Nonconvex SDP Problems of Structural Optimization

Michal Kočvara and Michael Stingl

UTIA Prague and University of Erlangen

Outline

- Structural design with stability, vibration control
- FMO—a particular case of structural design
- Solving nonconvex SDP by PENNON
- Examples

Structural design problems

MPEC:

$$\min_{\rho, u} F(\rho, u)$$

s.t.

$$\rho \in U_{\mathrm{ad}}$$

$$u \text{ solves } \mathcal{E}(\rho, u)$$

$F(\rho, u)$	...	cost functional (weight, stiffness, peak stress...)
ρ	...	design variable (thickness, material properties, shape...)
u	...	state variable (displacements, stresses)
U_{ad}	...	admissible designs

Structural design problems

WEIGHT versus STIFFNESS:

- W weight $\sum \rho_i$
- C stiffness (compliance) $f^T u$

Equilibrium constraint: u solves $\mathcal{E}(\rho, u) \longrightarrow \sum(\rho_i K_i)u = f$

Structural design problems

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$$\min C$$

s.t.

$$W \le \widehat{W}$$

equilibrium

Structural design problems

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$\min C$ s.t. $W \le \widehat{W}$ equilibrium	$\min W$ s.t. $C \le \widehat{C}$ equilibrium

S. Timoshenko:

Experience showed that structures like bridges or aircrafts may fail in some cases not on account of high stresses but owing to insufficient elastic stability.

Structural design with free vibration control

Three quantities to control:

- W weight $\quad \sum \rho_i$
- C stiffness (compliance) $\quad f^T u$
- λ min. eigenfrequency $\quad K(\rho)u = \lambda M(\rho)u$

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$\min C$	$\min W$	$\max \lambda$
s.t.	s.t.	s.t.
$W \le \widehat{W}$	$C \le \widehat{C}$	$W \le \widehat{W}$
$\lambda \ge \widehat{\lambda}$	$\lambda \ge \widehat{\lambda}$	$C \le \widehat{C}$
equilibrium	equilibrium	equilibrium

Three quantities to control:

- W weight $\sum \rho_i$
- C stiffness (compliance) $f^T u$
- λ critical buckling force $K(\rho)u = \lambda G(\rho, u)u$

$\min C$ s.t. $W \le \widehat{W}$, $\lambda \ge 1$, equilibrium	$\min W$ s.t. $C \le \widehat{C}$, $\lambda \ge 1$, equilibrium	$\max \lambda$ s.t. $W \le \widehat{W}$, $C \le \widehat{C}$, equilibrium

Structural design with stability control

Lowest (positive) eigenvalue of

$$K(\rho)u = \lambda G(\rho, u)u$$

(critical force) should be bigger than 1.

$$\min_{\rho,u} W(\rho)$$

s.t.

$$K(\rho)u = f$$

$$f^T u \leq \widehat{C}$$

$$\rho_i \geq 0, \quad i = 1, \ldots, m$$

$$\lambda \geq 1$$

Two standard tricks:

$$K(\rho) \succ 0, \qquad u = K(\rho)^{-1} f$$

$$f^T K(\rho)^{-1} f \le \widehat{C} \quad \Longleftrightarrow \quad \begin{pmatrix} \widehat{C} & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0$$

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$$\left.\begin{aligned} K(\rho)u = \lambda G(\rho, u)u \\ \lambda \ge 1 \end{aligned}\right\} \quad \Longleftrightarrow \quad K(\rho) - G(\rho, u) \succeq 0$$

$$\Longleftrightarrow \quad K(\rho) - \widetilde{G}(\rho) \succeq 0$$

$$\widetilde{G}(\rho) = G(\rho, K(\rho)^{-1} f)$$

Structural design with stability control

Formulated as SDP problem:

$$\min_{\rho} W(\rho)$$

subject to

$$K(\rho) - \widetilde{G}(\rho) \succeq 0$$

$$\begin{pmatrix} c & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0$$

$$\rho_i \geq 0, \quad i = 1, \ldots, m$$

where

$$K(\rho) = \sum \rho_i K_i, \qquad \widetilde{G}(\rho) = \sum \widetilde{G}_i$$

Free Material Optimization

Aim:

Given an amount of material, boundary conditions and external load f, find the material (distribution) so that the body is as stiff as possible under f.

