Bounded H^∞-calculus for generators of analytic contraction semigroups on L^p spaces

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Based on a collaboration with O. Dragičević (U. of Ljubljana)

LMS Midlands Regional Meeting and Workshop on
Interactions of Harmonic Analysis and Operator Theory
Birmingham, 13-16 September 2016

Analytic contraction semigroups

Suppose that $(T(t))_{t>0}$ is a C_0-semigroup on $L^p(\mu)$, $p \geqslant 1$, s.t.

$$\|T(t)f\|_p \leqslant \|f\|_p \quad \forall f \in L^p(\mu)$$

whenever $t > 0$ and $p \in [1, \infty]$.

Let $-\mathcal{A}_p$ denote its generator on $L^p(\mu)$, $1 \leqslant p < \infty$. Suppose that $\mathcal{A}_p$ is 1-1.

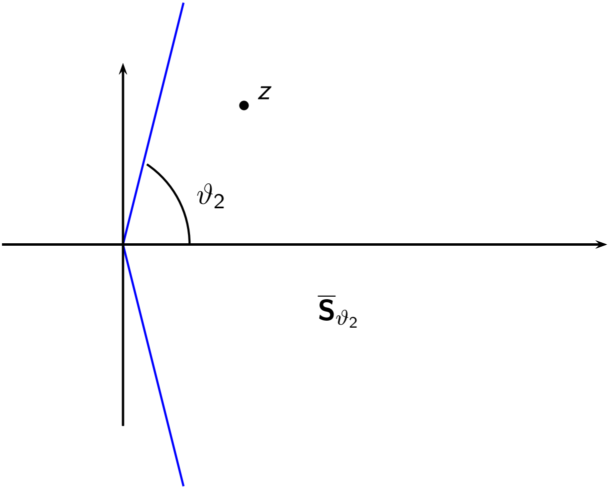
Suppose that the numerical range of $\mathcal{A}_2$ is contained in a sector of angle $< \pi/2$: let $\vartheta_2^* \in [0, \pi/2)$ be the smallest angle s.t.

$$|\mathrm{Im}\,\langle \mathcal{A}_2 f, f\rangle_{L^2}| \leqslant \tan\vartheta_2^* \cdot \mathrm{Re}\,\langle \mathcal{A}_2 f, f\rangle_{L^2}, \quad \forall f \in \mathrm{D}(\mathcal{A}_2)$$

For every $\vartheta \in [0, \pi/2)$, set

$$\vartheta^* := \pi/2 - \vartheta$$

- Lumer-Phillips theorem: $\|T(z)\|_2 \leqslant 1$, for all $z \in \overline{\mathbf{S}}_{\vartheta_2}$



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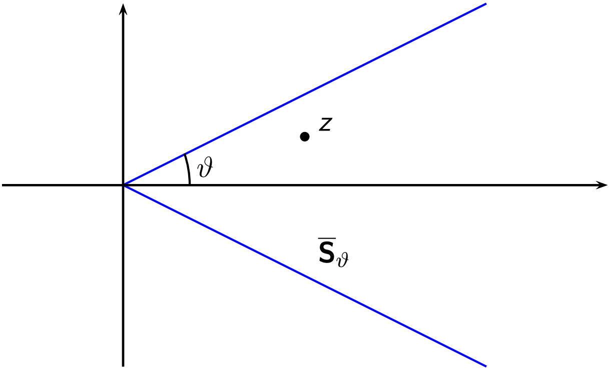
- Complex interpolation: for all $p \in (1, \infty)$ $\exists \vartheta > 0$ s.t. $\|T(z)\|_p \leqslant 1$, for all $z \in \overline{\mathbf{S}}_\vartheta$



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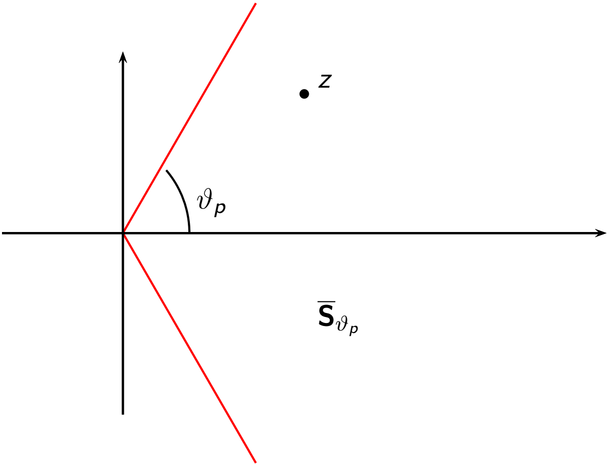

