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Structural Topology Optimization with Eigenvalues

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Abstract

The paper considers different problem formulations of topology optimization of discrete or discretized structures with eigenvalues as constraints or as objective functions. We study multiple load case formulations of minimum weight, minimum compliance problems and of the problem of maximizing the minimal eigenvalue of the structure including the effect of non-structural mass. The paper discusses interrelations of the problems and, in particular, shows how solutions of one problem can be derived from solutions of the other ones. Moreover, we present equivalent reformulations as semidefinite programming problems with the property that, for the minimum weight and minimum compliance problem, each local optimizer of these problems is also a global one. This allows for the calculation of guaranteed global optimizers of the original problems by the use of modern solution techniques of semidefinite programming. For the problem of maximization of the minimum eigenvalue we show how to verify the global optimality and present an algorithm for finding a tight approximation of a globally optimal solution. Numerical examples are provided for truss structures. Examples of both academic and larger size illustrate the theoretical results achieved and demonstrate the practical use of this approach. We conclude with an extension on multiple non-structural mass conditions.

1 Introduction

The subject of this paper is topology optimization of discrete and discretized structures with consideration of free vibrations of the optimal structure. Maximization of the fundamental eigenvalue of a structure is a classic problem of structural engineering. The (generalized) eigenvalue problem typically reads as

$$K(x)w = \lambda(M(x) + M_0)w$$

where $K(x)$ and $M(x)$ are symmetric and positive semidefinite matrices that continuously (often linearly) depend on the parameter x. The main difficulty brings the nonsmooth dependence of eigenvalues on this parameter. The problem has been treated in the engineering literature since the beginning of 70s; see the paper [16] and the overview [15] summarizing the early development. See also the recent book [17] for up-to-date bibliography on this subject. The general problem of eigenvalue optimization belongs also to classic problems of linear algebra. When the matrix $M(x) + M_0$ is positive definite for all x, then one can resort to the theory developed for the standard eigenvalue problem; see [11] for an excellent overview. Not many papers studying the

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dependence of the eigenvalues on a parameter are available for the general case when $M(x)+M_0$ is only positive semidefinite; see, e.g. [4, 18, 20].

We present three different formulations of the structural design problem. In the first one we minimize the volume of the structure subject to equilibrium conditions and compliance constraints. Additionally, we require that the fundamental natural frequency of the optimal structure is bigger than or equal to a certain threshold value. The second formulation is analogous, we just switch the volume and the compliance. In the third formulation we maximize the fundamental frequency, i.e., the minimum eigenvalue of certain generalized eigenvalue problem, subject to equilibrium conditions and constraints on the volume and the compliance. Using the semidefinite programming (SDP) framework, we formulate all three problems in a unified way; while the first two problems lead to linear SDP formulations that were already studied earlier ([14, 6]), the third problem leads to an SDP with a bilinear matrix inequality (BMI) constraint. This formulation, however straightforward, has never been used for the numerical solution of the problem, up to our knowledge. The reason for this was the lack of available SDP-BMI solvers. We solve the problem by a recently developed code PENBMI [7].

We further analyze the mutual relation of our three problems. We show that the problems are in certain sense equivalent. More precisely, taking a certain specific solution from the solution set of one problem, we get a solution of another problem with the same data. We also show that this equivalence does not hold for an arbitrary solution of the problem; this is also illustrated by several numerical examples.

An important property of the SDP reformulations of the minimum volume and minimum compliance problem is that each local minimum of any of these problems is also a global minimum. This is not readily seen from the original problem formulations and brings an important information to the designer. For the problem of maximization of the minimum eigenvalue we show how to verify the global optimality and present an algorithm for finding an ε-approximation of a globally optimal solution.

Numerical examples conclude the paper. They illustrate the various formulations and theorems developed in the paper and also demonstrate the solvability of the SDP formulations and thus their practical usefulness.

All formulations and theorems in the presentation are developed problems using the discrete structural models, the trusses. This is to keep the notation fixed and simple. The theory also applies to discretized structures, for instance, to the variable thickness sheet or the free material optimization problems [3].

We use standard notation; in particular the notation "$A \succeq 0$" means that the symmetric matrix A is positive semidefinite, and "$A \succ 0$" means that it is positive definite. For two symmetric matrices A, B the notation "$A \succeq B$" ("$A \succ B$") means that $A - B$ is positive semidefinite (positive definite). The $k \times k$ identity matrix is denoted by $I_{k\times k}$; $\ker(A)$ and $\mathrm{range}(A)$ denote the null space and the range space of a matrix A, respectively.

