Nonconvex SDP Problems of Structural Optimization

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Outline

- Structural design with stability, vibration control
- FMO—a particular case of structural design
- Solving nonconvex SDP by PENNON
- Examples
Structural design problems

MPEC:

\[
\min_{\rho, u} F(\rho, u)
\]

\[\text{s.t.} \quad \rho \in U_{ad} \]

\[u \text{ solves } E(\rho, u)\]

\[F(\rho, u) \quad \ldots \quad \text{cost functional (weight, stiffness, peak stress...)}\]

\[\rho \quad \ldots \quad \text{design variable (thickness, material properties, shape...)}\]

\[u \quad \ldots \quad \text{state variable (displacements, stresses)}\]

\[U_{ad} \quad \ldots \quad \text{admissible designs}\]
Structural design problems

WEIGHT versus STIFFNESS:

- $W$ weight $\sum \rho_i$
- $C$ stiffness (compliance) $f^T u$

Equilibrium constraint: $u$ solves $\mathcal{E}(\rho, u) \rightarrow \sum (\rho_i K_i) u = f$
Structural design problems

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\[
\begin{align*}
\min C \\
s.t. \\
W \leq \hat{W} \\
equilibrium
\end{align*}
\]
Structural design problems

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\begin{align*}
\text{min } C \\
\text{s.t. } W &\leq \hat{W} \\
\text{equilibrium} \\
\text{min } W \\
\text{s.t. } C &\leq \hat{C} \\
\text{equilibrium}
\end{align*}
\]
S. Timoshenko:

*Experience showed that structures like bridges or aircrafts may fail in some cases not on account of high stresses but owing to insufficient elastic stability.*
Structural design with free vibration control

Three quantities to control:

- $W$ weight $\sum \rho_i$
- $C$ stiffness (compliance) $f^T u$
- $\lambda$ min. eigenfrequency $K(\rho)u = \lambda M(\rho)u$
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Structural design with free vibration control

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\[
\begin{align*}
\text{min } C & \\
\text{s.t.} & \\
W & \leq \hat{W} \\
\lambda & \geq \hat{\lambda} \\
equilibrium & \\
\text{equilibrium} & \\
\text{min } W & \\
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\]
Structural design with free vibration control

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\text{equilibrium}
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\]

\[
\begin{align*}
\max \lambda \\
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C \leq \hat{C} \\
\text{equilibrium}
\end{align*}
\]
### Structural design with stability control

**Three quantities to control:**

- **$W$** weight \( \sum \rho_i \)
- **$C$** stiffness (compliance) \( f^T u \)
- **$\lambda$** critical buckling force \( K(\rho)u = \lambda G(\rho, u)u \)

<table>
<thead>
<tr>
<th>( \min C )</th>
<th>( \min W )</th>
<th>( \max \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>s.t. ( W \leq \hat{W} ) \hspace{1cm} ( \lambda \geq 1 ) equilibrium</td>
<td>s.t. ( C \leq \hat{C} ) \hspace{1cm} ( \lambda \geq 1 ) equilibrium</td>
<td>s.t. ( W \leq \hat{W} ) \hspace{1cm} ( C \leq \hat{C} ) equilibrium</td>
</tr>
</tbody>
</table>
Lowest (positive) eigenvalue of

\[
K(\rho)u = \lambda G(\rho, u)u
\]

(critical force) should be bigger than 1.

\[
\min_{\rho, u} W(\rho)
\]

s.t.

\[
K(\rho)u = f
\]

\[
f^T u \leq \hat{C}
\]

\[
\rho_i \geq 0, \quad i = 1, \ldots, m
\]

\[
\lambda \geq 1
\]
Two standard tricks:

\[ K(\rho) \succ 0, \quad u = K(\rho)^{-1} f \]

\[ f^T K(\rho)^{-1} f \leq \hat{C} \iff \begin{pmatrix} \hat{C} & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0 \]
Two standard tricks:

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\[ f^T K(\rho)^{-1} f \leq \hat{C} \iff \begin{pmatrix} \hat{C} & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0 \]

\[ K(\rho) u = \lambda G(\rho, u) u \iff \begin{cases} \lambda \geq 1 \\ K(\rho) - G(\rho, u) \succeq 0 \\ K(\rho) - \tilde{G}(\rho) \succeq 0 \\ \tilde{G}(\rho) = G(\rho, K(\rho)^{-1} f) \end{cases} \]
Structural design with stability control

Formulated as SDP problem:

\[
\min_{\rho} W(\rho)
\]

subject to

\[
K(\rho) - \tilde{G}(\rho) \succeq 0
\]

\[
\begin{pmatrix}
c & f^T \\
f & K(\rho)
\end{pmatrix} \succeq 0
\]

\[
\rho_i \geq 0, \quad i = 1, \ldots, m
\]

where

\[
K(\rho) = \sum \rho_i K_i, \quad \tilde{G}(\rho) = \sum \tilde{G}_i
\]
Aim:

Given an amount of material, boundary conditions and external load $f$, find the material (distribution) so that the body is as stiff as possible under $f$.

The design variables are the material properties at each point of the structure.
Free Material Optimization

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\[
\inf_{\mathbf{E} \succeq 0} \sup_{u \in U} \left( -\frac{1}{2} \int_{\Omega} \langle \mathbf{E} \mathbf{e}(u), \mathbf{e}(u) \rangle \, dx + \int_{\Gamma_2} f \cdot u \, dx \right)
\]

\[
\int tr(E) \, dx \leq 1
\]
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\inf_{\substack{E \succeq 0 \\ \int tr(E) dx \leq 1}} \sup_{u \in U} \left( -\frac{1}{2} \int_\Omega \langle E e(u), e(u) \rangle \, dx + \int_{\Gamma_2} f \cdot u \, dx \right)
\]

\[
\inf_{\rho \geq 0} \sup_{u \in U} \left( -\frac{1}{2} \int_\Omega \rho \langle e(u), e(u) \rangle \, dx + \int_{\Gamma_2} f \cdot u \, dx \right)
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$$\inf_{\rho \geq 0} \sup_{u \in U} \frac{1}{2} \int_{\Omega} \rho \langle e(u), e(u) \rangle \, dx + \int_{\Gamma_2} f \cdot u \, dx$$

$$\int \rho \, dx \leq 1$$

$$\inf_{\alpha \in \mathbb{R}, u \in U} \left\{ \alpha - f^T u \mid \alpha \geq \frac{m}{2} u^T A_i u \text{ for } i = 1, \ldots, m \right\}$$
FMO, example

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FMO, example
Formulated as SDP problem:

\[
\begin{align*}
\min_{\rho} & \quad W(\rho) \\
\text{subject to} & \quad \begin{pmatrix} c & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0 \\
& \quad \rho_i \geq 0, \quad i = 1, \ldots, m \\
& \quad K(\rho) - \tilde{G}(\rho) \succeq 0
\end{align*}
\]

where

\[
K(\rho) = \sum \rho_i K_i, \quad \tilde{G}(\rho) = \sum \tilde{G}_i
\]
PENNON for SDP

Problem:
\[ \min_{x \in \mathbb{R}^n} \{ b^T x : A(x) \preceq 0 \} \]
\[ A : \mathbb{R}^n \rightarrow \mathbb{S}_d \]

Notation:
\[ \langle A, B \rangle_{\mathbb{S}_d} := \text{tr} \left( A^T B \right) \text{ inner product on } \mathbb{S}_d \]
\[ \mathbb{S}_{d^+} = \{ A \in \mathbb{S}_d \mid A \text{ positive semidefinite} \} \]
\[ U \in \mathbb{S}_{d^+} \text{ matrix multiplier (dual variable)} \]
\[ \Phi_p \text{ penalty function on } \mathbb{S}_d \]
PENNON for SDP: algorithm