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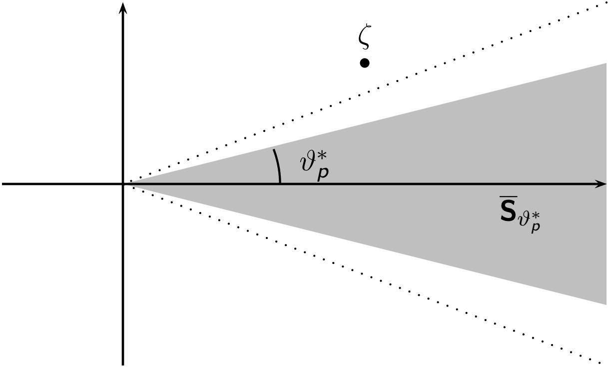
- $\mathcal{A}_p$ is sectorial with sectoriality angle $\omega(\mathcal{A}_p) \leqslant \vartheta_p^*$; i.e.

$$\sigma(\mathcal{A}_p) \subseteq \overline{\mathbf{S}}_{\vartheta_p^*} \quad \text{and} \quad \left\|(\zeta - \mathcal{A}_p)^{-1}\right\| \leqslant C(\vartheta)|\zeta|^{-1}, \quad \forall \zeta \in \mathbb{C}\setminus\overline{\mathbf{S}}_{\vartheta},\ \vartheta > \vartheta_p^*$$



Functional calculus (McIntosh 1986) Let $\vartheta > \omega(\mathcal{A}_p)$. For, say, $m \in H^\infty(\mathbf{S}_\vartheta)$ one can define the closed d.d. (possibly unbounded) operator $m(\mathcal{A}_p)$. For m with polynomial decay at 0 and ∞

$$m(\mathcal{A}_p)f = \frac{1}{2\pi i}\int_{\partial^+\mathbf{S}_\vartheta} m(\zeta)(\zeta - \mathcal{A}_p)^{-1} f \,\mathrm{d}\zeta$$

Definition (bounded H^∞-calculus)

We say that $\mathcal{A}_p$ has *bounded $H^\infty(\mathbf{S}_\vartheta)$-calculus* if

- $m \in H^\infty(\mathbf{S}_\vartheta) \Rightarrow m(\mathcal{A}_p) \in \mathcal{B}(L^p(\mu))$
- $\|m(\mathcal{A}_p)\| \leqslant C\|m\|_\infty, \quad \forall\ m \in H^\infty(\mathbf{S}_\vartheta)$

$$\omega_H(\mathcal{A}_p) := \inf\{\vartheta \in (\omega(\mathcal{A}_p), \pi) : \ \mathcal{A}_p \ \text{ has bounded } H^\infty(\mathbf{S}_\vartheta)\text{-calculus}\}$$

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- $\omega_H(\mathcal{A}_p) = \omega(\mathcal{A}_p)$? Unknown. (False on reflexive B. spaces)
- Sharp lower bound for ϑ_p?
- $\omega_H(\mathcal{A}_p) \leqslant \vartheta_p^*$?

Recall $\omega(\mathcal{A}_p) \leqslant \vartheta_p^*$, where $\vartheta_p = \pi/2 - \vartheta_p^*$ is the contractivity angle

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For generators of symmetric contraction semigroups we can answer the last two questions

Symmetric contraction semigroups

For $1 < p < \infty$ define

$$\phi_p^* = \arctan \frac{|p-2|}{2\sqrt{p-1}}, \quad \phi_p = \pi/2 - \phi_p^*$$

Theorem (Kriegler 2011)

For all symmetric contraction semigroups $\vartheta_p \geqslant \phi_p$; i.e.

$$\|T(z)\|_p \leqslant 1, \quad \forall z \in \overline{\mathbf{S}}_{\phi_p}$$

- Proved by Bakry (1989) for symmetric diffusion semigroups
- Proved by Liskevich and Perelmuter (1995) for sub-Markovian semigroups

The angle ϕ_p is sharp because it is sharp for the Euclidean heat semigroup on $\mathbb{R}^n$ (Bakry 1989 or Weissler 1979 + Epperson 1989)

Functional calculus for generators of symmetric contractions:

- Stein 1970:

$$\omega_H(\mathcal{A}_p) \leqslant \pi/2 \quad \forall \text{ Markovian gen.}$$

- Cowling 1983:

$$\omega_H(\mathcal{A}_p) \leqslant \pi|1/2 - 1/p| := \vartheta_p^C$$

- Kunstmann and Štrkalj 2003:

$$\omega_H(\mathcal{A}_p) \leqslant \vartheta_p^{K\check{S}} < \vartheta_p^C \quad \forall \text{ sub}-\text{Markovian gen.}$$

- Kriegler 2011: The condition that the generator is sub-Markovian can be removed