2 Problem definitions, relations

2.1 Basic notations, generalized eigenvalues

We consider a general mechanical structure, discrete or discretized by the finite element method. The number of members or finite elements is denoted by m, the total number of "free" degrees of freedom (i.e., not fixed by Dirichlet boundary conditions) by n. For a given set of n_ℓ (independent) load vectors

$$f_\ell \in \mathbb{R}^n, \quad f_\ell \neq 0, \qquad \ell = 1, \dots, n_\ell, \tag{1}$$

the structure should satisfy linear equilibrium equations

$$K(x)u_\ell = f_\ell, \qquad \ell = 1, \dots, n_\ell. \tag{2}$$

Here $K(x)$ is the stiffness matrix of the structure, depending on a design variable x. We will assume linear dependence of K on x,

$$K(x) = \sum_{i=1}^{m} x_i K_i \tag{3}$$

with $x_i K_i$ being the element stiffness matrices. Note that the stiffness matrix of element (member) e_i is typically defined as

$$x_i K_i = x_i P_i \widehat{K}_i P_i^T \tag{4}$$

where $P_i P_i^T$ is a projection from $\mathbb{R}^n$ to the space of element (member) degrees of freedom. In other words, $\widehat{K}_i$ is a matrix localized on the particular element, while K_i lives in the full space $\mathbb{R}^n$. Further,

$$x_i \widehat{K}_i = \int_{e_i} x_i B_i^T E_i B_i \, dV \, .$$

where the rectangular matrix B_i contains derivatives of shape functions of the respective degrees of freedom and E_i is a symmetric matrix containing information about material properties. To exclude pathological situations, we assume that

$$f_\ell \in \operatorname{range}\Big(\sum_{i=1}^{m} K_i\Big) \qquad \text{for all } \ell = 1, \dots, n_\ell \tag{5}$$

which means that there exists a material distribution $x \geq 0$ that can carry all loads f_ℓ (i.e., there exist corresponding $u_1, \dots, u_\ell$ satisfying (2)).

Similarly to the definition of $K(x)$, the mass matrix $M(x)$ of the structure is assumed to be given as

$$M(x) = \sum_{i=1}^{m} x_i M_i, \quad M_i = P_i \widehat{M}_i P_i^T \tag{6}$$

with element mass matrices

$$x_i \widehat{M}_i = \int_{e_i} x_i N_i^T N_i \, dV \, ; \tag{7}$$

here N_i contains the shape functions of the degrees of freedom associated with the i^{th} element.

The design variables $x \in \mathbb{R}^m$ represent, for instance, the thickness, cross-sectional area or material properties of the element. We will assume that

$$x_i \geq 0, \qquad i = 1, \dots, m \, .$$

Notice that the matrices $\widehat{K}_i$, $\widehat{M}_i$ have the properties $\widehat{K}_i \succeq 0$, $\widehat{M}_i \succ 0$, and thus $K(x) \succeq 0$, $M(x) \succeq 0$ for all $x \geq 0$. From a practical point of view, it is worth noticing that the element matrices K_i and M_i are very sparse with only nonzero elements corresponding to degrees of freedom of the i^{th} element. That means, for each i, the matrices K_i and M_i have the same nonzero structure. The matrices $K(x)$, $M(x)$, however, may be dense, in general.

We assume that the discretized structure is connected and the boundary conditions are such that $K(e) \succ 0$ and $M(e) \succ 0$, where e is the vector of all ones. The latter condition simply excludes rigid body movement for any $x > 0$.

In the sequel, we will sometimes collect the displacement vectors $u_1, \ldots, u_{n_\ell}$ for all the load cases in one vector

$$u = (u_1^T, \ldots, u_{n_\ell}^T)^T \in \mathbb{R}^{n \cdot n_\ell},$$

for simplification of the notation.