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$$\inf_{\substack{E \succcurlyeq 0 \\ \int tr(E)dx \leq 1}} \sup_{u \in U} -\frac{1}{2} \int_{\Omega} \langle E e(u), e(u) \rangle \, dx + \int_{\Gamma_2} f \cdot u \, dx$$

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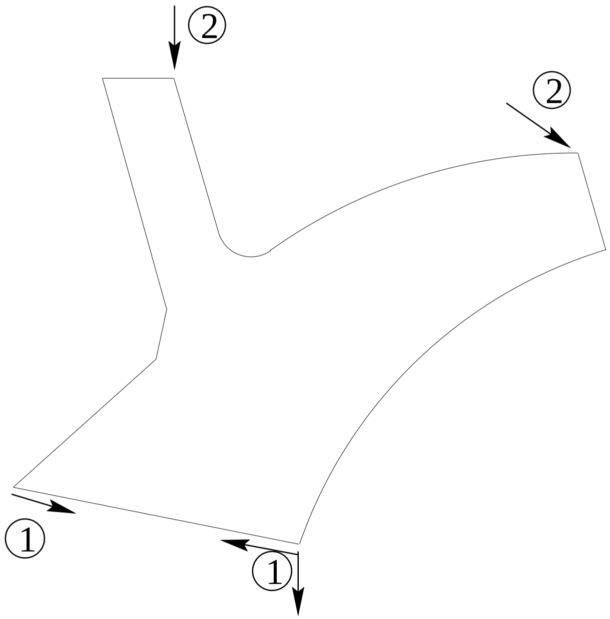
$$\inf_{\alpha\in\mathbb{R}, u\in U}\left\{\alpha - f^T u \,|\, \alpha \geq \frac{m}{2} u^T A_i\, u \quad \text{for} \quad i=1,\ldots,m\right\}$$

FMO, example



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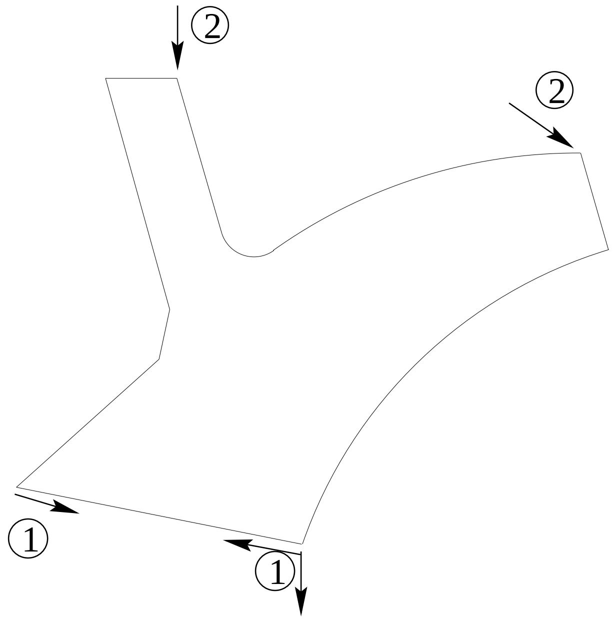