Generalized augmented Lagrangian algorithm for SDP:
(based on modified barrier method of R. Polyak, 1992)

We have

\[ \mathcal{A}(x) \preceq 0 \iff \Phi_p(\mathcal{A}(x)) \preceq 0 \]

and the corresponding **augmented Lagrangian**

\[ F(x, U, p) := f(x) + \langle U, \Phi_p(\mathcal{A}(x)) \rangle_{S_d} \]

**Algorithm:**

(i) Find \( x^{k+1} \) satisfying \( \| \nabla_x F(x, U^k, p^k) \| \leq \epsilon^k \)

(ii) \( U^{k+1} = D_{\mathcal{A}} \Phi_p(\mathcal{A}(x); U^k) \)

(iii) \( p^{k+1} < p^k \)

Best choice of \( \Phi \):

\[ \Phi(A) = (A - I)^{-1} - I \]
PENNON for SDP: theory

Based on Breitfeld-Shanno, 1993; generalized by M. Stingl, 2003

Assume:

1. \( f, A \in C^2 \)
2. \( x \in \Omega \) nonempty, bounded
3. Constraint Qualification

Then \( \exists \) an index set \( \mathcal{K} \) so that:

- \( x_k \to \hat{x}, \ k \in \mathcal{K} \)
- \( U_k \to \hat{U}, \ k \in \mathcal{K} \)
- \( (\hat{x}, \hat{U}) \) satisfies first-order optimality conditions
The reciprocal barrier function in SDP

$$\Phi(A) = (A - I)^{-1} - I$$

Hessian

$$\frac{\partial^2}{\partial x_i \partial x_j} \Phi(A(x)) =$$

$$(A(x) - I)^{-1} \frac{\partial A(x)}{\partial x_i} (A(x) - I)^{-1} \frac{\partial A(x)}{\partial x_j} (A(x) - I)^{-1}$$

$$+ (A(x) - I)^{-1} \frac{\partial^2 A(x)}{\partial x_i \partial x_j} (A(x) - I)^{-1}$$

$$+ (A(x) - I)^{-1} \frac{\partial A(x)}{\partial x_j} (A(x) - I)^{-1} \frac{\partial A(x)}{\partial x_i} (A(x) - I)^{-1}$$
FMO with stability constraint (nonconvex SDP)

\[ K(\rho) + G(\rho) \succeq 0 \]

\[ K(\rho) = \sum_{e=1}^{M} \rho_e K_e \]

\[ G(\rho) = \sum_{e=1}^{M} G_e\]

\[ G_e(\rho) = \sum_{k=1}^{K} B_{e,k}^T S_{e,k}(\rho) B_{e,k} \]

\[ S_{e,k}(\rho) = \begin{pmatrix} \sigma_1 & \sigma_3 \\ \sigma_3 & \sigma_2 \end{pmatrix} \]

\[ \sigma_{e,k}(\rho) = \rho_e \tilde{B}_{e,k}^T (K^{-1}(\rho)f) \]
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memory: \( O(M^2) \) (\( M = 500 \approx 64 \text{ MB} \))
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CPU: \( O(K^2 \times d^2 \times M^3) \) for one Hessian assembling
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memory: \( O(M^2) \) (\( M = 500 \approx 64 \text{ MB} \))

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All dense matrix-matrix multiplications implemented in BLAS
FMO with stability constraint (nonconvex SDP)

\[ K(\rho) + G(\rho) \succeq 0 \]

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Pentium 4, 2.4GHz, \( \sim 100 \) Newton steps:

400 elements \( \ldots \) 8 h 45 min, 1000 elements \( \ldots \) \( \sim 130 \) hours
Examples, FMO w. stability constraint

FMO with vibration constraint (linear SDP)
Examples, FMO w. stability constraint

FMO with vibration constraint (linear SDP)
Examples, FMO w. stability constraint

FMO with vibration constraint (linear SDP)