In this paper we do not rely on any other properties of stiffness and mass matrices than those outlined above. Therefore, the problem formulations and the conclusions apply to a broad class of problems, e.g., to the variable thickness sheet problem or the free material optimization problem [3]. For the sake of transparency, however, we concentrate on a particular class of discrete structures, namely trusses. A truss is an assemblage of pin-jointed uniform straight bars. The bars are subjected to only axial tension and compression when the truss is loaded at the joints. With a given load and a given set of joints at which the truss is fixed, the goal of the designer is to find a truss that is as light as possible and satisfies the equilibrium conditions. In the simplest, yet meaningful, approach, the number of the joints (nodes) and their position are kept fixed. The design variables are the bar volumes and the only constraints are the equilibrium equation and an upper bound on the weighted sum of the displacements of loaded nodes, so-called *compliance*. Recently, this model (or its equivalent reformulations) has been extensively analyzed in the mathematical and engineering literature (see, e.g., [2, 3] and the references therein).

In this article, we will additionally consider free vibrations of the optimal structure. The free vibrations are the eigenvalues of the generalized eigenvalue problem

$$K(x)w = \lambda(M(x) + M_0)w\,. \tag{8}$$

The matrix $M_0 \in \mathbb{R}^{n\times n}$ is assumed to be symmetric and positive semidefinite. It denotes the mass matrix of a given non-structural mass ("dead load"). For the sake of completeness, the choice $M_0 = 0$ is possible and will be treated in more detail below.

In the sequel we use the notation

$$X := \{x \in \mathbb{R}^m \mid x \geq 0,\ x \neq 0\}\ .$$

As a consequence of the construction of $K(x)$ and $M(x)$ we obtain our first result.

Lemma 2.1. *For each $x \in X$ it holds that*

$$\ker(M(x) + M_0) \subseteqq \ker(K(x))\ .$$

Proof. Let $u \in \mathbb{R}^n$ be in $\ker(M(x) + M_0)$. Then $u^T(M(x) + M_0)u = 0$, i.e. (cf. (6)),

$$0 = u^T\Big(\sum_{i=1}^m x_i P_i \widehat{M}_i P_i^T + M_0\Big)u = \sum_{i=1}^m x_i (P_i^T u)^T \widehat{M}_i (P_i^T u) + u^T M_0 u\ .$$

Because $\widehat{M}_i \succ 0$ for all i, and because $M_0 \succeq 0$, we conclude that

$$P_i^T u = 0 \qquad \text{for all } i \text{ such that } x_i > 0.$$

Hence, by the definition of $K(x)$ and by (4),

$$K(x)u = \sum_{i=1}^m x_i K_i u = \sum_{i=1}^m x_i P_i \widehat{K}_i P_i^T u = \sum_{i:\, x_i \neq 0} x_i P_i \widehat{K}_i P_i^T u = 0,$$

and the proof is complete. $\square$

We now want to define a function $\lambda_{\min}$ as the smallest eigenvalue λ of problem (8) for a given structure represented by $x \in X$. Before doing that, we mention the following dilemma in the generalized eigenvalue problem (8). If $x \in X$ is fixed and $(\lambda, w) \in \mathbb{R} \times \mathbb{R}^n$ is a solution of (8) with $w \neq 0$ but $w \in \ker(M(x) + M_0)$ then Lemma 2.1 shows that also $K(x)w = 0$. Hence (μ, w) is also a solution of (8) for arbitrary $\mu \in \mathbb{R}$. In this situation we say that this eigenvalue is *undefined*; otherwise it is *well-defined*. Because undefined eigenvalues are meaningless from the engineering point of view, we want to exclude them from our considerations. This leads to the following definition.

Definition 2.1. For any $x \in X$, let $\lambda_{\min}(x)$ denote the smallest well-defined eigenvalue of (8), i.e.,

$$\lambda_{\min}(x) = \min\{\lambda \mid \exists w \in \mathbb{R}^n : \text{ Eq. (8) holds for } (\lambda, w) \text{ and } w \notin \ker(M(x) + M_0)\}\,;$$

This defines a function $\lambda_{\min} : X \longrightarrow \mathbb{R} \cup \{+\infty\}$.

The next proposition collects basic properties of $\lambda_{\min}(\cdot)$.