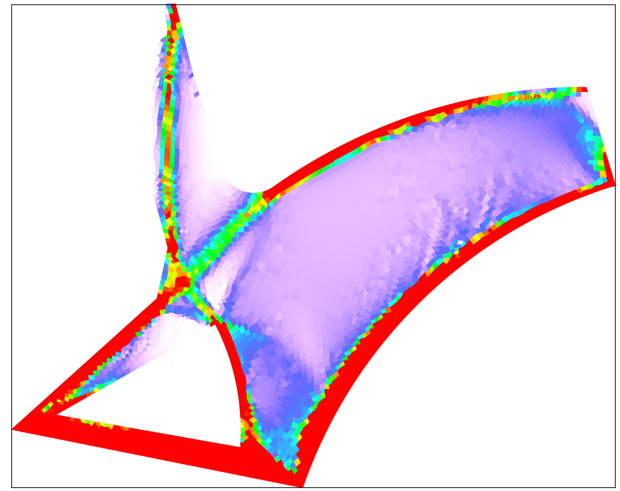

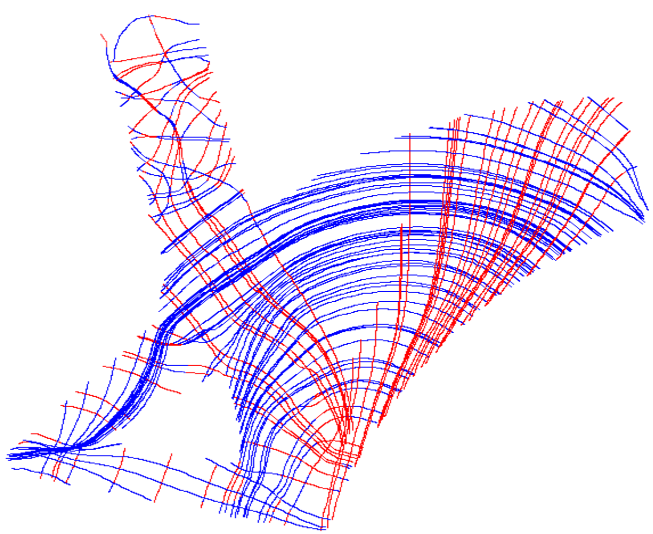

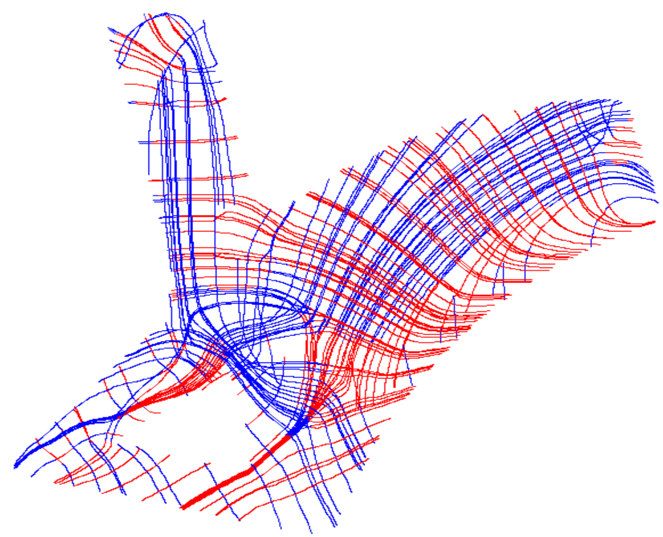

Structural design with stability control

Formulated as SDP problem:

$$\min_{\rho} W(\rho)$$

subject to

$$\begin{pmatrix} c & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0$$

$$\rho_i \geq 0, \quad i = 1, \ldots, m$$

$$K(\rho) - \widetilde{G}(\rho) \succeq 0$$

where

$$K(\rho) = \sum \rho_i K_i, \qquad \widetilde{G}(\rho) = \sum \widetilde{G}_i$$

PENNON for SDP

Problem:

$$\min_{x \in \mathbb{R}^n} \left\{ b^T x : \mathcal{A}(x) \preccurlyeq 0 \right\}$$

$$\mathcal{A} : \mathbb{R}^n \longrightarrow \mathbb{S}_d$$

Notation:

$\langle A, B \rangle_{\mathbb{S}_d}$ $:=$ tr $\left(A^T B\right)$ *inner product* on $\mathbb{S}_d$

$\mathbb{S}_{d_+}$ $= \{A \in \mathbb{S}_d \mid A$ *positive semidefinite*$\}$

$U \in \mathbb{S}_{d_+}$ *matrix multiplier (dual variable)*

Φ_p *penalty function* on $\mathbb{S}_d$

PENNON for SDP: algorithm

Generalized augmented Lagrangian algorithm for SDP:
(based on modified barrier method of R. Polyak, 1992)

We have

$$\mathcal{A}(x) \preccurlyeq 0 \Longleftrightarrow \Phi_p(\mathcal{A}(x)) \preccurlyeq 0$$

and the corresponding *augmented Lagrangian*

$$F(x, U, p) := f(x) + \langle U, \Phi_p(\mathcal{A}(x)) \rangle_{\mathbb{S}_d}$$

Algorithm:

(i) Find x^{k+1} satisfying $\|\nabla_x F(x, U^k, p^k)\| \leq \epsilon^k$

(ii) $U^{k+1} = D_{\mathcal{A}} \Phi_p(\mathcal{A}(x); U^k)$

(iii) $p^{k+1} < p^k$

Best choice of Φ: $\Phi(A) = (A - I)^{-1} - I$

PENNON for SDP: theory

Based on Breitfeld-Shanno, 1993; generalized by M. Stingl, 2003

Assume:

1. $f, \mathcal{A} \in C^2$
2. $x \in \Omega$ nonempty, bounded
3. Constraint Qualification

Then $\exists$ an index set $\mathcal{K}$ so that:

- $x_k \longrightarrow \hat{x}, \ k \in \mathcal{K}$
- $U_k \longrightarrow \widehat{U}, \ k \in \mathcal{K}$
- $(\hat{x}, \widehat{U})$ satisfies first-order optimality conditions

PENNON for SDP: Hessian

The reciprocal barrier function in SDP

$$\Phi(A) = (A - I)^{-1} - I$$

Hessian

$$\frac{\partial^2}{\partial x_i \partial x_j}\Phi(\mathcal{A}(x)) =$$

$$(\mathcal{A}(x) - I)^{-1}\frac{\partial \mathcal{A}(x)}{\partial x_i}(\mathcal{A}(x) - I)^{-1}\frac{\partial \mathcal{A}(x)}{\partial x_j}(\mathcal{A}(x) - I)^{-1}$$

$$+ (\mathcal{A}(x) - I)^{-1}\frac{\partial^2 \mathcal{A}(x)}{\partial x_i \partial x_j}(\mathcal{A}(x) - I)^{-1}$$

$$+ (\mathcal{A}(x) - I)^{-1}\frac{\partial \mathcal{A}(x)}{\partial x_j}(\mathcal{A}(x) - I)^{-1}\frac{\partial \mathcal{A}(x)}{\partial x_i}(\mathcal{A}(x) - I)^{-1}$$

PENNON fo SDP: complexity

FMO with stability constraint (nonconvex SDP)

$$K(\rho) + G(\rho) \succeq 0$$

$$K(\rho) = \sum_{e=1}^{M} \rho_e K_e$$

$$G(\rho) = \sum_{e=1}^{M} G_e \qquad G_e(\rho) = \sum_{k=1}^{K} B_{e,k}^T S_{e,k}(\rho) B_{e,k}$$

$$S_{e,k}(\rho) = \begin{pmatrix} \sigma_1 & \sigma_3 \\ \sigma_3 & \sigma_2 \end{pmatrix} \qquad \sigma_{e,k}(\rho) = \rho_e \widetilde{B}_{e,k}^T (K^{-1}(\rho) f)_e$$

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All dense matrix-matrix multiplications implemented in BLAS

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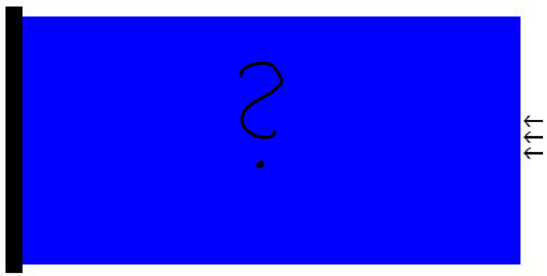
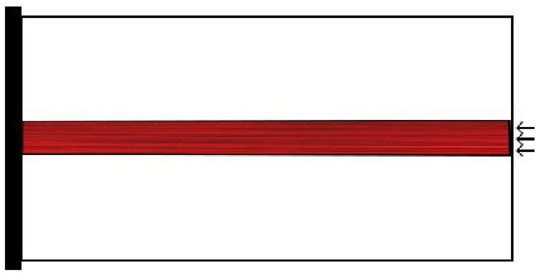
Pentium 4, 2.4GHz, ∼100 Newton steps:
400 elements . . . 8 h 45 min, 1000 elements . . . ∼130 hours

FMO with vibration constraint (linear SDP)





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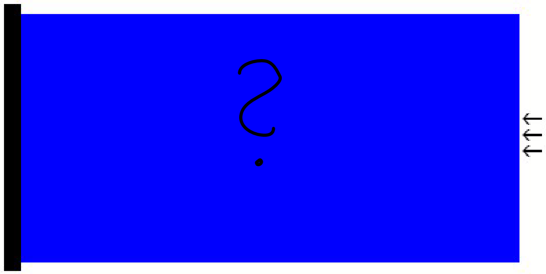