Note: $\phi_p^* < \vartheta_p^{K\check{S}} < \vartheta_p^{C}$, when $p \neq 2$

Theorem (C., Dragičević 2013)

For all generators of symmetric contraction semigroups

$$\omega_H(\mathcal{A}_p) \leqslant \phi_p^*$$

Moreover, for all $\vartheta > \phi_p^*$ and $m \in H^\infty(\mathbf{S}_\vartheta)$

$$\|m(\mathcal{A}_p)\| \lesssim (p^{9/4} \log p) \cdot (\vartheta - \phi_p^*)^{-2}$$

ϕ_p^* is sharp: by Epperson's theorem,

$$\omega_H(L_p) \geqslant \omega(L_p) = \phi_p^*,$$

where $L = \Delta + x \cdot \nabla$ on $\mathbb{R}^n$ endowed with the standard Gaussian measure.

García-Cuerva, Mauceri, Meda, Sjögren, Torrea 2001:
$\omega_H(L_p) = \phi_p^*$ with $\|m(L_p)\| \leqslant C(n, p)$

Nonsymmetric Ornstein-Uhlenbeck operators on $\mathbb{R}^n$

Let $Q, A \in \mathbb{R}^{n,n}$ s.t.

- Q is symmetric and positive definite
- $\sigma(A) \subset \mathbb{C}_+$

Set $S(t) = e^{-tA}$ and define

$$Q_t = \int_0^t S(u) Q S^*(u)\,\mathrm{d}u, \quad Q_\infty = \int_0^\infty S(u) Q S^*(u)\,\mathrm{d}u.$$

Lyapunov equation:

$$AQ_\infty + Q_\infty A^* = Q$$

Gaussian measure of covariance Q_t:

$$\mathrm{d}\gamma_t(x) = \frac{1}{(2\pi)^{n/2}(\det Q_t)^{1/2}} \exp\left(-\frac{\langle Q_t^{-1} x, x\rangle}{2}\right) \mathrm{d}x, \quad 1 < t \leqslant \infty$$

O-U semigroup via Kolmogorov's formula:

$$T(t)f(x) := \int_{\mathbb{R}^n} f(S(t)x + y)\,\mathrm{d}\gamma_t(y), \quad f \in C_b(\mathbb{R}^n)$$

- γ_∞ is the invariant measure
- $(T(t))_{t>0}$ extends to a positivity preserving semigroup of contractions on $L^p(\gamma_\infty)$, $1 \leqslant p \leqslant \infty$.

For $1 < p < \infty$, denote its generator on $L^p(\gamma_\infty)$ by $-\mathcal{L}_p$

The space $C_c^\infty(\mathbb{R}^n)$ is a core for $\mathcal{L}_p$ and

$$\mathcal{L}f(x) = -\frac{1}{2}\mathrm{div}\,(Q\nabla f)(x) + \langle \nabla f(x), Ax\rangle_{\mathbb{C}^n}, \quad f \in C_c^\infty(\mathbb{R}^n)$$

Denote by ∇_∞^* the formal adjoint of ∇ on $L^2(\gamma_\infty)$. Then,

$$\mathcal{L}f = \nabla_\infty^*(Q_\infty A^* \nabla f), \quad f \in C_c^\infty(\mathbb{R}^n)$$

$$B := Q_\infty A^*, \quad B_s := (B^* + B)/2, \quad B_a := (B - B^*)/2$$

By Lyapunov equation $B_s = Q/2 > 0$. Therefore,

$$|\mathrm{Im}\,\langle B\xi, \xi\rangle_{\mathbb{C}^n}| \leqslant \tan \vartheta_2^* \cdot \mathrm{Re}\,\langle B\xi, \xi\rangle_{\mathbb{C}^n},$$

where

$$\vartheta_2^* = \arctan \left\| B_s^{-1/2} B_a B_s^{-1/2} \right\|$$

The O-U operator $\mathcal{L}_2$ is associated with the sesquilinear form

$$\mathfrak{a}(f, g) = \int_{\mathbb{R}^n} \langle B\nabla f, \nabla g\rangle \,\mathrm{d}\gamma_\infty, \quad \mathrm{D}(\mathfrak{a}) = W^{1,2}(\gamma_\infty)$$

The form $\mathfrak{a}$ is d.d., closed and Kato-sectorial of angle ϑ_2^*.