Proposition 2.2. *(a)* $\lambda_{\min}(\cdot)$ *is finite and non-negative on* X *.*

(b) For all $x \in X$*,*

$$\lambda_{\min}(x) = \inf_{u:\,(M(x)+M_0)u\neq 0} \frac{u^T K(x) u}{u^T (M(x) + M_0) u}\,.$$

(c) For all $x \in X$*,*

$$\lambda_{\min}(x) = \sup\{\lambda \mid K(x) - \lambda(M(x) + M_0) \succeq 0\}\,.$$

(d) $\lambda_{\min}(\cdot)$ *is upper semicontinuous on* X*.*

(e) Let $\varepsilon > 0$ *be fixed. Then* $\lambda_{\min}(\cdot)$ *is continuous on* $X_\varepsilon := \{x \in \mathbb{R}^m \mid x \geq \varepsilon > 0\}$*.*

(f) $-\lambda_{\min}(\cdot)$ *is quasiconvex on* X*.*

Proof. For the proof of (a) and (b) let $x \in X$ be fixed, and let $K := K(x)$ and $M := M(x) + M_0$, for simplicity. Because M is symmetric, there exists an orthonormal basis $\{v_1, \ldots, v_r\} \subset \mathbb{R}^n$ of $\mathrm{range}(M)$ where $r = \mathrm{rank}(M)$. Consider the matrix $P := (v_1 \cdots v_r) \in \mathbb{R}^{n \times r}$ consisting column-wise of the vectors v_j. We state the generalized eigenvalue problem

$$P^T K P z = \lambda P^T M P z \tag{9}$$

with $z \in \mathbb{R}^n$.

First we show that $P^T M P$ is positive definite. To see this, let $z \neq 0$ be arbitrary, and assume that $z^T P^T M P z = 0$. Because M is positive semidefinite, this implies $Pz = 0$. But the columns of P are linearly independent, and hence we arrive at $z = 0$, a contradiction. This shows that all eigenvalues of (9) are well-defined, and (as often seen) problem (9) can be equivalently written as an ordinary eigenvalue problem

$$\widetilde{K} z = \lambda z \tag{10}$$

with $\widetilde{K} := (P^T M P)^{-1/2} P^T K P (P^T M P)^{-1/2}$.

Next we prove that λ is a well-defined eigenvalue of problem (8) if and only if it is an eigenvalue of problem (9) (and thus also an eigenvalue of $\widetilde{K}$ in (10)). First, let (λ, w) be a solution of (8) with $w \notin \ker(M)$. The latter property shows that there exist $w_1 \in \ker(M)$ and

$w_2 \in \mathrm{range}(M)$, $w_2 \neq 0$, such that $w = w_1 + w_2$. Inserting $w = w_1 + w_2$ into (8) gives $Kw_1 + Kw_2 = \lambda(Mw_1 + Mw_2)$, i.e.,

$$Kw_2 = \lambda M w_2 \tag{11}$$

due to Lemma 2.1. Notice that $w_2 \neq 0$, and thus (λ, w_2) is also a solution of (8). Because $w_2 \in \mathrm{range}(M)$, there exists $z \in \mathbb{R}^r$ such that $w_2 = Pz$. Hence, (11) becomes

$$KPz = \lambda MPz,$$

and multiplication by P^T from the left shows that (λ, z) is a solution of (9).

Vice versa, let (λ, z) be a solution of (9) with $z \neq 0$. Consider $w := Pz$. Because the columns of P form a basis of $\mathrm{range}(M)$, it is $w \neq 0$ and $w \in \mathrm{range}(M)$. Through the general identity $\mathrm{range}(M)^\perp = \ker(M^T) = \ker(M)$ we see that $w \notin \ker(M)$. Moreover, as z is a solution of (9), $P^T K w = \lambda P^T M w$ which we may multiply by P from the left to end up with

$$PP^T K w = \lambda PP^T M w. \tag{12}$$

Now, Lemma 2.1 shows that $\mathrm{range}(K) \subseteqq \mathrm{range}(M)$, i.e., $Kw \in \mathrm{range}(M)$. By construction, PP^T is a projection matrix onto $\mathrm{range}(M)$, and thus (12) becomes $Kw = \lambda Mw$. (Alternatively, notice that $P^T P = I_{r\times r}$. Hence, for each $\tilde{w} = P\tilde{z} \in \mathrm{range}(M)$, $PP^T\tilde{w} = PP^T P\tilde{z} = P\tilde{z} = \tilde{w}$.) As $w \notin \ker(M)$ this proves that λ is a well-defined eigenvalue of problem (8). Because $\widetilde{K} \succeq 0$, each eigenvalue λ in (10) is nonnegative, and we are done with the proof of (a).