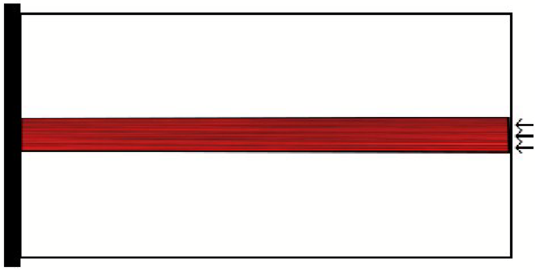

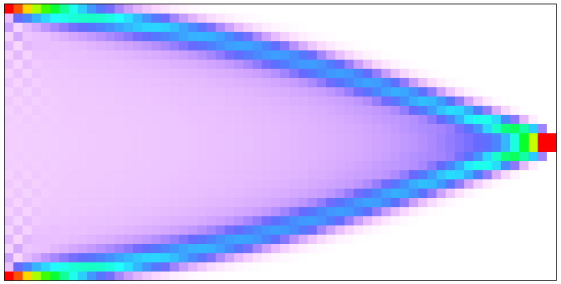

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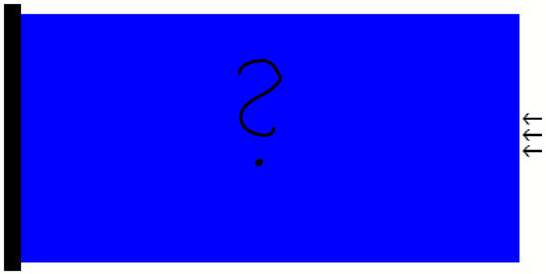
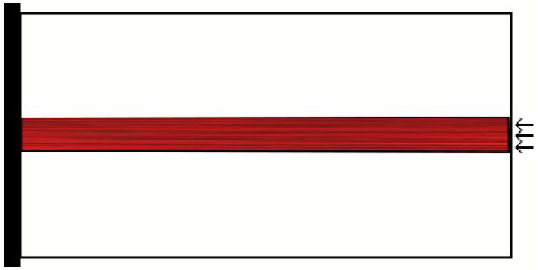
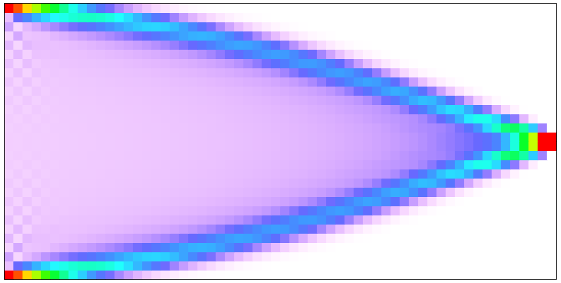






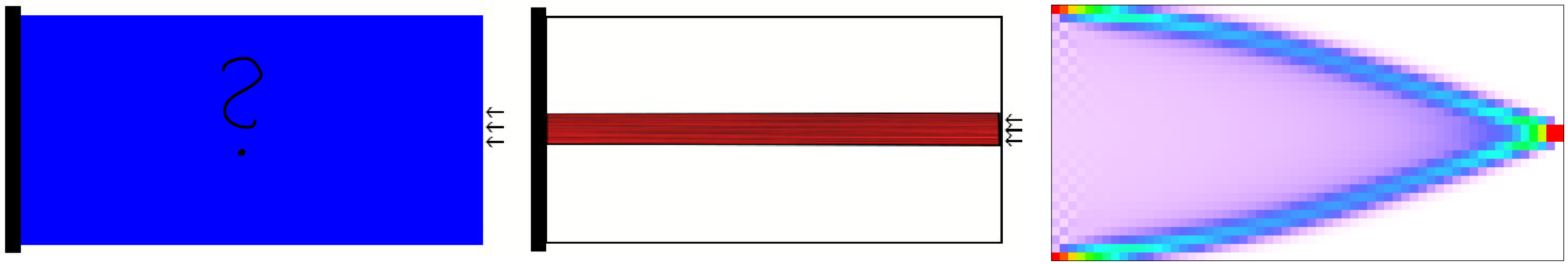
Linear SDP, SDPA input file (Pentium 4, 2.5 GHz):

problem	no. of variables	size of matrix
shmup-3	420	1801+840
shmup-4	800	3361+1600
shmup-5	1800	7441+3660

