\Phi : \mathbb{R}^n_+ \to \{X = (x,t); t > \varphi(x)\}$ defined by

$$\Phi(X) = (x, c_0 t + (\theta_t * \varphi)(x))$$

where $(\theta_t)_{t>0}$ is smooth compactly supported approximate identity and c_0 is large enough so that Φ is one to one.

Then the function $v = u \circ \Phi$ solves an elliptic equation in $\mathbb{R}^n_+$ with coefficients satisfying

The Carleson condition - motivation

Consider the boundary value problems associated with a smooth elliptic operator in the region above a graph $t = \varphi(x)$, for φ Lipschitz.

Consider a mapping $\Phi : \mathbb{R}^n_+ \to \{X = (x, t); t > \varphi(x)\}$ defined by

$$\Phi(X) = (x, c_0 t + (\theta_t * \varphi)(x))$$

where $(\theta_t)_{t>0}$ is smooth compactly supported approximate identity and c_0 is large enough so that Φ is one to one.

Then the function $v = u \circ \Phi$ solves an elliptic equation in $\mathbb{R}^n_+$ with coefficients satisfying

The Carleson condition - motivation

$$\delta(X)^{-1}\left(\operatorname{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2$$

is the density of a Carleson measure on Ω.

Definition

A measure μ in Ω is Carleson if there exists a constant $C = C(r_0)$ such that for all $r \le r_0$ and $Q \in \partial\Omega$,

$$\mu(B(Q,r)\cap\Omega) \le C\sigma(B(Q,r)\cap\partial\Omega).$$

The best possible C is the Carleson norm. When μ is Carleson we write $\mu \in \mathcal{C}$.

The Carleson condition - motivation

$$\delta(X)^{-1}\left(\mathrm{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2$$

is the density of a Carleson measure on Ω.

Definition

A measure μ in Ω is Carleson if there exists a constant $C = C(r_0)$ such that for all $r \leq r_0$ and $Q \in \partial\Omega$,

$$\mu(B(Q,r) \cap \Omega) \leq C\sigma(B(Q,r) \cap \partial\Omega).$$

The best possible C is the Carleson norm. When μ is Carleson we write $\mu \in \mathcal{C}$.

If $\lim_{r_0 \to 0} C(r_0) = 0$, then we say that the measure μ satisfies the vanishing Carleson condition, and we write $\mu \in \mathcal{C}_V$.

The Carleson condition - motivation

$$\delta(X)^{-1}\left(\mathrm{osc}_{B(X,\delta(X)/2)}a_{ij}\right)^2$$

is the density of a Carleson measure on Ω.

Definition

A measure μ in Ω is Carleson if there exists a constant $C = C(r_0)$ such that for all $r \leq r_0$ and $Q \in \partial\Omega$,

$$\mu(B(Q,r)\cap\Omega) \leq C\sigma(B(Q,r)\cap\partial\Omega).$$

The best possible C is the Carleson norm. When μ is Carleson we write $\mu \in \mathcal{C}$.

If $\lim\limits_{r_0\to 0} C(r_0) = 0$, then we say that the measure μ satisfies the vanishing Carleson condition, and we write $\mu \in \mathcal{C}_V$.

Results for Dirichlet problem $(D)_p$

Kenig-Pipher, 2001 If

$$\delta(X)^{-1}\left(\mathrm{osc}_{B(X,\delta(X)/2)}a_{ij}\right)^2$$

is a density of Carleson measure on a Lipschitz domain Ω then $(D)_p$ is solvable for some (large) $p<\infty$.

M.D-Pipher-Petermichl, 2007 For any $p\in(1,\infty)$ there exists $C=C(p)>0$ such that if the Carleson norm bounded is less than $C(p)$ and the Lipschitz constant L of the domain Ω is smaller than $C(p)$ then $(D)_p$ is solvable.

Results for Dirichlet problem $(D)_p$

Kenig-Pipher, 2001 If

$$\delta(X)^{-1}\left(\mathrm{osc}_{B(X,\delta(X)/2)}a_{ij}\right)^2$$

is a density of Carleson measure on a Lipschitz domain Ω then $(D)_p$ is solvable for some (large) $p<\infty$.

M.D-Pipher-Petermichl, 2007 For any $p\in(1,\infty)$ there exists $C=C(p)>0$ such that if the Carleson norm bounded is less than $C(p)$ and the Lipschitz constant L of the domain Ω is smaller than $C(p)$ then $(D)_p$ is solvable.

Results for Neumann and Regularity problems in 2D

M.D-Rule, 2010 Let $\Omega \subset \mathbb{R}^2$.

Results for Neumann and Regularity problems in 2D

M.D-Rule, 2010 Let $\Omega \subset \mathbb{R}^2$.
For any $p \in (1, \infty)$ there exists $C = C(p) > 0$ such that if the coefficients of A satisfy a variant of our Carleson condition with norm less than $C(p)$ and the Lipschitz constant L of the domain Ω is smaller than $C(p)$ then $(R)_p$ and $(N)_p$ are solvable.

Results for Neumann and Regularity problems in 2D

M.D-Rule, 2010 Let $\Omega \subset \mathbb{R}^2$.
For any $p \in (1, \infty)$ there exists $C = C(p) > 0$ such that if the coefficients of A satisfy a variant of our Carleson condition with norm less than $C(p)$ and the Lipschitz constant L of the domain Ω is smaller than $C(p)$ then $(R)_p$ and $(N)_p$ are solvable.

Main theorem-Regularity

Let $1 < p < \infty$ and let Ω be a Lipschitz domain with Lipschitz norm L. Let

$$\delta(X)^{-1}\left(\operatorname{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2$$

be the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$. Then there exists $\varepsilon = \varepsilon(\Lambda, n, p) > 0$ such that if $\max\{L, C(r_0)\} < \varepsilon$ then the $(R)_p$ regularity problem

$$\begin{cases} Lu & = 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} & = f \quad \text{on } \partial\Omega \\ \widetilde{N}(\nabla u) & \in L^p(\partial\Omega) \end{cases}$$

is solvable for all f with $\|\nabla_T f\|_{L^p(\partial\Omega)} < \infty$.