To finish the proof of (b), we use formulation (10) and the Rayleigh quotient to see that

$$\lambda_{\min}(x) = \inf_{z\neq 0} \frac{z^T \widetilde{K} z}{z^T z}.$$

Inserting the definition of $\widetilde{K}$, and using the substitutions $\tilde{z} := (P^T MP)^{-1/2} z$ and $w := P\tilde{z}$, we conclude

$$\begin{aligned}
\lambda_{\min}(x) &= \inf_{z\neq 0} \frac{z^T (P^T MP)^{-1/2} P^T KP (P^T MP)^{-1/2} z}{z^T z} &(13)\\
&= \inf_{\tilde{z}\neq 0} \frac{\tilde{z}^T P^T KP\tilde{z}}{\tilde{z}^T P^T MP\tilde{z}} \\
&= \inf_{w\in \mathrm{range}(M)\colon w\neq 0} \frac{w^T K w}{w^T M w}. &(14)
\end{aligned}$$

Now, for each $\tilde{u}$ with $M\tilde{u} \neq 0$ there exist $\tilde{v} \in \ker(M)$ and $\tilde{w} \in \ker(M)^\perp = \mathrm{range}(M)$ such that $\tilde{u} = \tilde{v} + \tilde{w}$. Hence, by Lemma 2.1,

$$\frac{\tilde{u}^T K \tilde{u}}{\tilde{u}^T M \tilde{u}} = \frac{\tilde{w}^T K \tilde{w}}{\tilde{w}^T M \tilde{w}}.$$

Thus we can continue (13) to (14) with

$$\lambda_{\min}(x) = \inf_{w\in \mathrm{range}(M)\colon w\neq 0} \frac{w^T K w}{w^T M w} = \inf_{u\colon Mu\neq 0} \frac{u^T K u}{u^T M u},$$

which proves (b).

(c): Let us first show the "$\geq$" part. Take an arbitrary λ satisfying $K(x) - \lambda(M(x) + M_0) \succeq 0$, i.e.,

$$u^T K(x) u - \lambda u^T (M(x) + M_0) u \geq 0 \quad \forall u \neq 0\,.$$

Consider u with $(M(x)+M_0)u \neq 0$; then we have

$$\frac{u^T K(x) u}{u^T (M(x)+M_0) u} \geq \lambda \,.$$

Because λ and u were arbitrary, we can write "inf" in front of the fraction and "sup" in front of λ and the inequality remains valid.

The proof of the "$\leq$" part is similar: Let

$$\tilde{\lambda} := \inf_{u:\,(M(x)+M_0)u\neq 0} \frac{u^T K(x) u}{u^T (M(x)+M_0) u}.$$

Then

$$\begin{aligned}
\tilde{\lambda} \leq \frac{u^T K(x) u}{u^T (M(x)+M_0) u} & \qquad \forall u : (M(x)+M_0)u \neq 0 \\
\iff u^T K u - \tilde{\lambda} u^T (M(x)+M_0) u \geq 0 & \qquad \forall u : (M(x)+M_0)u \neq 0 \\
\iff u^T K u - \tilde{\lambda} u^T (M(x)+M_0) u \geq 0 & \qquad \forall u \in \mathbb{R}^n \ \ \text{(see Lemma 2.1)} \\
\iff K(x) - \tilde{\lambda}(M(x)+M_0) \succeq 0 & \\
\iff \tilde{\lambda} \leq \sup\{\lambda \mid K(x) - \lambda(M(x)+M_0) \succeq 0\} \,. &
\end{aligned}$$

(d): Let $\bar{x} \in \mathbb{R}^m$, $\bar{x} \geq 0$, and let $\{x^k\}_k$ be an arbitrary sequence such that $x^k \to \bar{x}$. We want to show that $\limsup_{x\to\bar{x}} \lambda_{\min}(x) \leq \lambda_{\min}(\bar{x})$. Take a subsequence $\{x^k_j\}_j$ of $\{x^k\}_k$ such that

$$\lim_{j\to\infty} \lambda_{\min}(x^k_j) = \bar{\lambda} := \limsup_{x\to\bar{x}} \lambda_{\min}(x) \,.$$

By definition,

$$K(x^k_j) - \lambda_{\min}(x^k_j)(M(x^k_j) + M_0) \succeq 0 \quad \forall j$$

and, passing with j to the infinity, we get

$$K(\bar{x}) - \bar{\lambda}(M(\bar{x}) + M_0) \succeq 0 \,,$$

using the continuous dependence of $K(x)$ and $M(x)$ on x and closedness of the cone of positive semidefinite matrices. Hence

$$\bar{\lambda} \leq \sup\{\lambda \mid K(\bar{x} - \lambda(M(\bar{x}) + M_0) \succeq 0\} = \lambda_{\min}(\bar{x})$$

and we are done.

