Some mathematical legacy from Alan McIntosh

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Alan McIntosh’s fields of contributions:

Functional analysis
Partial differential equations
Functional calculus
Multilinear analysis
Singular integrals
Hardy spaces
Clifford algebra for multidimensional analysis
Geometry
Dirac operators

Interaction between fields and topics

Algebraic way of doing analysis

Challenges (Kato's conjecture)

Depth

Conciseness

Kindness

Friendship

Generosity

(Un)confidence

Passion for science

End of June 2016:

“'The only {new} maths I have done these few months is to get my son Keith and my math son Andreas working together on a first order approach to Maxwell scattering on Photovoltaic solar cells. It is common to roughen the back surface of the cell with small pyramids on roughly light scale and Keith wants to produce fast models of the reflection and refraction from incoming waves that can be accessed from his website quickly. I would love to be more involved.”

PhD Thesis in 1966 at UC Berkeley, 85 journal articles. Career at Macquarie University, Australian National University (Australia). 2002 Moyal Medal, 2015 Hannan Medal in the mathematical sciences awarded by the Australian academy of sciences.

The Kato problem

Early 1960: T. Kato studied fractional powers of maximal accretive operators on a Hilbert space. Maximal accretive means dense domain, that the numerical values (Au, u) are contained in the right half complex plane and that the resolvent is invertible. He proved that domains of A^α and $A^{*\alpha}$ coincide when $0 \leq \alpha < 1/2$. J.L. Lions proved shortly after this is wrong when $\alpha = 1/2$.

For operators A coming from regularly accretive forms (ie $\beta(u, v) = (Au, v)$: the form β is represented by a maximal accretive operator A), Kato remarked $\alpha = 1/2$ (the most interesting case for applications) was still unknown.

McIntosh (PhD topic on forms) found a counterexample (1972): $H = \ell^2(\mathbb{Z})$, $A = D(I + zB)D$ for some $z \in \mathbb{C}$, with $De_j = 2^j e_j$ and $Be_j = \sum b_n e_{n+j}$ where the Fourier series $\sum b_n e^{in\tau}$ agrees with the periodic sawtooth $\tau/\pi - 1$ on $(0, 2\pi)$.

The Kato problem II

McIntosh stressed that the counterexample tells us nothing about the case of operators coming from PDE's, the ones that motivated Kato. He formulated the "Kato square root conjecture" (For the story, Kato always claimed this conjecture is not due to him). Here is the simplest case: Consider

$$Lu = \sum -\partial_j(a_{jk}(x)\partial_k u), \quad x \in \mathbb{R}^n.$$

a_{jk} are complex, L^∞ coefficients with bound Λ defined on $\mathbb{R}^n$, with for some $\lambda > 0$

$$\lambda|\xi|^2 \leq \mathrm{Re} \sum a_{jk}\xi_k\overline{\xi_j}, \quad \xi \in \mathbb{C}^n.$$

The conjecture: prove with C depending only on n, λ, Λ,

$$\|\sqrt{L}u\|_{L^2(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^2(\mathbb{R}^n)},$$

McIntosh's multilinear approach

One can normalize the matrix (a_{jk}) to be $I - B$ with $\|B\| < 1$. Then $L = -\mathrm{div}(I - B)\nabla$,

$$\sqrt{L}u = -\frac{2}{\pi}\int_0^\infty (1 - t^2\mathrm{div}(I - B)\nabla)^{-1}\mathrm{div}(I - B)\nabla u\, dt$$

and doing a Neumann series expansion wrt to B

$$\sqrt{L}u = -\frac{2}{\pi}\int_0^\infty Q_t(\nabla u)\frac{dt}{t} - \frac{2}{\pi}\sum_{k\geq 0}\int_0^\infty Q_t(B(I-P_t))^k BP_t(\nabla u)\frac{dt}{t}$$

with $Q_t = -t\mathrm{div}(1 - t^2\mathrm{div}\nabla)^{-1}$, $P_t = (1 - t^2\mathrm{div}\nabla)^{-1}$.

Problem: Prove that each term is bounded in L^2 by $c_k\|B\|_\infty^{k+1}\|\nabla u\|_2$ and control c_k to sum when $\|B\|_\infty < 1$.

McIntosh's multilinear approach II

$$\sqrt{L}u = -\frac{2}{\pi}\int_0^\infty Q_t(\nabla u)\frac{dt}{t} - \frac{2}{\pi}\sum_{k\geq 0}\int_0^\infty Q_t(B(I-P_t))^k BP_t(\nabla u)\frac{dt}{t}$$

• constant term (wrt B) is a classical operator, equal to $\sqrt{-\Delta}u$.
• term of order 1 in 1d is already hard to control: related to the Calderon commutator $[g, \sqrt{-\Delta}]$ with $g' = B$. The $L^2(\mathbb{R})$ boundedness of it was obtained in 1965 (using complex methods).
• Subsequent terms related to iterated commutators whose boundedness was proved by Coifman-Meyer (1977). Main tools were singular integral operators and in particular Carleson measure/BMO estimates. McIntosh proposed this multilinear scheme to them leading to the solution of the Kato conjecture in 1d due to some algebraic miracles to obtain convergence (1981). For multidimensions, the methods have limitations and no one knows how to control c_k directly.