Main theorem-Regularity

Let $1 < p < \infty$ and let Ω be a Lipschitz domain with Lipschitz norm L. Let

$$\delta(X)^{-1}\left(\operatorname{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2$$

be the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$. Then there exists $\varepsilon = \varepsilon(\Lambda, n, p) > 0$ such that if $\max\{L, C(r_0)\} < \varepsilon$ then the $(R)_p$ regularity problem

$$\begin{cases} Lu & = 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} & = f \quad \text{on } \partial\Omega \\ \widetilde{N}(\nabla u) & \in L^p(\partial\Omega) \end{cases}$$

is solvable for all f with $\|\nabla_T f\|_{L^p(\partial\Omega)} < \infty$. Moreover, there exists a constant $C = C(\Lambda, n, a, p) > 0$ such that

$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|\nabla_T f\|_{L^p(\partial\Omega)}.$$

Main theorem-Regularity

Let $1 < p < \infty$ and let Ω be a Lipschitz domain with Lipschitz norm L. Let

$$\delta(X)^{-1}\left(\operatorname{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2$$

be the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$. Then there exists $\varepsilon = \varepsilon(\Lambda, n, p) > 0$ such that if $\max\{L, C(r_0)\} < \varepsilon$ then the $(R)_p$ regularity problem

$$\begin{cases} Lu & = 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} & = f \quad \text{on } \partial\Omega \\ \widetilde{N}(\nabla u) & \in L^p(\partial\Omega) \end{cases}$$

is solvable for all f with $\|\nabla_T f\|_{L^p(\partial\Omega)} < \infty$. Moreover, there exists a constant $C = C(\Lambda, n, a, p) > 0$ such that

$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|\nabla_T f\|_{L^p(\partial\Omega)}.$$

Main theorem-Neumann

Let $1 < p < \infty$ and let Ω be a Lipschitz domain with Lipschitz norm L. Let

$$\delta(X)^{-1}\left(\mathrm{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2$$

be the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$. Then there exists $\varepsilon = \varepsilon(\Lambda, n, p) > 0$ such that if $\max\{L, C(r_0)\} < \varepsilon$ then the $(N)_p$ Neumann problem

$$\begin{cases} Lu & = 0 \quad \text{in } \Omega \\ A\nabla u \cdot \nu & = f \quad \text{on } \partial\Omega \\ \widetilde{N}(\nabla u) & \in L^p(\partial\Omega) \end{cases}$$

is solvable for all $f \in L^p(\partial\Omega)$.

Main theorem-Neumann

Let $1 < p < \infty$ and let Ω be a Lipschitz domain with Lipschitz norm L. Let

$$\delta(X)^{-1}\left(\mathrm{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2$$

be the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$. Then there exists $\varepsilon = \varepsilon(\Lambda, n, p) > 0$ such that if $\max\{L, C(r_0)\} < \varepsilon$ then the $(N)_p$ Neumann problem

$$\begin{cases} Lu & = 0 \quad \text{in } \Omega \\ A\nabla u \cdot \nu & = f \quad \text{on } \partial\Omega \\ \widetilde{N}(\nabla u) & \in L^p(\partial\Omega) \end{cases}$$

is solvable for all $f \in L^p(\partial\Omega)$. Moreover, there exists a constant $C = C(\Lambda, n, a, p) > 0$ such that

$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}.$$

Main theorem-Neumann

Let $1 < p < \infty$ and let Ω be a Lipschitz domain with Lipschitz norm L. Let

$$\delta(X)^{-1}\left(\operatorname{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2$$

be the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$. Then there exists $\varepsilon = \varepsilon(\Lambda, n, p) > 0$ such that if $\max\{L, C(r_0)\} < \varepsilon$ then the $(N)_p$ Neumann problem

$$\begin{cases} Lu & = 0 \quad \text{in } \Omega \\ A\nabla u \cdot \nu & = f \quad \text{on } \partial\Omega \\ \widetilde{N}(\nabla u) & \in L^p(\partial\Omega) \end{cases}$$

is solvable for all $f \in L^p(\partial\Omega)$. Moreover, there exists a constant $C = C(\Lambda, n, a, p) > 0$ such that

$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}.$$

Regularity problem - solving $p = 2$ is enough

M.D-Kirsch, 2012 Assume that $(R)_1$ (Regularity problem in Hardy-Sobolev space) is solvable.

Regularity problem - solving $p = 2$ is enough

M.D-Kirsch, 2012 Assume that $(R)_1$ (Regularity problem in Hardy-Sobolev space) is solvable.

Then for $p \in (1, \infty)$ we have

$$(R)_p \Longleftrightarrow (D^*)_{p'}.$$

Regularity problem - solving $p = 2$ is enough

M.D-Kirsch, 2012 Assume that $(R)_1$ (Regularity problem in Hardy-Sobolev space) is solvable.

Then for $p \in (1, \infty)$ we have

$$(R)_p \Longleftrightarrow (D^*)_{p'}.$$

Here $\frac{1}{p} + \frac{1}{p'} = 1$ and (D^*) is the Dirichlet problem for the adjoint operator L^*.

Regularity problem - solving $p = 2$ is enough

M.D-Kirsch, 2012 Assume that $(R)_1$ (Regularity problem in Hardy-Sobolev space) is solvable.

Then for $p \in (1, \infty)$ we have

$$(R)_p \Longleftrightarrow (D^*)_{p'}.$$

Here $\frac{1}{p} + \frac{1}{p'} = 1$ and (D^*) is the Dirichlet problem for the adjoint operator L^*. Since $(R)_2$ implies $(R)_1$ we get solvability of $(R)_p$ for small Carleson norm from solvability of $(R)_2$.

Regularity problem - solving $p = 2$ is enough

M.D-Kirsch, 2012 Assume that $(R)_1$ (Regularity problem in Hardy-Sobolev space) is solvable.