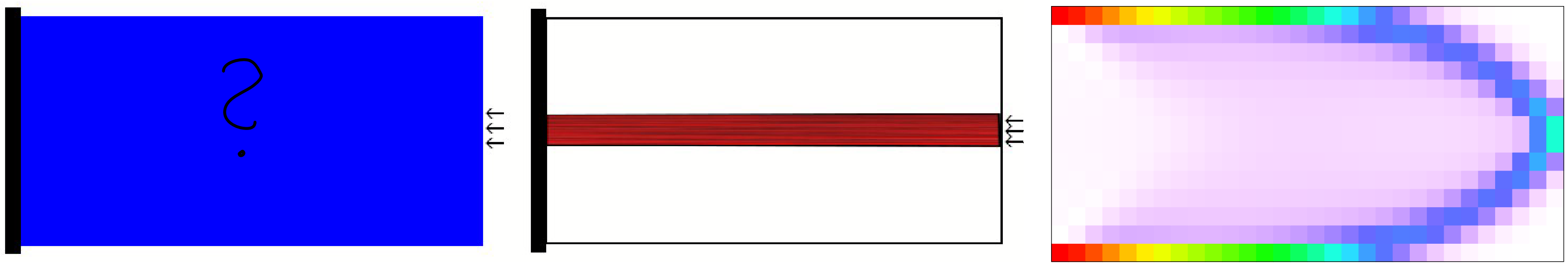
problem	PENNON	SDPT3	SDPA	DSDP	CSDP	SeDuMi
shmup-3	381	417	497	439	1395	23322
shmup-4	2095	2625	2952	2798	5768	$>$127320
shmup-5	14149	23535	m	fail	m	m

Examples, FMO w. stability constraint

FMO with vibration constraint (linear SDP)



FMO with stability constraint (nonlinear SDP)



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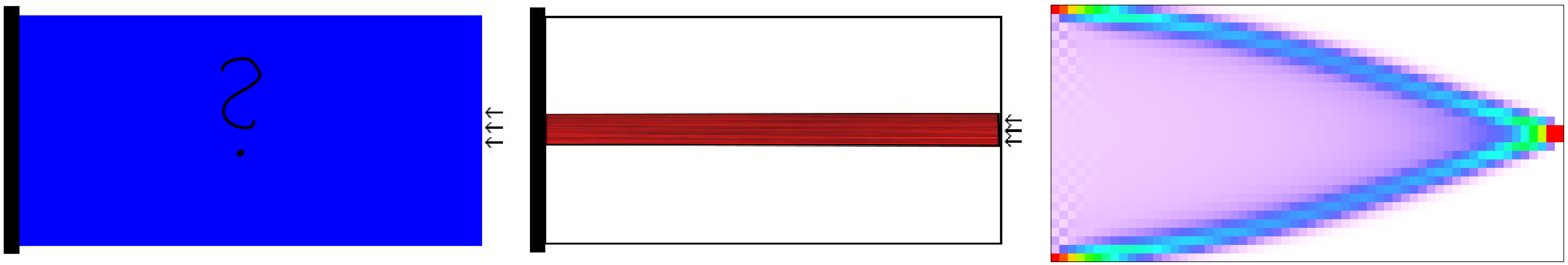

shmup3 (420 elements) . . . 6 min 20 sec

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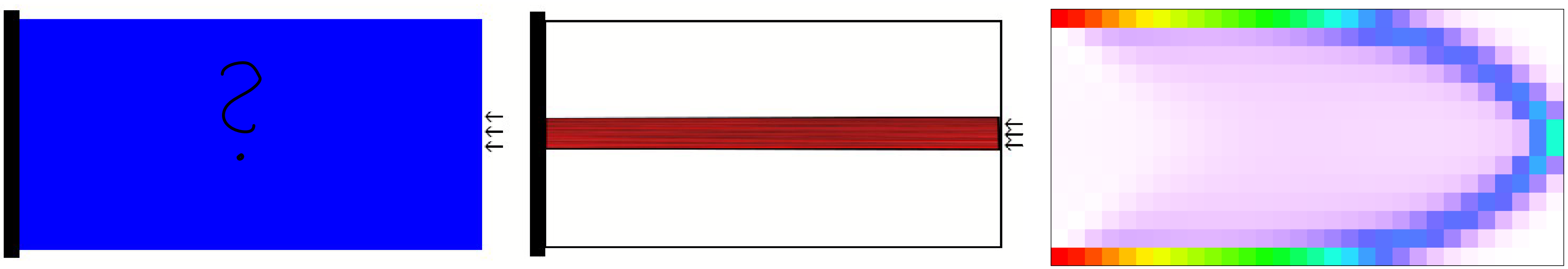