Hence the O-U semigroup is analytic and contractive on $\overline{\mathbf{S}}_{\vartheta_2}$

For $p \in (1, \infty)$, define

$$\boxed{\text{Recall: } \mathcal{L}_2 = \nabla_\infty^* B \nabla}$$

$$\vartheta_p = \operatorname{arccot} \frac{\sqrt{(p-2)^2 + p^2 (\tan \vartheta_2^*)^2}}{2\sqrt{p-1}}$$

Theorem (Chill, Fašangová, Metafune, Pallara 2005)

- $\|T(z)\|_p \leqslant 1, \ \forall z \in \overline{\mathbf{S}}_{\vartheta_p}$
- $\|T(z)\|_p \leqslant C$ *for all* $z \in \mathbf{S}_\vartheta \Rightarrow \vartheta \leqslant \vartheta_p$

- $\vartheta_p = \vartheta_{p/(p-1)}$
- $\mathcal{L}_2$ is self-adjoint iff $B = B^*$ iff $\vartheta_2^* = 0$. In this case $\vartheta_p = \phi_p$
- By the second item in the theorem $\omega(\mathcal{L}_p) = \vartheta_p^*$

Maas and van Neerven 2007: Analogous result for analytic O-U semigroups on abstract Wiener spaces

Theorem (C., Dragičević 2016)

Sharp bounded H^∞-calculus for O-U operators:

$$\omega_H(\mathcal{L}_p) = \vartheta_p^* = \omega(\mathcal{L}_p),$$

for every $p \in (1, \infty)$.

- For symmetric O-U operators $\vartheta_p^* = \phi_p^*$
- Nondegeneracy condition $\mathrm{N}(Q) = \{0\}$ can be removed and replaced by analyticity on L^2 of the semigroup
- Analogous result for generators of analytic O-U semigroups on abstract Wiener spaces

  van Neerven, Maas, Goldys, Chojnowska-Michalik, Fuhrman, Da Prato, Lunardi...

$\boxed{\text{NOTATION: } q = p/(p-1)}$

Nazarov-Treil Bellman function: Fix $p > 2$ and $\delta > 0$

$$\mathcal{Q}(\zeta,\eta) := |\zeta|^p + |\eta|^q + \delta \begin{cases} |\zeta|^2|\eta|^{2-q} & ; \ |\zeta|^p \leqslant |\eta|^q \\ \dfrac{2}{p}|\zeta|^p + \left(\dfrac{2}{q} - 1\right)|\eta|^q & ; \ |\zeta|^p \geqslant |\eta|^q \end{cases}$$

- $\mathcal{Q}(\zeta,\eta) \lesssim (|\zeta|^p + |\eta|^q)$
- $\mathcal{Q} \in C^1(\mathbb{R}^4)$

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- $\mathcal{Q}(\zeta,\eta) \lesssim (|\zeta|^p + |\eta|^q)$
- $\mathcal{Q} \in C^1(\mathbb{R}^4)$ and $\mathcal{Q} \in C^2(\mathbb{R}^4 \setminus \Upsilon_0)$, where

$$\Upsilon_0 = \{(\zeta,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2\,; (\eta = 0) \vee (|\zeta|^p = |\eta|^q)\}$$

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- $|\partial_\zeta \mathcal{Q}(\zeta,\eta)| \lesssim \max\{|\zeta|^{p-1}, |\eta|\}, \quad |\partial_\eta \mathcal{Q}(\zeta,\eta)| \lesssim |\eta|^{q-1}$

$$\partial_\zeta = (\partial_{\zeta_1} - i\partial_{\zeta_2}) \quad \text{and} \quad \partial_\eta = (\partial_{\eta_1} - i\partial_{\eta_2})$$

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- Nontrivial convexity properties...this will come next

Our approach to bounded H^∞-calculus

The target is to prove $\omega_H(\mathcal{L}_p) = \omega(\mathcal{L}_p) = \vartheta_p^*$, where ϑ_p is the contractivity angle of the O-U semigroup

Cowling, Doust, McIntosh and Yagi 1996: it is enough to prove

$$\int_0^\infty \left| \int_{\mathbb{R}^n} \mathcal{L}\, T(te^{\pm i\vartheta}) f \, \overline{T^*(te^{\mp i\vartheta}) g} \, \mathrm{d}\gamma_\infty \right| \mathrm{d}t \leqslant C \|f\|_p \|g\|_q, \quad \forall \vartheta < \vartheta_p$$

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The "classical" approach to such bilinear inequality is based on square functions; e.g.

$$\mathcal{G}_\vartheta(f)(\cdot) = \left(\int_0^\infty \left| (t\mathcal{L})^{1/2} T(te^{\pm i\vartheta}) f(\cdot) \right|^2 \frac{\mathrm{d}t}{t} \right)^{1/2}$$

Problem: how to prove the boundedness of $\mathcal{G}_\vartheta$ on $L^p(\gamma_\infty)$?