Then for $p \in (1, \infty)$ we have

$$(R)_p \Longleftrightarrow (D^*)_{p'}.$$

Here $\frac{1}{p} + \frac{1}{p'} = 1$ and (D^*) is the Dirichlet problem for the adjoint operator L^*. Since $(R)_2$ implies $(R)_1$ we get solvability of $(R)_p$ for small Carleson norm from solvability of $(R)_2$.

Reduction to differentiable coefficients

The idea comes from [DPP]. For a matrix A satisfying our Carleson condition with ellipticity constant Λ one can find (by mollifying coefficients of A) a new matrix $\widetilde{A}$ with same ellipticity constant Λ such that $\widetilde{A}$ satisfies that

Reduction to differentiable coefficients

The idea comes from [DPP]. For a matrix A satisfying our Carleson condition with ellipticity constant Λ one can find (by mollifying coefficients of A) a new matrix $\widetilde{A}$ with same ellipticity constant Λ such that $\widetilde{A}$ satisfies that

$$\sup\{\delta(X)|(\nabla\widetilde{A})(Y)|^2;\ Y \in B(X,\delta(X)/2)\}$$

Reduction to differentiable coefficients

The idea comes from [DPP]. For a matrix A satisfying our Carleson condition with ellipticity constant Λ one can find (by mollifying coefficients of A) a new matrix $\widetilde{A}$ with same ellipticity constant Λ such that $\widetilde{A}$ satisfies that

$$\sup\{\delta(X)|(\nabla\widetilde{A})(Y)|^2;\ Y \in B(X,\delta(X)/2)\}$$

and

$$\sup\{\delta(X)^{-1}|(A-\widetilde{A})(Y)|^2;\ Y \in B(X,\delta(X)/2)\}$$

Reduction to differentiable coefficients

The idea comes from [DPP]. For a matrix A satisfying our Carleson condition with ellipticity constant Λ one can find (by mollifying coefficients of A) a new matrix $\widetilde{A}$ with same ellipticity constant Λ such that $\widetilde{A}$ satisfies that

$$\sup\{\delta(X)|(\nabla\widetilde{A})(Y)|^2;\ Y \in B(X,\delta(X)/2)\}$$

and

$$\sup\{\delta(X)^{-1}|(A-\widetilde{A})(Y)|^2;\ Y \in B(X,\delta(X)/2)\}$$

are (small) Carleson norms.

Reduction to differentiable coefficients

The idea comes from [DPP]. For a matrix A satisfying our Carleson condition with ellipticity constant Λ one can find (by mollifying coefficients of A) a new matrix $\widetilde{A}$ with same ellipticity constant Λ such that $\widetilde{A}$ satisfies that

$$\sup\{\delta(X)|(\nabla\widetilde{A})(Y)|^2;\ Y \in B(X,\delta(X)/2)\}$$

and

$$\sup\{\delta(X)^{-1}|(A-\widetilde{A})(Y)|^2;\ Y \in B(X,\delta(X)/2)\}$$

are (small) Carleson norms. The solvability of the regularity problem for perturbed operators has been studied in Kenig-Pipher.

Reduction to differentiable coefficients

The idea comes from [DPP]. For a matrix A satisfying our Carleson condition with ellipticity constant Λ one can find (by mollifying coefficients of A) a new matrix $\widetilde{A}$ with same ellipticity constant Λ such that $\widetilde{A}$ satisfies that

$$\sup\{\delta(X)|(\nabla\widetilde{A})(Y)|^2;\ Y \in B(X, \delta(X)/2)\}$$

and

$$\sup\{\delta(X)^{-1}|(A-\widetilde{A})(Y)|^2;\ Y \in B(X, \delta(X)/2)\}$$

are (small) Carleson norms. The solvability of the regularity problem for perturbed operators has been studied in Kenig-Pipher.

$p = 2$ and Square function

Main goal is to establish and two estimates:

$$\|S(\nabla u)\|^2_{L^2} \lesssim \text{boundary data } + \varepsilon \|\widetilde{N}(\nabla u)\|^2_{L^2}$$

$p = 2$ and Square function

Main goal is to establish and two estimates:

$$\|S(\nabla u)\|^2_{L^2} \lesssim \text{boundary data } + \varepsilon \|\widetilde{N}(\nabla u)\|^2_{L^2}$$

$$\|S(\nabla u)\|^2_{L^2} \approx \|\widetilde{N}(\nabla u)\|^2_{L^2}.$$

$p=2$ and Square function

Main goal is to establish and two estimates:

$$\|S(\nabla u)\|^2_{L^2} \lesssim \text{boundary data } + \varepsilon \|\widetilde{N}(\nabla u)\|^2_{L^2}$$

$$\|S(\nabla u)\|^2_{L^2} \approx \|\widetilde{N}(\nabla u)\|^2_{L^2}.$$

The Square function

For any $v : \Omega \to \mathbb{R}$ we consider

$$S(v)(Q) = \left(\int_{\Gamma(Q)} |\nabla v(X)|^2 \delta(x)^{2-n} \, dX \right)^{1/2},$$

for all $Q \in \partial\Omega$. Observe that

$$\|S(v)\|_{L^2}^2 \approx \int_{\Omega} |\nabla v(X)|^2 \delta(X) \, dX.$$

The Square function

For any $v : \Omega \to \mathbb{R}$ we consider

$$S(v)(Q) = \left(\int_{\Gamma(Q)} |\nabla v(X)|^2 \delta(x)^{2-n} \, dX \right)^{1/2},$$

for all $Q \in \partial\Omega$. Observe that

$$\|S(v)\|_{L^2}^2 \approx \int_{\Omega} |\nabla v(X)|^2 \delta(X) \, dX.$$

First estimate

Key ingredient: Deal separately with

$$\|S(\nabla_T u)\|_{L^2}^2$$

First estimate

Key ingredient: Deal separately with

$\|S(\nabla_T u)\|^2_{L^2}$ and $\|S(A\nabla u \cdot \nu)\|^2_{L^2}$.

First estimate

Key ingredient: Deal separately with

$\|S(\nabla_T u)\|^2_{L^2}$ and $\|S(A\nabla u \cdot \nu)\|^2_{L^2}$.

