Holomorphic functions which preserve holomorphic semigroups

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Heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta_x u \qquad (x \in \Omega \subseteq \mathbb{R}^d,\ t \geq 0), \\ u(t,x) &= 0 \qquad (x \in \partial\Omega), \\ u(0,x) &= f(x) \qquad (x \in \Omega). \end{aligned}$$

Te operator and boundary conditions can be varied in many ways.

Regard u as a function of t with values in a Banach space X of functions, for example $L^p(\Omega)$ or $C(\overline{\Omega})$. Rewrite the equation as an ODE for u:

$$u'(t) = \Delta u(t) \quad (t \geq 0), \qquad u(0) = f.$$

Abstract Cauchy problem

$$u'(t) = -Au(t) \quad (t \geq 0), \qquad u(0) = x,$$

where $u : [0, \infty) \to X$, A is a closed, densely defined operator on X, and $x \in X$. Solution should be

$$u(t) = e^{-tA}x.$$

What does e^{-tA} mean? Is it a bounded operator?

Second-order problem

$$u''(t) = -Au(t) \quad (t \geq 0), \qquad u(0) = x,\ u'(0) = y$$

should have solution

$$u(t) = \cos\left(t\sqrt{A}\right)x + \sin\left(t\sqrt{A}\right)\left(\sqrt{A}\right)^{-1}y.$$

Holomomorphic semigroups

Bounded holomorphic C_0-semigroup on X:

$$T : \Sigma_\theta := \{z \in \mathbb{C} : |\arg z| < \theta\} \to \mathcal{B}(X), \quad \text{holomorphic}$$

$$T(z_1 + z_2) = T(z_1)T(z_2), \qquad \lim_{z \to 0} \|T(z)x - x\| = 0,$$

$$\sup\{\|T(z)\| : z \in \Sigma_\theta\} < \infty,$$

Sectorial operator on X:

$$A : D(A) \subset X \to X, \quad \text{linear, } D(A) \text{ dense}, \qquad 0 < \theta < \pi,$$

$$\sigma(A) \subset \overline{\Sigma}_\theta, \qquad \|(\lambda + A)^{-1}\| \le \frac{C_\theta}{|\lambda|} \quad (\lambda \in \Sigma_{\pi-\theta})$$

Sectorial angle of A: the infimum ω_A of all such $\theta \in (0, \pi)$

Generators

The following are equivalent:

- $-A$ generates a bounded holomorphic semigroup T,
- A is sectorial with $\omega_A < \pi/2$.

$$T(z) = \exp(-zA) = \frac{1}{2\pi i} \int_\gamma e^{-\lambda z} (\lambda - A)^{-1} \, d\lambda \qquad (z \in \Sigma_{\pi/2-\omega_A})$$

The solutions of the abstract Cauchy problem extend holomorphically in t, so in particular they are in $C^\infty(0,\infty)$ for arbitrary initial data.

Examples: many second-order elliptic differential operators

More generally, a (bounded) C_0-*semigroup* is a (bounded) function $T : [0,\infty) \to \mathcal{B}(X)$ satisfying

$$T(t_1 + t_2) = T(t_1)T(t_2), \qquad \lim_{t\to 0} \|T(t)x - x\| = 0.$$

The *generator* is the closed, densely defined operator $-A$ on X where

$$(\lambda + A)^{-1}x = \int_0^\infty e^{-\lambda t} T(t)x \, dt \qquad (x \in X, \lambda \in \mathbb{C}_+ = \Sigma_{\pi/2}).$$

Then $T(\cdot)x$ is a classical solution of the abstract Cauchy problem if $x \in D(A)$.

If $-A$ generates a bounded C_0-semigroup then A is sectorial with $\omega_A \le \pi/2$. The converse is not true.

Subordination

Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup of sub-probability measures on $[0,\infty)$, so

$$\mu_{t_1} * \mu_{t_2} = \mu_{t_1+t_2}, \qquad \mu_t \to \delta_0 \text{ vaguely as } t \to 0.$$

They arise as transition probabilities of certain stochastic processes.

Then there is a (unique) function $f : (0,\infty) \to (0,\infty)$ such that

$$\int_0^\infty e^{-s\lambda}\, d\mu_t(s) = e^{-tf(\lambda)} \qquad (\lambda > 0, t \geq 0).$$

The functions f that arise in this way are known as *Bernstein functions* or by various other names including *Laplace exponents*.

Soon we will replace λ by an operator A in this formula.

Bernstein functions

A function $f : (0,\infty) \to (0,\infty)$ is a Bernstein function if and only if f is C^∞ and

$$(-1)^{n-1} f^{(n)}(t) \ge 0 \qquad (n \ge 1,\ t > 0).$$

Equivalently, there is a positive measure ν and $a, b \ge 0$ such that f has the Lévy–Khintchine representation

$$f(\lambda) = a + b\lambda + \int_0^\infty \left(1 - e^{-s\lambda}\right)\, d\nu(s), \qquad \int_0^\infty \frac{s}{1+s} d\nu(s) < \infty.$$

Then f extends, by the same formula, to a holomorphic function $f : \mathbb{C}_+ \to \mathbb{C}_+$.