We use a different technique based on heat-flow monotonicity and convexity of $\mathcal{Q}$

Consider the functional

$$\mathcal{E}(t) = \int_{\mathbb{R}^n} \mathcal{Q}\left(T(te^{\pm i\vartheta})f,\, T^*(te^{\mp i\vartheta})g\right) \,\mathrm{d}\gamma_\infty$$

The properties of $\mathcal{Q}$ ensure that $\mathcal{E}$ is "regular" and satisfies the initial condition

$$\mathcal{E}(0) \lesssim (\|f\|_p^p + \|g\|_q^q)$$

Suppose that

$$-\mathcal{E}'(t) \gtrsim \left|\int_{\mathbb{R}^n} \mathcal{L}\, T(te^{\pm i\vartheta})(f)\, \overline{T^*(te^{\mp i\vartheta})(g)}\, \mathrm{d}\gamma_\infty\right| \qquad (1)$$

Then, by integrating both sides of (1) from 0 to ∞, we obtain the desired bilinear embedding

(replace (f, g) with $(\lambda f, \lambda^{-1} g)$ and minimize in $\lambda > 0$)

Condition (1) reduces to:

$$\left|\int_{\mathbb{R}^n} \mathcal{L}f\overline{g}\,\mathrm{d}\gamma_\infty\right| \lesssim \mathrm{Re}\int_{\mathbb{R}^n}\Big(e^{\pm i\vartheta}(\partial_\zeta\mathcal{Q})(f,g)\mathcal{L}f + e^{\mp i\vartheta}(\partial_\eta\mathcal{Q})(f,g)\mathcal{L}^*g\Big)\mathrm{d}\gamma_\infty$$

for all $f \in \mathrm{D}(\mathcal{L}_p)$ and all $g \in \mathrm{D}(\mathcal{L}_q^*)$ and all $\vartheta < \vartheta_p$

Note that

$$\mathcal{Q}(\zeta,0) \approx |\zeta|^p, \quad \mathcal{Q}(0,\eta) \approx |\eta|^q$$

Therefore, the integral condition implies dissipativity:

$$\mathrm{Re}\Big(e^{\pm i\vartheta}\int_{\mathbb{R}^n}\bar{f}|f|^{p-2}\mathcal{L}f\ \mathrm{d}\gamma_\infty\Big) \geqslant 0, \quad \mathrm{Re}\Big(e^{\pm i\vartheta}\int_{\mathbb{R}^n}\bar{g}|g|^{q-2}\mathcal{L}^*g\ \mathrm{d}\gamma_\infty\Big) \geqslant 0$$

A general result (C., Dragičević) Let $\mathcal{A}$ be a closed, d.d. and 1-1 operator on $L^p(\mu)$. Suppose that there exists $\vartheta \in [0, \pi/2)$ s.t.

$$\left|\int_\Omega \mathcal{A} f \overline{g}\, \mathrm{d}\mu\right| \lesssim \mathrm{Re} \int_\Omega \Big(e^{\pm i\vartheta} (\partial_\zeta \mathcal{Q})(f,g) \mathcal{A} f + e^{\mp i\vartheta} (\partial_\eta \mathcal{Q})(f,g) \mathcal{A}^* g \Big) \mathrm{d}\mu$$

for all $f \in \mathrm{D}(\mathcal{A})$ and all $g \in \mathrm{D}(\mathcal{A}^*)$. Then,

(i) $-\mathcal{A}$ is the gen. of an analytic contr. semigroup on $L^p(\mu)$ of angle ϑ

(ii) $-\mathcal{A}^*$ is the gen. of an analytic contr. semigroup on $L^q(\mu)$ of angle ϑ

(iii) $\omega_H(\mathcal{A}) \leqslant \vartheta^*$

Problem: Find pointwise conditions on second-order partial derivatives of $\mathcal{Q}$ such that the integral inequality above holds

We are able to solve the problem when:

- $\mathcal{A}$ is the negative generator of a symmetric contraction (2013)
- $\mathcal{A}$ has a special form; e.g. $\mathcal{A} = \mathcal{L}$ is a O-U operator (2016)

Generalised convexity of the Bellman function $\mathcal{Q}$

Given $D \in \mathbb{C}^{n,n}$ define

$$\mathcal{M}(D) = \begin{bmatrix} \mathrm{Re}\, D & -\mathrm{Im}\, D \\ \mathrm{Im}\, D & \mathrm{Re}\, D \end{bmatrix} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$$

For $v \in \mathbb{R}^4$, $\omega = (\alpha, \beta) \in \mathbb{R}^{4n}$, and $D, E \in \mathbb{C}^{n,n}$ define

$$H_{\mathcal{Q}}^{(D,E)}[v;\omega] := \langle (\mathrm{Hess}\,\mathcal{Q}(v) \otimes I_{\mathbb{R}^n})\omega, [\mathcal{M}(D) \oplus \mathcal{M}(E)]\,\omega \rangle_{\mathbb{R}^4}.$$