We establish a local estimate for $\|S(\nabla_T u)\|^2_{L^2}$. In local coordinates we might assume that $\nabla_T u = (\partial_1 u, \partial_2 u, \ldots, \partial_{n-1} u)$ and $\nabla_\nu u = \partial_n u$.

First estimate

Key ingredient: Deal separately with

$\|S(\nabla_T u)\|^2_{L^2}$ and $\|S(A\nabla u\cdot\nu)\|^2_{L^2}$.

We establish a local estimate for $\|S(\nabla_T u)\|^2_{L^2}$. In local coordinates we might assume that $\nabla_T u = (\partial_1 u, \partial_2 u, \dots, \partial_{n-1} u)$ and $\nabla_\nu u = \partial_n u$.

For $w_k = \partial_k u$, $i = k, 2, \dots, n-1$ we use the fact that

$$\|S(w_k)\|^2_{L^2} \approx \int_{\mathbb{R}^{n-1}} \frac{a_{ij}}{a_{nn}} \partial_i w_k \partial_j w_k \, t \, dx \, dt.$$

First estimate

Key ingredient: Deal separately with

$\|S(\nabla_T u)\|^2_{L^2}$ and $\|S(A\nabla u \cdot \nu)\|^2_{L^2}$.

We establish a local estimate for $\|S(\nabla_T u)\|^2_{L^2}$. In local coordinates we might assume that $\nabla_T u = (\partial_1 u, \partial_2 u, \dots, \partial_{n-1} u)$ and $\nabla_\nu u = \partial_n u$.

For $w_k = \partial_k u$, $i = k, 2, \dots, n-1$ we use the fact that

$$\|S(w_k)\|^2_{L^2} \approx \int_{\mathbb{R}^{n-1}} \frac{a_{ij}}{a_{nn}} \partial_i w_k \partial_j w_k \, t \, dx \, dt.$$

Next, we integrate by parts. Several type of terms arise, in particular:

First estimate

Key ingredient: Deal separately with

$\|S(\nabla_T u)\|^2_{L^2}$ and $\|S(A\nabla u \cdot \nu)\|^2_{L^2}$.

We establish a local estimate for $\|S(\nabla_T u)\|^2_{L^2}$. In local coordinates we might assume that $\nabla_T u = (\partial_1 u, \partial_2 u, \dots, \partial_{n-1} u)$ and $\nabla_\nu u = \partial_n u$.

For $w_k = \partial_k u$, $i = k, 2, \dots, n-1$ we use the fact that

$$\|S(w_k)\|^2_{L^2} \approx \int_{\mathbb{R}^{n-1}} \frac{a_{ij}}{a_{nn}} \partial_i w_k \partial_j w_k \, t \, dx \, dt.$$

Next, we integrate by parts. Several type of terms arise, in particular:

Terms of the estimate:

$$\iint_{\mathbb{R}^n} \frac{1}{a_{nn}} w_k (L w_k) t \, dxdt$$

Here we use the fact that for $k < n$ the PDE w_k satisfies has relatively “nice” right hand side.

Terms of the estimate:

$$\iint_{\mathbb{R}^n} \frac{1}{a_{nn}} w_k (L w_k) t \, dxdt$$

Here we use the fact that for $k < n$ the PDE w_k satisfies has relatively "nice" right hand side. However, the estimate fails for $k = n$.

Terms of the estimate:

$$\iint_{\mathbb{R}^n} \frac{1}{a_{nn}} w_k (L w_k) t \, dxdt$$

Here we use the fact that for $k < n$ the PDE w_k satisfies has relatively “nice” right hand side.However, the estimate fails for $k = n$.

$$\iint_{\mathbb{R}^n} w_k (\partial_j w_k) \frac{a_{nj}}{a_{nn}} \, dxdt$$

Terms of the estimate:

$$\iint_{\mathbb{R}^n} \frac{1}{a_{nn}} w_k (L w_k) t \, dxdt$$

Here we use the fact that for $k < n$ the PDE w_k satisfies has relatively "nice" right hand side.However, the estimate fails for $k = n$.

$$\iint_{\mathbb{R}^n} w_k (\partial_j w_k) \frac{a_{nj}}{a_{nn}} \, dxdt$$

For $j = n$ this gives us a boundary term $-\frac{1}{2} \int_{\mathbb{R}^{n-1}} w_k^2 \, dx$.

Terms of the estimate:

$$\iint_{\mathbb{R}^n} \frac{1}{a_{nn}} w_k (L w_k) t \, dxdt$$

Here we use the fact that for $k < n$ the PDE w_k satisfies has relatively "nice" right hand side.However, the estimate fails for $k = n$.

$$\iint_{\mathbb{R}^n} w_k (\partial_j w_k) \frac{a_{nj}}{a_{nn}} \, dxdt$$

For $j = n$ this gives us a boundary term $-\frac{1}{2} \int_{\mathbb{R}^{n-1}} w_k^2 \, dx$.

Terms of the estimate:

$$\iint_{\mathbb{R}^n} |w_k||\nabla w_k||\nabla A| t \, dxdt$$

This is the error term where the Carleson condition on the coefficients is used. This term has an estimate

Terms of the estimate:

$$\iint_{\mathbb{R}^n} |w_k||\nabla w_k||\nabla A| t \, dxdt$$

This is the error term where the Carleson condition on the coefficients is used. This term has an estimate

$$\|(\nabla A)^2 t\|_{Carl} \|S(w_k)\|_{L^2} \|N(\nabla u)\|_{L^2}.$$

Terms of the estimate:

$$\iint_{\mathbb{R}^n} |w_k||\nabla w_k||\nabla A| t \, dxdt$$

This is the error term where the Carleson condition on the coefficients is used. This term has an estimate

$$\|(\nabla A)^2 t\|_{Carl} \|S(w_k)\|_{L^2} \|N(\nabla u)\|_{L^2}.$$

For the remaining estimate of the square function of the co-normal derivative we use the original equation $Lu = 0$ together with the established estimate for $\|S(\nabla_T u)\|^2_{L^2}$.