Examples: $f(z) = z^\alpha$ where $0 \le \alpha \le 1$, $\quad f(z) = \log(1+z)$, $f(z) = 1 - e^{-z}$.

Now we will replace λ or z by an operator A.

Subordinate semigroups

Suppose that $-A$ generates a bounded C_0-semigroup T, and let f be a Bernstein function, with convolution semigroup (μ_t). We can define a "subordinate" bounded C_0-semigroup T_f by:

$$T_f(t)x = \int_0^\infty T(s)x \, d\mu_t(s) \qquad (x \in X).$$

We can define an operator $f_0(A) : D(A) \to X$ by using the Levy-Khintchine representation

$$f_0(A)x = ax + bAx + \int_0^\infty (x - T(s)x) \, d\nu(s) \qquad (x \in D(A)),$$

and then take $f(A)$ to be the closure of $f_0(A)$.

Then $-f(A)$ is the generator of T_f (Bochner (1949), Phillips, Nelson)

Question: If T is a bounded holomorphic semigroup and f is a Bernstein function, is T_f also holomorphic?

Functional calculus

We assume that A is sectorial, and (for convenience) that A has dense range. Then A is injective, and $A^{-1} : \mathrm{Ran}(A) \to X$ is sectorial of the same angle.

Let $\theta > \omega_A$. For many holomorphic $f : \Sigma_\theta \to \mathbb{C}$, one can define $f(A)$ as a closed operator. There are several different methods, but they are all consistent, and have reasonably good functional calculus properties.

- Fractional powers A^α (Balakrishnan)
- Complete Bernstein functions (Hirsch)
- Bernstein functions (Bochner, Phillips, Schilling et al)
- Holomorphic functions with at most polynomial growth as $|z| \to \infty$ and $|z| \to 0$ (McIntosh, Haase)

If $f(A) \in \mathcal{B}(X)$ for all $f \in H^\infty(\Sigma_\theta)$, then f has *bounded H^∞-calculus* on Σ_θ.

Theorem (Cowling-Doust-McIntosh-Yagi)

Suppose that A is sectorial with dense range. The following are equivalent:

(i) *A has bounded H^∞-calculus on some sector Σ_ψ;*

(ii) *There exists $\theta \in (\omega_A, \pi)$ such that, for each $x \in X$ and $x^* \in X^*$,*

$$\int_{\partial\Sigma_\theta} \left|\langle A(\lambda - A)^{-2}x, x^*\rangle\right| \, |d\lambda| < \infty.$$

Many classes of differential operators have bounded H^∞-calculus.

There exist sectorial operators (of angle 0, on Hilbert space) which do not have bounded H^∞-calculus.

Answer for Bernstein functions

Theorem

If A is sectorial with $\omega_A < \pi/2$ and f is a Bernstein function, then $f(A)$ is sectorial with $\omega_{f(A)} \leq \omega_A$.

Balakrishnan (1960): Fractional powers: $\omega_{A^\alpha} = \alpha\omega_A$

Hirsch (1972): complete Bernstein functions (sectoriality, no angle)

Berg et al (1993): angle for complete Bernstein functions, partial results for some other Bernstein functions

Gomilko and Tomilov (2015): all Bernstein functions, $\omega_{f(A)} \leq \omega_A$.

Questions

Given sectorial A and holomorphic f, when is $f(A)$ sectorial?

More specifically,

Q1. For which f is $f(A)$ sectorial (with $\omega_{f(A)} \le \omega_A$) for all sectorial A?

Q2. For which A is $f(A)$ sectorial for all suitable f?

Q1 might be considered for the class of all Banach spaces X, or just for Hilbert spaces or some other class.

The set of functions f as in Q1 is closed under sums, positive scalar multiples, reciprocals and composition.

$\mathcal{NP}_+$-functions

For Q1, f should be

- holomorphic on $\mathbb{C}_+ = \Sigma_{\pi/2}$
- map $\mathbb{C}_+$ to $\mathbb{C}_+$
- map $(0,\infty)$ to $(0,\infty)$

Such a function is a *positive real* function (Cauer, Brune; Brown) or an $\mathcal{NP}_+$-function. Any $\mathcal{NP}_+$-function maps Σ_θ into Σ_θ for each $\theta \in (0,\pi/2)$.

$\mathcal{NP}_+$ is closed under sums, positive scalar multiples, reciprocals, compositions. It consists of the functions of the form

$$f(z) = \int_{-1}^{1} \frac{2z}{(1+z^2)+t(1-z^2)}\, d\mu(t)$$

for some finite positive Borel measure μ on $[-1,1]$. So estimates for the integrand which are uniform in t provide estimates for $|f(z)|$ subject to $f(1)=1$.

Question 2

For which A is $f(A)$ sectorial for all $f \in \mathcal{NP}_+$?

Theorem

Let A be a sectorial operator on a Banach space X with dense range and $\omega_A < \pi/2$, and let $\theta \in (\omega_A, \pi/2)$. Consider the following statements.