$H_{\mathcal{Q}}^{(D,E)}[v;\cdot]$ is the quadratic form on $\mathbb{R}^{4n}$ associated with the matrix

$$\begin{bmatrix} \mathrm{Re}\, D & -\mathrm{Im}\, D & & \\ \mathrm{Im}\, D & \mathrm{Re}\, D & & \\ & & \mathrm{Re}\, E & -\mathrm{Im}\, E \\ & & \mathrm{Im}\, E & \mathrm{Re}\, E \end{bmatrix}^T (\mathrm{Hess}\,\mathcal{Q}(v) \otimes I_{\mathbb{R}^n})$$

Let $B \in \mathbb{R}^{n,n}$ be strictly accretive. Then its numerical range angle ϑ_2^* belongs to $[0, \pi/2)$. Recall the notation

$$\vartheta_p = \operatorname{arccot} \frac{\sqrt{(p-2)^2 + p^2(\tan\vartheta_2^*)^2}}{2\sqrt{p-1}}$$

Theorem (C., Dragičević 2016)

Let $p \geqslant 2$. For every $0 \leqslant \vartheta < \vartheta_p$ there exist $\delta, a_0 > 0$ such that, if $\mathcal{Q}$ is the Bellman function associated with δ, and

$$C \in \{e^{i\vartheta}B, e^{-i\vartheta}B, e^{i\vartheta}B^*, e^{-i\vartheta}B^*\},$$

then

$$H_{\mathcal{Q}}^{(C,C^*)}[v;\omega] \geqslant a_0 \cdot \left\|\sqrt{B_s}\tilde{\alpha}\right\|_{\mathbb{C}^n} \left\|\sqrt{B_s}\tilde{\beta}\right\|_{\mathbb{C}^n},$$

for all $v \in \mathbb{R}^4 \setminus \Upsilon_0$ and $\omega = (\alpha, \beta) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$.

The constant a_0 is dimension-free.

If $\alpha = (\alpha_1, \alpha_2)$, then $\tilde{\alpha} := \alpha_1 + i\alpha_2$

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then

$$H^{(C,C^*)}_{\mathcal{Q}*\psi_\varepsilon}[v;\omega] \geqslant a_0 \cdot \left\|\sqrt{B_s}\tilde{\alpha}\right\|_{\mathbb{C}^n} \left\|\sqrt{B_s}\tilde{\beta}\right\|_{\mathbb{C}^n},$$

for all $v \in \mathbb{R}^4$ and $\omega = (\alpha, \beta) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$.

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- For proving the universal sharp multiplier theorem for generators of symmetric contractions, it is enough to consider the case $B = I_{\mathbb{R}}$; i.e. $C = e^{\pm i\vartheta} I_{\mathbb{C}}$ (C., Dragičević 2013)

- The theorem also holds in infinite dimension: replace $B \in \mathbb{R}^{n,n}$ with a strictly accretive bounded operator acting on a real separable Hilbert space $\mathcal{H}$.
  This is relevant for studying bounded H^∞-calculus for O-U operators on abstract Wiener spaces.
  In this case $\mathcal{H}$ is the RKHS associated with the diffusion Q.

For general accretive $C \in \mathbb{C}^{n,n}$? This is the subject of a forthcoming paper in collaboration with O. Dragičević.

Applications to semigroup contractivity and functional calculus for divergence-form operators with complex symbols.

Sharp bounded H^∞-calculus for O-U generator $\mathcal{L}$

Recall that $\mathcal{L}f = \nabla_\infty^*(B\nabla f)$, $f \in C_c^\infty(\mathbb{R}^n)$, where $B = Q_\infty A^*$.

The target is to prove $\omega_H(\mathcal{L}_p) = \vartheta_p^*$.

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We reduced the problem to prove the integral inequality $(\forall \vartheta < \vartheta_p)$

$$\left| \int \mathcal{L}f\overline{g}\,\mathrm{d}\gamma_\infty \right| \lesssim \mathrm{Re} \int \left(e^{\pm i\vartheta}(\partial_\zeta \mathcal{Q})(f,g)\mathcal{L}f + e^{\mp i\vartheta}(\partial_\eta \mathcal{Q})(f,g)\mathcal{L}^*g \right) \mathrm{d}\gamma_\infty$$

for all $f \in \mathrm{D}(\mathcal{L}_p)$ and all $g \in \mathrm{D}(\mathcal{L}_q^*)$.

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for all $f, g \in C_c^\infty(\mathbb{R}^n)$.