Terms of the estimate:

$$\iint_{\mathbb{R}^n} |w_k||\nabla w_k||\nabla A| t \, dxdt$$

This is the error term where the Carleson condition on the coefficients is used. This term has an estimate

$$\|(\nabla A)^2 t\|_{Carl} \|S(w_k)\|_{L^2} \|N(\nabla u)\|_{L^2}.$$

For the remaining estimate of the square function of the co-normal derivative we use the original equation $Lu = 0$ together with the established estimate for $\|S(\nabla_T u)\|_{L^2}^2$.

Second estimate: $\|S(\nabla u)\|^2_{L^2} \approx \|N(\nabla u)\|^2_{L^2}$

Difficult direction: $\|N(\nabla u)\|^2_{L^2} \lesssim \|S(\nabla u)\|^2_{L^2}$.

Second estimate: $\|S(\nabla u)\|^2_{L^2} \approx \|N(\nabla u)\|^2_{L^2}$

Difficult direction: $\|N(\nabla u)\|^2_{L^2} \lesssim \|S(\nabla u)\|^2_{L^2}$.

The proof relies on a stopping time argument of Kenig-Pipher-Toro.

Second estimate: $\|S(\nabla u)\|_{L^2}^2 \approx \|N(\nabla u)\|_{L^2}^2$

Difficult direction: $\|N(\nabla u)\|_{L^2}^2 \lesssim \|S(\nabla u)\|_{L^2}^2$.

The proof relies on a stopping time argument of Kenig-Pipher-Toro.

Neumann problem - solving for $p = 2$ and induction for integer p

We do not know an analogue of the result **M.D-Kirsch, 2012**.

Strategy:

Neumann problem - solving for $p = 2$ and induction for integer p

We do not know an analogue of the result **M.D-Kirsch, 2012**.

Strategy:

Solve the Neumann problem for $p = 2$ (using solvability of the $(R)_2$).

Neumann problem - solving for $p = 2$ and induction for integer p

We do not know an analogue of the result **M.D-Kirsch, 2012**.

Strategy:

Solve the Neumann problem for $p = 2$ (using solvability of the $(R)_2$).

Next, establish solvability of $(N)_p$ using $(R)_p$ and induction (for p integer $p \geq 3$).

Neumann problem - solving for $p = 2$ and induction for integer p

We do not know an analogue of the result **M.D-Kirsch, 2012**.

Strategy:

Solve the Neumann problem for $p = 2$ (using solvability of the $(R)_2$).

Next, establish solvability of $(N)_p$ using $(R)_p$ and induction (for p integer $p \geq 3$).

The induction step:

Let $p \geq 2$ be an integer, $0 \leq k \leq p-2$ and integer and u be a solution to $Lu = \mathrm{div} A\nabla u = 0$. Then there exists $\varepsilon > 0$ such that if the Carleson norm of the coefficients $C(r_0) < \varepsilon$ then for some $K = K(\Omega, \Lambda, n, \varepsilon, m, k) > 0$

$$\begin{aligned} & \iint_{\Omega_r} |\nabla_T u|^{p-k-2}|H|^k|\nabla H|^2\delta(X)\,dX \qquad (1) \\ \leq\ & (p-k-2)K\iint_{\Omega_{2r}} |\nabla_T u|^{p-k-3}|H|^{k+1}|\nabla H|^2\delta(X)\,dX \\ & + C(r)\iint_{\Omega\setminus\Omega_r} |\nabla u|^p\,dX \\ & + K\int_{\partial\Omega} |H|^p\,dx. \end{aligned}$$

Here $H = a_{ni}\partial_i u$.

The induction step:

Let $p \geq 2$ be an integer, $0 \leq k \leq p-2$ and integer and u be a solution to $Lu = \mathrm{div} A\nabla u = 0$. Then there exists $\varepsilon > 0$ such that if the Carleson norm of the coefficients $C(r_0) < \varepsilon$ then for some $K = K(\Omega, \Lambda, n, \varepsilon, m, k) > 0$

$$\begin{aligned} & \iint_{\Omega_r} |\nabla_T u|^{p-k-2} |H|^k |\nabla H|^2 \delta(X)\, dX \\ \leq \;& (p-k-2)K \iint_{\Omega_{2r}} |\nabla_T u|^{p-k-3} |H|^{k+1} |\nabla H|^2 \delta(X)\, dX \\ & + C(r) \iint_{\Omega \setminus \Omega_r} |\nabla u|^p \, dX \\ & + K \int_{\partial\Omega} |H|^p \, dx. \end{aligned} \tag{1}$$

Here $H = a_{ni}\partial_i u$. This is the co-normal derivative in $\mathbb{R}^n_+$.

The induction step:

Let $p \geq 2$ be an integer, $0 \leq k \leq p-2$ and integer and u be a solution to $Lu = \mathrm{div} A\nabla u = 0$. Then there exists $\varepsilon > 0$ such that if the Carleson norm of the coefficients $C(r_0) < \varepsilon$ then for some $K = K(\Omega, \Lambda, n, \varepsilon, m, k) > 0$

$$\begin{aligned} & \iint_{\Omega_r} |\nabla_T u|^{p-k-2}|H|^k|\nabla H|^2\delta(X)\,dX \qquad (1) \\ \leq\; & (p-k-2)K\iint_{\Omega_{2r}} |\nabla_T u|^{p-k-3}|H|^{k+1}|\nabla H|^2\delta(X)\,dX \\ & + C(r)\iint_{\Omega\setminus\Omega_r} |\nabla u|^p\,dX \\ & + K\int_{\partial\Omega} |H|^p\,dx. \end{aligned}$$

Here $H = a_{ni}\partial_i u$. **This is the co-normal derivative in $\mathbb{R}^n_+$.**

Open problem

Is the following analogue of result of **Kenig-Pipher, 2001** true?

Open problem

Is the following analogue of result of **Kenig-Pipher, 2001** true?

If

$$\delta(X)^{-1}\left(\operatorname{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2 \tag{2}$$

is a density of Carleson measure on a Lipschitz domain Ω then $(R)_p$

Open problem

Is the following analogue of result of **Kenig-Pipher, 2001** true?