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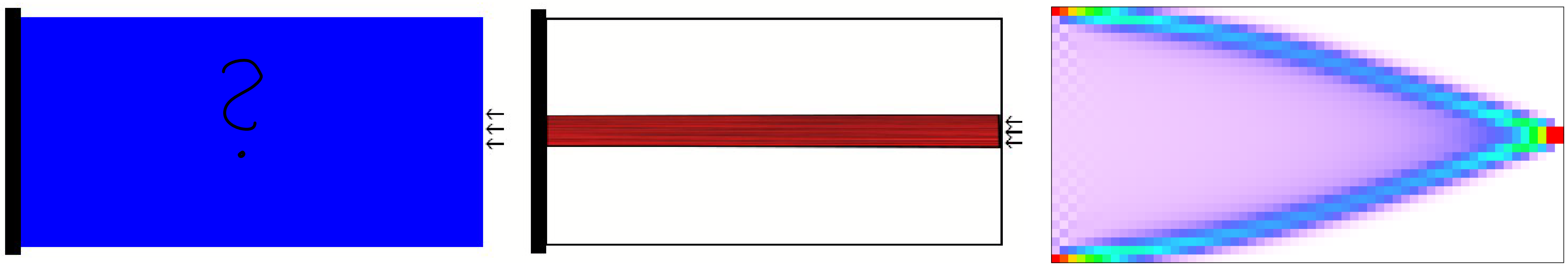

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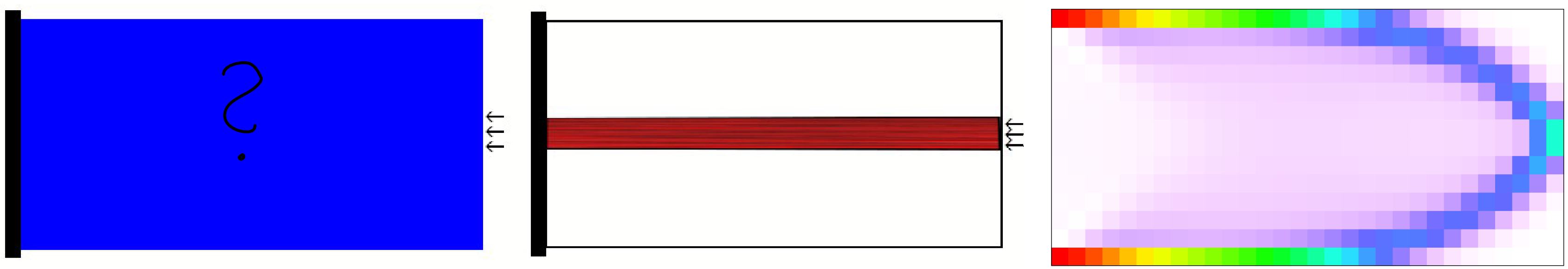

shmup3 (420 elements) . . . 8 hours

shmup3 with no SDP constraints (convex NLP) . . . 1 sec

Conclusions (so far)

PENNON algorithm works well for nonconvex SDP
–accurate solution within 60–100 internal iterations–
(more experience from BMI problems and truss design)

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FIRST-ORDER METHOD

Hessian free methods

Use conjugate gradient method for solving the Newton system

Use finite difference formula for Hessian-vector products:

$$\nabla^2 F(x_k)v \approx \frac{\nabla F(x_k + hv) - \nabla F(x_k)}{h}$$

with $h = (1 + \|x_k\|_2 \sqrt{\varepsilon})$

Complexity: Hessian-vector product = gradient evaluation
need for Hessian-vector-product type preconditioner

Limited accuracy (4–5 digits)

Can CG + approx. Hessian help?

Partly. . .

No preconditioning, approx. Hessian:
as many gradient evaluations as CG steps (good)
CG with no preconditioning inefficient (bad)

Nonlinear SDP—FMO with stability constraints

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Evaluation of exact diagonal as expensive as evaluation of full Hessian
Evaluation of approx. diagonal . . . . . .