(i) *A has bounded H^∞-calculus on Σ_θ.*

(ii) *For every $f \in \mathcal{NP}_+$, $f(A)$ is a sectorial operator of angle (at most) ω.*

(iii) *For every $f \in \mathcal{NP}_+$, $-f(A)$ is the generator of a bounded C_0-semigroup.*

(iv) *A has bounded H^∞-calculus on $\mathbb{C}_+$.*

Then

$$\text{(i)} \implies \text{(ii)} \implies \text{(iii)} \iff \text{(iv)}.$$

If X is a Hilbert space, all four properties are equivalent.

Limits at 0 and ∞

Let $f \in \mathcal{NP}_+$, and let

$$f(0+) = \lim_{t \to 0+} f(t), \qquad f(\infty) = \lim_{t \to \infty} f(t)$$

if these limits exist in $[0, \infty]$.

Proposition

Let $f \in \mathcal{NP}_+$ be a function such that $f(\infty)$ does not exist in $[0, \infty]$, and let X be a Banach space with a conditional basis. There exists a sectorial operator A on X, with angle 0, such that $-f(A)$ does not generate a C_0-semigroup.

So we restrict attention to $\mathcal{NP}_+$-functions for which $f(0+)$ and $f(\infty)$ exist.

A resolvent formula for scalar functions

Let $f \in \mathcal{NP}_+$ and assume that $f(\infty)$ exists. Let $q > 2$, $z \in \Sigma_{\pi/q}$, $\lambda \in \Sigma_{\pi-\pi/q}$. Then

$$
\begin{aligned}
&(\lambda + f(z))^{-1} \\
&= \frac{1}{\lambda + f(\infty)} + \frac{q}{\pi} \int_0^\infty \frac{\operatorname{Im} f(te^{i\pi/q})\, t^{q-1}}{(\lambda + f(te^{i\pi/q}))(\lambda + f(te^{-i\pi/q}))(t^q + z^q)}\, dt,
\end{aligned}
$$

where the integral may be improper.

We would like to replace λ by a sectorial operator A, but does the integral converge in any sense? Can it be estimated in a way which shows that $f(A)$ is sectorial?

Resolvent formula for operators

For a sectorial operator A, we want the formula

$$\begin{aligned}&(\lambda + f(A))^{-1} \\ &= \frac{1}{\lambda + f(\infty)} + \frac{q}{\pi}\int_0^\infty \frac{\operatorname{Im} f(te^{i\pi/q})\, t^{q-1}}{(\lambda + f(te^{i\pi/q}))(\lambda + f(te^{-i\pi/q}))}(t^q + A^q)^{-1}\,dt.\end{aligned}$$

Theorem

Assume that $f \in \mathcal{NP}_+$, and

$$\int_0^\infty \frac{|\operatorname{Im} f(te^{i\beta})|}{(r + f(t))^2}\,\frac{dt}{t} \le \frac{C_\beta}{r}, \qquad (r > 0,\ \beta \in (0, \pi/2)). \qquad (\mathcal{E})$$

1. *$f(0+)$ and $f(\infty)$ exist.*
2. *If A is sectorial of angle $\omega_A < \pi/2$, then the resolvent formula above holds and $f(A)$ is sectorial of angle at most ω_A.*

Another condition

The condition $(\mathcal{E})$ on f is preserved by sums, positive scalar multiples, reciprocals, and $f \mapsto f(1/z)$.

$f \in \mathcal{NP}_+$ satisfies $(\mathcal{D})$ if, for each $\beta \in (0, \pi/2)$ there exist $a, b, c, a', b', c' > 0$ such that

- f is monotonic on $(0, a/b)$ and $|\operatorname{Im} f(te^{i\beta})| \le ct|f'(bt)|$ for $t < a/b$, and
- f is monotonic on $(a'/b', \infty)$ and $|\operatorname{Im} f(te^{i\beta})| \le c't|f'(b't)|$ for $t > a'/b'$.

Theorem

1. *Any Bernstein function satisfies $(\mathcal{D})$, with $a = b = a' = b' = 1$.*
2. *Assume that f satisfies $(\mathcal{D})$. Then f satisfies $(\mathcal{E})$. Hence $f(A)$ is sectorial whenever A is sectorial with $\omega_A < \pi/2$.*

z and $1 - e^{-z}$ are both Bernstein functions, and so are their square roots. Their geometric mean $\sqrt{z(1 - e^{-z})}$ is not Bernstein, but it is $\mathcal{NP}_+$ and it satisfies $(\mathcal{D})$.

In fact, if $f_1, \dots, f_n$ are Bernstein, and the product $f_1 \cdots f_n$ is $\mathcal{NP}_+$ then the product satisfies $(\mathcal{D})$.

In particular the geometric mean of any number of Bernstein functions satisfies $(\mathcal{D})$.

If f is Bernstein and $\alpha \in (0, 1)$, then

$$g_\alpha(z) := [f(z^\alpha)]^{1/\alpha}$$

is $\mathcal{NP}_+$ and satisfies $(\mathcal{D})$. If $\alpha \in (0, 1/2]$ then g_α is Bernstein, but this is not known for $\alpha \in (1/2, 1)$.