If

$$\delta(X)^{-1}\left(\operatorname{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2 \qquad (2)$$

is a density of Carleson measure on a Lipschitz domain Ω then $(R)_p$ (and $(N)_p$)

Open problem

Is the following analogue of result of **Kenig-Pipher, 2001** true?

If

$$\delta(X)^{-1}\left(\operatorname{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2 \tag{2}$$

is a density of Carleson measure on a Lipschitz domain Ω then $(R)_p$ (and $(N)_p$) boundary value problems are solvable for some $p > 1$.

Open problem

Is the following analogue of result of **Kenig-Pipher, 2001** true?

If

$$\delta(X)^{-1}\left(\operatorname{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2 \tag{2}$$

is a density of Carleson measure on a Lipschitz domain Ω then $(R)_p$ (and $(N)_p$) boundary value problems are solvable for some $p > 1$.

Hint of solution for the Regularity problem:

Open problem

Is the following analogue of result of **Kenig-Pipher, 2001** true?

If

$$\delta(X)^{-1}\left(\operatorname{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2 \qquad (2)$$

is a density of Carleson measure on a Lipschitz domain Ω then $(R)_p$ (and $(N)_p$) boundary value problems are solvable for some $p > 1$.

Hint of solution for the Regularity problem: Prove that $(R)_1$ holds under the assumption (1).

Open problem

Is the following analogue of result of **Kenig-Pipher, 2001** true?

If

$$\delta(X)^{-1}\left(\operatorname{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2 \qquad (2)$$

is a density of Carleson measure on a Lipschitz domain Ω then $(R)_p$ (and $(N)_p$) boundary value problems are solvable for some $p > 1$.

Hint of solution for the Regularity problem: Prove that $(R)_1$ holds under the assumption (1). Then by **[M.D-Kirsch, 2012]** $(R)_{1+\varepsilon}$ holds for some small $\varepsilon > 0$.

Open problem

Is the following analogue of result of **Kenig-Pipher, 2001** true?

If

$$\delta(X)^{-1}\left(\operatorname{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2 \tag{2}$$

is a density of Carleson measure on a Lipschitz domain Ω then $(R)_p$ (and $(N)_p$) boundary value problems are solvable for some $p > 1$.

Hint of solution for the Regularity problem: Prove that $(R)_1$ holds under the assumption (1). Then by **[M.D-Kirsch, 2012]** $(R)_{1+\varepsilon}$ holds for some small $\varepsilon > 0$.

Systems with real coefficients

(joint work with Marius Mitrea and Sukjung Hwang).

Systems with real coefficients

(joint work with Marius Mitrea and Sukjung Hwang).

Consider the following PDE system

$$\mathcal{L}u = \left[\partial_i \left(A_{ij}^{\alpha\beta}(x)\partial_j u_\beta\right) + B_i^{\alpha\beta}(x)\partial_i u_\beta\right]_\alpha$$

for $i, j \in \{0, \dots, n-1\}$ and $\alpha, \beta \in \{1, \dots, N\}$., where $u = (u_\alpha)$ is a vector valued function.

Systems with real coefficients

(joint work with Marius Mitrea and Sukjung Hwang).

Consider the following PDE system

$$\mathcal{L}u = \left[\partial_i \left(A_{ij}^{\alpha\beta}(x)\partial_j u_\beta\right) + B_i^{\alpha\beta}(x)\partial_i u_\beta\right]_\alpha$$

for $i, j \in \{0, \ldots, n-1\}$ and $\alpha, \beta \in \{1, \ldots, N\}$., where $u = (u_\alpha)$ is a vector valued function.

The goal is to solve the Dirichlet, Neumann and Regularity problem of the system of N equations $\mathcal{L}u = 0$.

Systems with real coefficients

(joint work with Marius Mitrea and Sukjung Hwang).

Consider the following PDE system

$$\mathcal{L}u = \left[\partial_i \left(A_{ij}^{\alpha\beta}(x)\partial_j u_\beta\right) + B_i^{\alpha\beta}(x)\partial_i u_\beta\right]_\alpha$$

for $i, j \in \{0, \dots, n-1\}$ and $\alpha, \beta \in \{1, \dots, N\}$., where $u = (u_\alpha)$ is a vector valued function.

The goal is to solve the Dirichlet, Neumann and Regularity problem of the system of N equations $\mathcal{L}u = 0$.

Systems with real coefficients

Assumptions: Tensors A, B real valued A satisfying strong ellipticity and bounded.

Outline of objectives:

- L^2 Dirichlet problem

Systems with real coefficients

Assumptions: Tensors A, B real valued A satisfying strong ellipticity and bounded.

Outline of objectives:

- L^2 Dirichlet problem **done for symmetric operators**

Systems with real coefficients

Assumptions: Tensors A, B real valued A satisfying strong ellipticity and bounded.

Outline of objectives:

- L^2 Dirichlet problem **done for symmetric operators**
- L^2 Regularity problem

Systems with real coefficients

Assumptions: Tensors A, B real valued A satisfying strong ellipticity and bounded.

Outline of objectives:

- L^2 Dirichlet problem **done for symmetric operators**
- L^2 Regularity problem **good progress, seems doable**

Systems with real coefficients

Assumptions: Tensors A, B real valued A satisfying strong ellipticity and bounded.

Outline of objectives:

- L^2 Dirichlet problem **done for symmetric operators**
- L^2 Regularity problem **good progress, seems doable**
- L^p Dirichlet problem for $2-\varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon$

Systems with real coefficients

Assumptions: Tensors A, B real valued A satisfying strong ellipticity and bounded.

Outline of objectives:

- L^2 Dirichlet problem **done for symmetric operators**
- L^2 Regularity problem **good progress, seems doable**
- L^p Dirichlet problem for $2-\varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon$ **in progress**

Systems with real coefficients

Assumptions: Tensors A, B real valued A satisfying strong ellipticity and bounded.