Only L-BFGS preconditioner can be used — but it isn't really efficient

Conclusions, part II

Hessian-free SDP:

- First promising results, more testing (and coding) needed

Solving vibration problem as GEVP

Another option (vibration problems):
Solve the maximum eigenvalue problem formulated as GEVP:

λ min. eigenfrequency of $K(\rho)u = \lambda M(\rho)u$

$$\max \lambda$$

s.t.

$$W \le \widehat{W}$$

$$C \le \widehat{C}$$

equilibrium

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equilibrium

$$\max_{\rho,\lambda} \lambda$$

s.t.

$$K(\rho) - \lambda M(\rho) \succeq 0$$

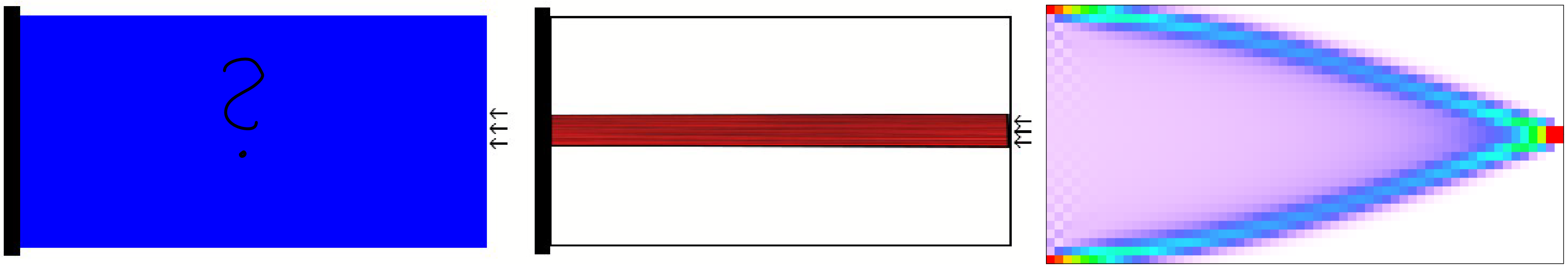
$$\sum \rho_i \leq \widehat{W}$$

$$\rho_i \geq 0, \quad i = 1, \ldots, m$$

$$\begin{pmatrix} \widehat{C} & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0$$

(quasiconvex) SDP problem with BMI constraints — solve by PENBMI

FMO with vibration constraint (linear SDP)



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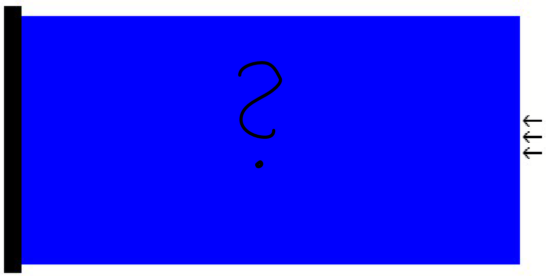
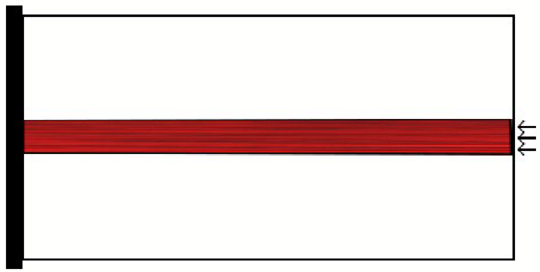
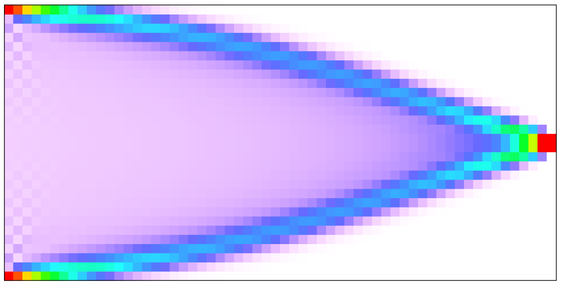
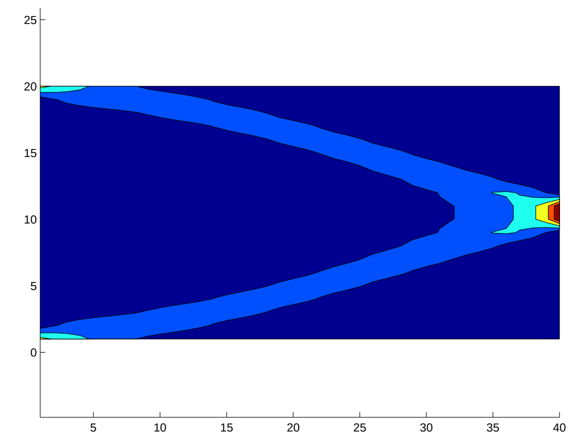






FMO with vibration constraint: BMI formulation)



THE END