Cauchy integrals

Kenig and Meyer (1985) wrote an article entitled “Kato’s square of accretive operators and Cauchy operators on Lipschitz curves are the same”.

Actually, this was known to McIntosh as early as the proof of the Kato conjecture in 1981 and the simultaneous proof of the $L^2(\mathbb{R})$ boundedness of the Cauchy operator on arbitrary Lipschitz curves (Calderón’s conjecture).

A convenient parametrization allows to do the same kind of multilinear series as for square roots.

McIntosh 1989 review on the Kato problem:

‘We see in retrospect the difficulties faced by anyone who tried to solve the Kato problem for elliptic sesquilinear forms in the 60’s and 70’s. The term of order 1 in the simplest case and in 1d is at least as difficult to estimate as the Calderón commutator integral! Of course, the 80’s have seen a great deal of progress in the estimation of such integrals.”

But it was not yet enough development to obtain the multidimensional Kato conjecture.

outgrowth: the Tb theorem

• In 1983, David-Journé proved a criterion for L^2 boundedness of singular integrals T called the $T1$ theorem. The necessary and sufficient condition is that $T1, T^*1$ be in the space BMO along with some mild control of the operator on the diagonal.

• McIntosh and Meyer (1985) realized that the proof of Kato problem also yield the complex interpolation result $[\dot{H}^1(\mathbb{R}^n), b\dot{H}^{-1}(\mathbb{R}^n)]_{1/2} = L^2(\mathbb{R}^n)$ when b is a bounded and accretive function on $\mathbb{R}^n$. This allowed them to formulate a Tb theorem, the first of this nature: they proved that $T1 = 0$ and $T^*b = 0$ plus a (different) diagonal condition suffices to conclude for the boundedness of the singular integral T.

• This theorem has been extended in many ways: different settings, different operators with different proofs (not relying on the solution of the Kato problem), more general conditions. One of these extensions was precisely devised toward the solution of the Kato conjecture on $\mathbb{R}^n$ (A., Hofmann, Lacey, McIntosh, Tchamitchian, 2002)

outgrowth: layer potential on Lipschitz surfaces

There is a multidimensional analogue of the Cauchy operator on Lipschitz curves: this requires to use the setup of Clifford algebras and the concept of monogenic functions replacing that of holomorphic functions. A function on an open set of $\mathbb{R}^{n+1}$ taking values in the Clifford algebra with unit e_0 and generated by $e_1, \ldots, e_n$ is left monogenic if f annihilates the Dirac operator $Df \equiv \sum_{k=0}^{n} \partial_k f_S \, e_k e_S = 0$. The function $k(x) = c_n \frac{\bar{x}}{|x|^{n+1}}$, is called the Cauchy-Clifford kernel: it is left (and right) monogenic away from 0. "Convolution" with k on Lipschitz surfaces Γ is called the Cauchy-Clifford operator. $T(b)$ method allows to prove it is bounded on $L^p(\Gamma, dS)$, $1 < p < \infty$. This was first proved by Li, McIntosh, Semmes without this technology (extending some method of Coifman, Jones, Semmes with square function estimates of "à la" Kenig). The scalar part of the CC operator is the double layer potential on the domain bounded by Γ, useful to solve boundary value problems

One of McIntosh's definitive contribution for posterity is the theory of bounded holomorphic functional calculi for non self-adjoint operators. Indeed, trying to solve the Kato problem, one looses self-adjointness but one keeps the spectral fact that the spectrum is contained in a sector of the complex plane (sectorial) or a symmetric double sector (bisectorial) together with invertibility of the resolvent with appropriate estimates. Let T be such an operator, assumed bisectorial.

$H^\infty(S_\mu)$ functional calculus for T on Hilbert space H is the existence of a unique Banach algebra homomorphism

$$\Phi_\mu : \quad H^\infty(S_\mu) \quad \to \quad \mathcal{B}(\overline{\mathsf{R}(T)})$$

with $\Phi_\mu(1) = 1$, $\Phi_\mu((1+iz)^{-1}) = (1+iT)^{-1}$ restricted to $\overline{\mathsf{R}(T)}$ and that is continuous in the following sense: uniformly convergent sequences on compact sets of S_μ are mapped to strongly convergent sequences of operators.

Uniqueness implies consistency with change of angle μ. Notation $\Phi_\mu(\varphi) = \varphi(T)$. Boundedness means

$$\|\varphi(T)h\| \lesssim \|\varphi\|_\infty \|h\|, \quad \forall \varphi \in H^\infty(S_\mu), \forall h \in \overline{\mathsf{R}(T)}.$$

Theorem: Boundedness holds iff the square function estimate

$$\int_0^\infty \|\lambda T(1+\lambda^2 T^2)^{-1} h\|^2 \, \frac{d\lambda}{\lambda} \sim \|h\|^2, \quad \forall h \in \overline{\mathsf{R}(T)}$$

and the one for T^* hold. Choice of $z(1+z^2)^{-1}$ irrelevant.