Outline of objectives:

- L^2 Dirichlet problem **done for symmetric operators**
- L^2 Regularity problem **good progress, seems doable**
- L^p Dirichlet problem for $2-\varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon$ **in progress**
- Neuman problem, Regularity for other p's

Systems with real coefficients

Assumptions: Tensors A, B real valued A satisfying strong ellipticity and bounded.

Outline of objectives:

- L^2 Dirichlet problem **done for symmetric operators**
- L^2 Regularity problem **good progress, seems doable**
- L^p Dirichlet problem for $2-\varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon$ **in progress**
- Neuman problem, Regularity for other p's ??????

Systems with real coefficients

Assumptions: Tensors A, B real valued A satisfying strong ellipticity and bounded.

Outline of objectives:

- L^2 Dirichlet problem **done for symmetric operators**
- L^2 Regularity problem **good progress, seems doable**
- L^p Dirichlet problem for $2-\varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon$ **in progress**
- Neuman problem, Regularity for other p's **??????**

Systems with real coefficients

Assumptions: Tensors A, B real valued A satisfying strong ellipticity and bounded.

Outline of objectives:

- L^2 Dirichlet problem **done for symmetric operators**
- L^2 Regularity problem **good progress, seems doable**
- L^p Dirichlet problem for $2 - \varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon$ **in progress**
- Neuman problem, Regularity for other p's **??????**

Scalar equation with complex coefficients

(joint work withJill Pipher) - work in progress.

Scalar equation with complex coefficients

(joint work withJill Pipher) - work in progress.

Such PDE can be seen as a **real** system with a **skew-symmetric** tensors $A_{ij}^{\alpha\beta}$.

Scalar equation with complex coefficients

(joint work withJill Pipher) - work in progress.

Such PDE can be seen as a **real** system with a **skew-symmetric** tensors $A_{ij}^{\alpha\beta}$.

The usual ellipticity assumption:

$$\lambda|\xi|^2 \leq Re \sum_{i,j=0}^{n-1} A_{ij}\xi_i\overline{\xi_j} \leq \Lambda|\xi|^2$$

for $\xi \in \mathbb{C}^n$.

Scalar equation with complex coefficients

(joint work withJill Pipher) - work in progress.

Such PDE can be seen as a **real** system with a **skew-symmetric** tensors $A_{ij}^{\alpha\beta}$.

The usual ellipticity assumption:

$$\lambda|\xi|^2 \leq Re \sum_{i,j=0}^{n-1} A_{ij}\xi_i\overline{\xi_j} \leq \Lambda|\xi|^2$$

for $\xi \in \mathbb{C}^n$. Can be rewritten as (for all $\chi, \eta \in \mathbb{R}^n$)

$$|\chi|^2 + |\eta|^2 \approx \langle Re\, A\,\chi, \chi\rangle + \langle Re\, A\,\eta, \eta\rangle + \langle (Im\, A - Im\, A^t)\chi, \eta\rangle.$$

Scalar equation with complex coefficients

(joint work withJill Pipher) - work in progress.

Such PDE can be seen as a **real** system with a **skew-symmetric** tensors $A_{ij}^{\alpha\beta}$.

The usual ellipticity assumption:

$$\lambda|\xi|^2 \leq Re \sum_{i,j=0}^{n-1} A_{ij}\xi_i\overline{\xi_j} \leq \Lambda|\xi|^2$$

for $\xi \in \mathbb{C}^n$. Can be rewritten as (for all $\chi, \eta \in \mathbb{R}^n$)

$$|\chi|^2 + |\eta|^2 \approx \langle Re\, A\,\chi, \chi\rangle + \langle Re\, A\,\eta, \eta\rangle + \langle (Im\, A - Im\, A^t)\chi, \eta\rangle.$$

Scalar equation with complex coefficients

Again, under this assumption the L^2 theory as described above **works**.

New idea:

Scalar equation with complex coefficients

Again, under this assumption the L^2 theory as described above **works**.

New idea: Direct method for L^p solvability

Scalar equation with complex coefficients

Again, under this assumption the L^2 theory as described above **works**.

New idea: Direct method for L^p solvability (in real case this is done in [DPP]).

Scalar equation with complex coefficients

Again, under this assumption the L^2 theory as described above **works**.

New idea: Direct method for L^p solvability (in real case this is done in [DPP]).

New ingredient: Concept of L^p dissipativity of Cialdea and Maz'ya.

Scalar equation with complex coefficients

Again, under this assumption the L^2 theory as described above **works**.

New idea: Direct method for L^p solvability (in real case this is done in [DPP]).

New ingredient: Concept of L^p dissipativity of Cialdea and Maz'ya.
C+M define this in the context of parabolic PDEs.

Scalar equation with complex coefficients

Again, under this assumption the L^2 theory as described above **works**.

New idea: Direct method for L^p solvability (in real case this is done in [DPP]).

New ingredient: Concept of L^p dissipativity of Cialdea and Maz'ya. C+M define this in the context of parabolic PDEs. If $\mathcal{L}$ is L^p dissipative then $u_t - \mathcal{L}u = 0$ satisfies

Scalar equation with complex coefficients

Again, under this assumption the L^2 theory as described above **works**.

New idea: Direct method for L^p solvability (in real case this is done in [DPP]).

New ingredient: Concept of L^p dissipativity of Cialdea and Maz'ya. C+M define this in the context of parabolic PDEs. If $\mathcal{L}$ is L^p dissipative then $u_t - \mathcal{L}u = 0$ satisfies

$$\|u(t)\|_{L^p} \leq \|u(0)\|_{L^p}, \quad t \geq 0.$$

Scalar equation with complex coefficients

Again, under this assumption the L^2 theory as described above **works**.

New idea: Direct method for L^p solvability (in real case this is done in [DPP]).

New ingredient: Concept of L^p dissipativity of Cialdea and Maz'ya. C+M define this in the context of parabolic PDEs. If $\mathcal{L}$ is L^p dissipative then $u_t - \mathcal{L}u = 0$ satisfies

$$\|u(t)\|_{L^p} \leq \|u(0)\|_{L^p}, \quad t \geq 0.$$

C+M come with **algebraic** condition that allows to check when $\mathcal{L}$ is L^p dissipative.