Linear SDP, SDPA input file (Pentium 4, 2.5 GHz):

<table>
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<tr>
<th>problem</th>
<th>no. of variables</th>
<th>size of matrix</th>
</tr>
</thead>
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<td>1801+840</td>
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<tr>
<td>shmup-4</td>
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<td>3361+1600</td>
</tr>
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<td>7441+3660</td>
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<tr>
<th>problem</th>
<th>PENNON</th>
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<th>SDPA</th>
<th>DSDP</th>
<th>CSDP</th>
<th>SeDuMi</th>
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<td>23535</td>
<td>m</td>
<td>fail</td>
<td>m</td>
<td>m</td>
</tr>
</tbody>
</table>
Examples, FMO w. stability constraint

FMO with vibration constraint (linear SDP)

FMO with stability constraint (nonlinear SDP)
Examples, FMO w. stability constraint

FMO with **vibration** constraint (**linear** SDP)

 shmup3 (420 elements) . . . 6 min 20 sec

FMO with **stability** constraint (**nonlinear** SDP)

 shmup3 (420 elements) . . . 8 hours
Examples, FMO w. stability constraint

FMO with vibration constraint (linear SDP)

shmup3 (420 elements) . . . 6 min 20 sec

FMO with stability constraint (nonlinear SDP)

shmup3 (420 elements) . . . 8 hours

shmup3 with no SDP constraints (convex NLP) . . . 1 sec
Conclusions (so far)

PENNON algorithm works well for nonconvex SDP
–accurate solution within 60–100 internal iterations–
(more experience from BMI problems and truss design)
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complexity of second-order method too high
(for “large” problems)
Conclusions (so far)

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– accurate solution within 60–100 internal iterations–
(more experience from BMI problems and truss design)

complexity of second-order method too high
(for “large” problems)

FIRST-ORDER METHOD
Hessian free methods

Use conjugate gradient method for solving the Newton system

Use finite difference formula for Hessian-vector products:

$$\nabla^2 F(x_k)v \approx \frac{\nabla F(x_k + hv) - \nabla F(x_k)}{h}$$

with $h = (1 + \|x_k\|_2 \sqrt{\varepsilon})$

Complexity: Hessian-vector product = gradient evaluation
need for Hessian-vector-product type preconditioner

Limited accuracy (4–5 digits)
Can CG + approx. Hessian help?

Partly . . .

No preconditioning, approx. Hessian: 
as many gradient evaluations as CG steps (good) 
CG with no preconditioning inefficient (bad)
Nonlinear SDP—FMO with stability constraints

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Partly . . .

No preconditioning, approx. Hessian:
as many gradient evaluations as CG steps (good)
CG with no preconditioning inefficient (bad)

Evaluation of exact diagonal as expensive as evaluation of full Hessian
Evaluation of approx. diagonal . . . .

Only L-BFGS preconditioner can be used — but it isn’t really efficient
Hessian-free SDP:
- First promising results, more testing (and coding) needed
Another option (vibration problems):
Solve the maximum eigenvalue problem formulated as GEVP:

\[ \lambda \ \text{min. eigenfrequency of} \ K(\rho)u = \lambda M(\rho)u \]

\[
\begin{align*}
\max & \quad \lambda \\
\text{s.t.} & \quad W \leq \hat{W} \\
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\[
\begin{align*}
\max_{\lambda, \rho} & \quad \lambda \\
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& \quad C \leq \hat{C} \\
& \quad \rho_i \geq 0, \quad i = 1, \ldots, m \\
& \quad \left( \hat{C} \ f^T \ K(\rho) \right) \succeq 0
\end{align*}
\]

(quasiconvex) SDP problem with BMI constraints — solve by PENBMI
Solving vibration problem as GEVP (example)

FMO with *vibration* constraint (*linear* SDP)
FMO with vibration constraint (linear SDP)

FMO with vibration constraint: BMI formulation
THE END