Main application to Kato conjecture

$$D := \begin{bmatrix} 0 & -\mathrm{div} \\ \nabla & 0 \end{bmatrix}, \quad B := \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}, \quad A = (a_{jk}).$$

$T := BD$ on $L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$ is bisectorial.

$$(BD)^2 = \begin{bmatrix} -\mathrm{div} A\nabla & 0 \\ 0 & -A\nabla \mathrm{div} \end{bmatrix}, \quad \sqrt{(BD)^2} = \begin{bmatrix} \sqrt{L} & 0 \\ 0 & \sqrt{M} \end{bmatrix}.$$

Let $\mathrm{sgn}(z) = 1$ if $\mathrm{Re}\, z > 0$ and -1 if $\mathrm{Re}\, z < 0$. If BD has a bounded holomorphic functional calculus on L^2, $\mathrm{sgn}(BD)$ is a bounded involution on $\overline{\mathrm{R}(BD)}$. Since $\sqrt{(BD)^2} = \mathrm{sgn}(BD)BD$, we get for $u \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$ under appropriate domain assumptions that,

$$\|\sqrt{L}u\|_2 = \left\| \sqrt{(BD)^2} \begin{bmatrix} u \\ 0 \end{bmatrix} \right\|_2 \approx \left\| BD \begin{bmatrix} u \\ 0 \end{bmatrix} \right\|_2 = \|A(\nabla u)\|_2 \approx \|\nabla u\|_2.$$

Approach developed successfully by Axelsson, Keith, McIntosh (2007): reproves Kato and gives more.

Extending classical theory

Assume T is bisectorial with BHFC. Set $\chi^\pm = 1_{\pm \operatorname{Re} z>0}$ and $H_T^\pm$, the image of $\overline{\mathsf{R}(T)}$ under $\chi^\pm(T)$. These two operators are bounded complementary projectors, hence one has the spectral splitting $\overline{\mathsf{R}(T)} = H_T^+ \oplus H_T^-$. The extension

$$(Ch)(\lambda) = \begin{cases} e^{-\lambda T}\chi^+(T)h, & \lambda > 0 \\ e^{-\lambda T}\chi^-(T)h, & \lambda < 0 \end{cases}$$

solves the equation

$$\partial_\lambda Ch + TCh = 0, \quad h \in \overline{\mathsf{R}(T)}, \quad \lambda \neq 0$$

and has limits $\chi^\pm(T)h$ when $\lambda \to 0^\pm$.
If $T = \frac{1}{i}\frac{d}{dz}$ on a Lipschitz curve, these is the Cauchy extension of h (Plemelj formulas) and the spectral spaces are the well-known holomorphic Hardy spaces. In the 1980's, McIntosh had worked out the case of T being the Clifford-Dirac operator D on a Lipschitz surface.

Hardy spaces I

Here too, McIntosh's vision allowed to make substantial progress in a theory that has proved useful and popular.

Originally Hardy spaces on the unit circle arise as traces of some holomophic function spaces on the unit disk.

In the 1960's and 1970's, Stein, Weiss, Fefferman, Coifman, Latter and others developed a theory of spaces freeing its dependence to holomorphy (hence to the *d* bar operator), valid in any dimension.

As universal these spaces may be, they were not adapted to certain problems: we had to go back to the relation between space and operator.

Hardy spaces II

With Duong and McIntosh, we introduced (2003) a family of Hardy spaces adapted to a sectorial operator using the concept of tent spaces. In the years the theory grew by making the hypotheses (setting and operator) as minimal as possible. For example, with McIntosh and Russ, we developed a Hardy space theory on a doubling complete Riemannian manifold adapted to the Hodge-Dirac operator $D = d + d^*$. Important to have "generic" functions called molecules (or atoms). McIntosh said they should be in the range of D with some further localization properties. It looked too simple to be correct in general for me but it turned out to be the right point of view as he proved first in articles with his student Lou (in Euclidean situation with differential forms of some degree)

A strong legacy: first order operators

McIntosh advocated for the systematic use of first order operators for second order problems because they are simpler to treat. This was also the motivation of the pioneers (Dirac, Clifford, Maxwell...)
In his publications, one clearly sees the evolution of his thoughts on the matter building from different topics over the years and merging into a conceptual approach applicable in many different situations. Let us list some of them that he dealt with

- The Dirac operator D in Clifford setting.
- The matrix D in the Kato problem.
- The Maxwell-Dirac operator $D + ke_4$ in 3d.
- The Hodge-Dirac operator $d + d^*$ in Riemannian geometry.
- Stokes operator in Lipschitz domains.

What could explain the long time (more than one century) before the actual success of these first order methods is the lack of analytic methods to treat rough situations until very recently.
It is precisely, the modern *Tb* theory that was developed following the insight of McIntosh that allows for results under perturbations of these models with non smooth coefficients or in non smooth geometries.

This is certainly a line of thoughts to keep following in the future.

May you rest in peace, Alan. We'll miss you.