Scalar equation with complex coefficients

Again, under this assumption the L^2 theory as described above **works**.

New idea: Direct method for L^p solvability (in real case this is done in [DPP]).

New ingredient: Concept of L^p dissipativity of Cialdea and Maz'ya. C+M define this in the context of parabolic PDEs. If $\mathcal{L}$ is L^p dissipative then $u_t - \mathcal{L}u = 0$ satisfies

$$\|u(t)\|_{L^p} \leq \|u(0)\|_{L^p}, \quad t \geq 0.$$

C+M come with **algebraic** condition that allows to check when $\mathcal{L}$ is L^p dissipative.

Scalar equation with complex coefficients

We call $\mathcal{L}$ with complex coefficients **strongly** L^p dissipative if

$$\langle Re\,A\,\lambda,\lambda\rangle+\langle Re\,A\,\eta,\eta\rangle+\left\langle\left(\sqrt{\frac{p'}{p}}Im\,A-\sqrt{\frac{p}{p'}}Im\,A^t\right)\lambda,\eta\right\rangle\approx|\lambda|^2+|\eta|^2.$$

This is (when $p=2$ just the usual ellipticity condition for the complex matrices).

Scalar equation with complex coefficients

We call $\mathcal{L}$ with complex coefficients **strongly** L^p dissipative if

$$\langle Re\, A\,\lambda,\lambda\rangle+\langle Re\, A\,\eta,\eta\rangle+\left\langle\left(\sqrt{\frac{p'}{p}}\,Im\, A-\sqrt{\frac{p}{p'}}\,Im\, A^t\right)\lambda,\eta\right\rangle\approx|\lambda|^2+|\eta|^2.$$

This is (when $p=2$ just the usual ellipticity condition for the complex matrices).

We have the following:

Scalar equation with complex coefficients

We call $\mathcal{L}$ with complex coefficients **strongly** L^p dissipative if

$$\langle Re\, A\,\lambda,\lambda\rangle+\langle Re\, A\,\eta,\eta\rangle+\left\langle\left(\sqrt{\frac{p'}{p}}Im\, A-\sqrt{\frac{p}{p'}}Im\, A^t\right)\lambda,\eta\right\rangle\approx|\lambda|^2+|\eta|^2.$$

This is (when $p=2$ just the usual ellipticity condition for the complex matrices).

We have the following: Let $A\in L^\infty(\Omega)$ be a matrix that is uniformly elliptic. Then there exists $p_0\in[1,2)$ ($p_0=1$ if and only if $Im\, A=0$) such that for all $p\in(p_0,p_0')$

Scalar equation with complex coefficients

We call $\mathcal{L}$ with complex coefficients **strongly** L^p dissipative if

$$\langle Re\,A\,\lambda,\lambda\rangle+\langle Re\,A\,\eta,\eta\rangle+\left\langle\left(\sqrt{\tfrac{p'}{p}}\,Im\,A-\sqrt{\tfrac{p}{p'}}\,Im\,A^t\right)\lambda,\eta\right\rangle\approx|\lambda|^2+|\eta|^2.$$

This is (when $p=2$ just the usual ellipticity condition for the complex matrices).

We have the following: Let $A\in L^\infty(\Omega)$ be a matrix that is uniformly elliptic. Then there exists $p_0\in[1,2)$ ($p_0=1$ if and only if $Im\,A=0$) such that for all $p\in(p_0,p_0')$ the **above algebraic condition** holds.

Scalar equation with complex coefficients

We call $\mathcal{L}$ with complex coefficients **strongly** L^p dissipative if

$$\langle Re\, A\,\lambda,\lambda\rangle+\langle Re\, A\,\eta,\eta\rangle+\left\langle\left(\sqrt{\tfrac{p'}{p}}\,Im\, A-\sqrt{\tfrac{p}{p'}}\,Im\, A^t\right)\lambda,\eta\right\rangle\approx|\lambda|^2+|\eta|^2.$$

This is (when $p=2$ just the usual ellipticity condition for the complex matrices).

We have the following: Let $A\in L^\infty(\Omega)$ be a matrix that is uniformly elliptic. Then there exists $p_0\in[1,2)$ ($p_0=1$ if and only if $Im\, A=0$) such that for all $p\in(p_0,p_0')$ the **above algebraic condition** holds.

Scalar equation with complex coefficients

Our calculations show that for p for which this algebraic condition holds one can establish an estimates:

$$\|S_p(u)\|^2_{L^p} \lesssim \text{boundary data} + \varepsilon\|\widetilde{N}(u)\|^p_{L^p}$$

Scalar equation with complex coefficients

Our calculations show that for p for which this algebraic condition holds one can establish an estimates:

$$\|S_p(u)\|^2_{L^p} \lesssim \text{boundary data} + \varepsilon \|\widetilde{N}(u)\|^p_{L^p}$$

and

$$\|S_p(\nabla u)\|^p_{L^p} \approx \|\widetilde{N}(\nabla u)\|^p_{L^p}.$$

Scalar equation with complex coefficients

Our calculations show that for p for which this algebraic condition holds one can establish an estimates:

$$\|S_p(u)\|^2_{L^p} \lesssim \text{boundary data} + \varepsilon\|\widetilde{N}(u)\|^p_{L^p}$$

and

$$\|S_p(\nabla u)\|^p_{L^p} \approx \|\widetilde{N}(\nabla u)\|^p_{L^p}.$$

Hence L^p Dirichlet solvability follows !!!!

Scalar equation with complex coefficients

Our calculations show that for p for which this algebraic condition holds one can establish an estimates:

$$\|S_p(u)\|^2_{L^p} \lesssim \text{boundary data} + \varepsilon \|\widetilde{N}(u)\|^p_{L^p}$$

and

$$\|S_p(\nabla u)\|^p_{L^p} \approx \|\widetilde{N}(\nabla u)\|^p_{L^p}.$$

Hence L^p Dirichlet solvability follows !!!!