(e): By construction, $M(x) \succ 0$ for $x \in X_\varepsilon$. Then the pencil $(K(x), M(x)+M_0)$ is definite and we can apply general theory saying that the eigenvalues of (8) depend continuously on parameter x ([4, 20]).

(f): For each $u : (M(x)+M_0)u \neq 0$, the function $u \mapsto \dfrac{u^T K(x) u}{u^T (M(x)+M_0) u}$ is a linear-fractional function in $(K(x), (M(x)+M_0)$, hence a quasilinear function in variables $(K(x), (M(x)+M_0))$ (see [5]), and thus in x. Using point (b), we conclude that $-\lambda_{\min}(x)$ is quasiconvex in x, because it is the supremum of a family of quasilinear (and thus quasiconvex) functions (here we use the fact that $-\inf g(x) = \sup -g(x)$). $\square$

Remark 2.3. The projection PP^T defined in the above proof takes, in fact, a particularly simple structure. Assume that $x \in X$ is given and that $\ker(M(x)) \subset \ker(M_0)$. Denote by $\mathcal{B} \subseteqq \{1,\ldots,n\}$ the degrees of freedom associated only with elements j such that $x_j = 0$ and by $\mathcal{A}$ its complement. With $k := |\mathcal{A}|$ we assume without restriction that $\mathcal{A} = \{1,\ldots,k\}$, and $\mathcal{B} = \{k+1,\ldots,n\}$. Then $K(x)$ and $M(x)+M_0$ can be partitioned as follows:

$$K(x) = \begin{pmatrix} K_{\mathcal{A}\mathcal{A}} & K_{\mathcal{A}\mathcal{B}} \\ K_{\mathcal{B}\mathcal{A}} & K_{\mathcal{B}\mathcal{B}} \end{pmatrix}, \quad M(x)+M_0 = \begin{pmatrix} M_{\mathcal{A}\mathcal{A}} & M_{\mathcal{A}\mathcal{B}} \\ M_{\mathcal{B}\mathcal{A}} & M_{\mathcal{B}\mathcal{B}} \end{pmatrix}.$$

Clearly, $K_{\mathcal{A}\mathcal{A}} \succeq 0$; further (see Appendix A) $M_{\mathcal{A}\mathcal{A}} \succ 0$, and, by Lemma 2.1, $K_{\mathcal{A}\mathcal{B}} = K_{\mathcal{B}\mathcal{A}}^T = M_{\mathcal{A}\mathcal{B}} = M_{\mathcal{B}\mathcal{A}}^T = 0$ and $K_{\mathcal{B}\mathcal{B}} = M_{\mathcal{B}\mathcal{B}} = 0$ (as, e.g., $K_{\mathcal{B}\mathcal{B}} = \sum_{i:x_i=0} x_i K_i$). By this, each eigenvalue $\lambda_{\mathcal{A}}$ of the problem

$$K_{\mathcal{A}\mathcal{A}} w = \lambda_{\mathcal{A}} M_{\mathcal{A}\mathcal{A}} w$$

is a well-defined eigenvalue of problem (8). $\Diamond$

For a general $x \in X$ we cannot obtain more than upper semicontinuity of $\lambda_{\min}(\cdot)$ (cf. Prop. 2.2(d)). The following example shows that $\lambda_{\min}(\cdot)$ may be discontinuous at the boundary of X, when certain components of x are equal to zero.

Example 2.4. Consider the truss depicted in Figure 1. Let the truss be symmetric w.r.t. its horizontal axis, so consider only two design variables, x_1 and x_2. The corresponding stiffness and