Generalised Cauchy-Schwarz inequality:

$$|\langle Bz, w\rangle_{\mathbb{C}^n}| \leqslant (1 + \tan \vartheta_2^*) \left\|\sqrt{B_s}z\right\| \left\|\sqrt{B_s}w\right\|.$$

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for all $f, g \in C_c^\infty(\mathbb{R}^n)$.

The inequality above now follows from our convexity theorem.

Let $B \in \mathbb{R}^{n,n}$ be strictly accretive. Then its numerical range angle ϑ_2^* belongs to $[0, \pi/2)$. Recall the notation

$$\vartheta_p = \operatorname{arccot} \frac{\sqrt{(p-2)^2 + p^2(\tan \vartheta_2^*)^2}}{2\sqrt{p-1}}$$

Theorem (C., Dragičević 2016)

Let $p \geqslant 2$. For every $0 \leqslant \vartheta < \vartheta_p$ there exist $\delta, a_0 > 0$ such that, if $\mathcal{Q}$ is the Bellman function associated with δ, and

$$C \in \{e^{i\vartheta}B, e^{-i\vartheta}B, e^{i\vartheta}B^*, e^{-i\vartheta}B^*\},$$

then

$$H_{\mathcal{Q}}^{(C,C^*)}[v;\omega] \geqslant a_0 \cdot \left\| \sqrt{B_s}\tilde{\alpha} \right\|_{\mathbb{C}^n} \left\| \sqrt{B_s}\tilde{\beta} \right\|_{\mathbb{C}^n},$$

for all $v \in \mathbb{R}^4 \setminus \Upsilon_0$ and $\omega = (\alpha, \beta) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$.

The constant a_0 is dimension-free.

If $\alpha = (\alpha_1, \alpha_2)$, then $\tilde{\alpha} := \alpha_1 + i\alpha_2$

Convexity of power functions

We extrapolate the theorem from the "generalised convexity" of power functions:

$$F_r(\zeta) := |\zeta|^r, \quad \zeta \in \mathbb{R}^2, \quad 0 \leqslant r < \infty$$

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Note that

$$\mathcal{Q} = \begin{cases} (1 + 2\delta/p)F_p \otimes \mathbf{1} + [1 + \delta(1 - 2/p)]\mathbf{1} \otimes F_q, & \text{if} \quad |\zeta|^p \geqslant |\eta|^q \\ \\ F_p \otimes \mathbf{1} + \mathbf{1} \otimes F_q + \delta F_2 \otimes F_{2-q}, & \text{if} \quad |\zeta|^p \leqslant |\eta|^q \end{cases}$$

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For $C \in \mathbb{C}^{n,n}$, $v \in \mathbb{R}^4 \setminus \Upsilon_0$ and $\omega = (\alpha, \beta) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$

$$H_{\mathcal{Q}}^{(C,C^*)}[v;\omega] = \begin{cases} (1 + 2\delta/p) H_{F_p}^{C}[\zeta;\alpha] + [1 + \delta(1 - 2/p)] H_{F_q}^{C^*}[\eta;\beta] \\ \\ H_{F_p}^{C}[\zeta;\alpha] + H_{F_q}^{C^*}[\eta;\beta] + \delta H_{F_2 \otimes F_{2-q}}^{(C,C^*)}[v;\omega] \end{cases}$$

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Where for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{2n}$

$$H_{F_p}^{C}[\zeta;\alpha] := \left\langle \mathrm{Hess}(F_p)(\zeta) \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \begin{bmatrix} \mathrm{Re}\, C & -\mathrm{Im}\, C \\ \mathrm{Im}\, C & \mathrm{Re}\, C \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \right\rangle_{\mathbb{R}^{2n}}$$

The map $C \mapsto H^C_{F_p}[s;\alpha]$ is $\mathbb{R}$-linear.

In particular, for $B \in \mathbb{R}^{n,n}$,

$$H^{e^{\pm i\vartheta}B}_{F_p}[\zeta;\alpha] = \cos\vartheta \cdot H^B_{F_p}[\zeta;\alpha] \pm \sin\vartheta \cdot H^{iB}_{F_p}[\zeta;\alpha]$$

Suppose now that $B \in \mathbb{R}^{n,n}$ is strictly accretive. Recall that ϑ_2^* denotes its numerical range angle and

$$\vartheta_p = \operatorname{arccot} \frac{\sqrt{(p-2)^2 + p^2(\tan\vartheta_2^*)^2}}{2\sqrt{p-1}}$$

Lemma

For all $\alpha \in \mathbb{R}^{2n}$ we have

$$|H^{iB}_{F_p}[\zeta;\xi]| \leqslant \cot\vartheta_p \cdot H^B_{F_p}[\zeta;\alpha]$$

The lemma can be deduced from (sharp) analyticity of O-U semigroup on $L^p(\gamma_\infty)$.

Set

$$\Delta(p,\vartheta) = \frac{\sin(\vartheta_p - \vartheta)}{\sin\vartheta_p}$$

The lemma implies that

$$H_{F_p}^{e^{\pm i\vartheta}B}[\zeta;\alpha] \geqslant \Delta(p,\vartheta) H_{F_p}^{B}[\zeta;\alpha],$$

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Lemma

Let $\xi \in \mathbb{R}^{2n}$. Then for all $\zeta \in \mathbb{R}^2$ we have

$$H_{F_p}^{I}[\zeta;\alpha] \geqslant p|\zeta|^{p-2}\,\|\alpha\|^2$$

It follows that for every $\vartheta < \vartheta_p$ the quadratic form $H_{F_p}^{e^{\pm i\vartheta}B}[\zeta;\cdot]$ is strictly positive definite...

The symmetric case: nonlocal operators

Let $T(t) = \exp(-t\mathcal{A})$, $t > 0$ be a symmetric contraction semigroup on (Ω, μ).

Recall that the goal is to prove

$$\omega_H(\mathcal{A}_p) \leqslant \phi_p^* = \arcsin|1 - 2/p|, \quad p \in (1, \infty)$$

We use the generalised convexity of $\mathcal{Q}$ with $C = e^{\pm i\vartheta} I_{\mathbb{C}}$, $\vartheta < \phi_p$

Equivalent to study the quadratic form on $\mathbb{R}^4$ associated with

$$\mathcal{R}_\vartheta(\mathcal{Q}) := \frac{1}{2}\left(\begin{bmatrix} \mathcal{O}_\vartheta^T & 0 \\ 0 & \mathcal{O}_{-\vartheta}^T \end{bmatrix} \cdot \mathrm{Hess}(\mathcal{Q}) + \mathrm{Hess}(\mathcal{Q}) \cdot \begin{bmatrix} \mathcal{O}_\vartheta & 0 \\ 0 & \mathcal{O}_{-\vartheta} \end{bmatrix}\right),$$

where $\mathcal{O}_\vartheta :=$ Rotation of angle ϑ in $\mathbb{R}^2$

We reduced the problem to prove the integral inequality

$$\left|\int_\Omega \mathcal{A}(f)\overline{g}\,\mathrm{d}\mu\right| \lesssim \int_\Omega \mathrm{Re}\,(e^{\pm i\vartheta}\partial_\zeta \mathcal{Q}(f,g)\mathcal{A}f + e^{\mp i\vartheta}\partial_\eta \mathcal{Q}(f,g)\mathcal{A}g)\,\mathrm{d}\mu,$$

for all $f \in \mathrm{D}(\mathcal{A}_p)$, $g \in \mathrm{D}(\mathcal{A}_q)$, and all $\vartheta < \phi_p$.

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for all $f \in \mathrm{D}(\mathcal{A}_p)$, $g \in \mathrm{D}(\mathcal{A}_q)$, and all $\vartheta < \phi_p$.

(i) The Integral inequality is true for the two-point generator:

$$\mathcal{A} = \mathcal{G} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

on $\mathbb{C}^2 = L^\infty(\{a,b\}, \nu_{a,b})$, $\nu_{a,b} = (\delta_a + \delta_b)/2$

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$$2\int_{\{a,b\}} \mathcal{G}u \cdot v\,\mathrm{d}\nu_{a,b} = [u(a) - u(b)] \cdot [v(a) - v(b)]$$

The integral inequality for $\mathcal{G}$ follows by the mean value theorem and positiveness of $\mathcal{R}_\vartheta(\mathcal{Q})$.

We reduced the problem to prove the integral inequality

$$\left|\int_\Omega (I - T(t))(f)\overline{g}\,\mathrm{d}\mu\right| \lesssim \int_\Omega \mathrm{Re}\,(e^{\pm i\vartheta}\partial_\zeta \mathcal{Q}(f,g)(I - T(t))f + \cdots)\,\mathrm{d}\mu,$$

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(ii) **Suppose** that $(T(t))_{t>0}$ is Markovian and it has a kernel $k_t(x,y)$, $t>0$.

In this case we have the following representation formula:

$$\int_\Omega (I-T(t))(u)v \,\mathrm{d}\mu = \int_{\Omega\times\Omega} \left(\int_{\{x,y\}} \mathcal{G}u \cdot v \,\mathrm{d}\nu_{x,y} \right) k_t(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)$$

and the integral inequality follows from that for $\mathcal{G}$ (recall that $k_t \geqslant 0$)

(iii) In the general case we have to "adapt" the representation formula

Thank you for your attention!