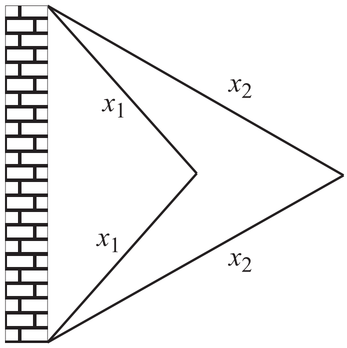

Figure 1: Example showing possible discontinuity of $\lambda_{\min}$

mass matrix have the following form (where rounded values are displayed for better illustration)

$$K(x) = \begin{pmatrix} x_1 \cdot 2 & 0 & 0 & 0 \\ 0 & x_1 \cdot 2 & 0 & 0 \\ 0 & 0 & x_2 \cdot 1.28 & 0 \\ 0 & 0 & 0 & x_2 \cdot 0.32 \end{pmatrix}$$

$$M(x) = \begin{pmatrix} x_1 \cdot 2.83 & 0 & 0 & 0 \\ 0 & x_1 \cdot 2.83 & 0 & 0 \\ 0 & 0 & x_2 \cdot 4.47 & 0 \\ 0 & 0 & 0 & x_2 \cdot 4.47 \end{pmatrix}$$

The corresponding (unordered) eigenvalues are

$$\lambda = \begin{pmatrix} \frac{2}{2.83}\frac{x_1}{x_1} \\ \frac{2}{2.83}\frac{x_1}{x_1} \\ \frac{1.28}{4.47}\frac{x_2}{x_2} \\ \frac{0.32}{4.47}\frac{x_2}{x_2} \end{pmatrix}$$

The function $\lambda_{\min}$ has then the following values

$$\begin{aligned}
\lambda_{\min}(x) &= \frac{0.32}{4.47} \approx 0.07 \quad \text{for } x_2 > 0 \\
\lambda_{\min}(x) &= \frac{2}{2.83} \approx 0.71 \quad \text{for } x_2 = 0
\end{aligned}$$

and is thus discontinuous at $x_2 = 0$. The reason for the discontinuity lies in the fact that, when $x_2 = 0$ the eigenvalue $\frac{0.32}{4.47}\frac{x_2}{x_2}$ becomes undefined and $\lambda_{\min}$ "jumps" to what was before the second smallest eigenvalue. $\Diamond$

Remark 2.5. Example 4.5 will indicate that $\lambda_{\min}(\cdot)$ may not even be Lipschitz continuous near the boundary of X. $\Diamond$

2.2 The original formulations

We first give three formulations of the truss design problem that are well-known in the engineering literature. These formulations are obtained by just "writing down" the primal requirements and natural constraints.

The minimum volume problem In the traditional formulation of the truss topology problem, one minimizes the weight of the truss subject to equilibrium conditions and constraints on the smallest eigenfrequency.

$$\min_{x\in\mathbb{R}^m,\, u\in\mathbb{R}^{n\cdot n_\ell}} \sum_{i=1}^m x_i \tag{P$_{\text{vol}}$}$$

subject to

$$\begin{aligned}
&\left(\sum_{i=1}^m x_i K_i\right) u_\ell = f_\ell, \quad \ell = 1,\dots,n_\ell \\
&f_\ell^T u_\ell \le \overline{\gamma}, \quad \ell = 1,\dots,n_\ell \\
&x_i \ge 0, \quad i = 1,\dots,m \\
&\lambda_{\min}(x) \ge \overline{\lambda}\,.
\end{aligned}$$

Here $\overline{\gamma}$ is a given upper bound on the compliance of the optimal structure and $\overline{\lambda} > 0$ is a given threshold eigenvalue. Objective function of this problem is the function $(x,u) \mapsto \sum x_i$. Notice that the eigenvalue constraint is discontinuous (see Example 2.4); this (and not only this) makes the problem rather difficult.

The minimum compliance problem In this formulation one minimizes the worst-case compliance (maximizes the stiffness) of the truss subject to equilibrium conditions and constraints on the minimum eigenfrequency.

$$\min_{x\in\mathbb{R}^m, u\in\mathbb{R}^{n\cdot n_\ell}} \max_{1\le\ell\le n_\ell} f_\ell^T u_\ell \tag{P$_{\rm compl}$}$$

subject to

$$\left(\sum_{i=1}^m x_i K_i\right) u_\ell = f_\ell, \quad \ell = 1,\dots,n_\ell$$

$$\sum_{i=1}^m x_i \le \overline{V}$$

$$x_i \ge 0, \quad i = 1,\dots,m$$

$$\lambda_{\min}(x) \ge \overline{\lambda}\,.$$

Here $\overline{V} > 0$ is an upper bound on the volume of the optimal structure and, again, $\overline{\lambda} > 0$ is a given threshold eigenvalue. For this problem, the objective function is the nonsmooth function $(x,u) \mapsto \max_{1\le\ell\le n_\ell} f_\ell^T u_\ell$. Again, notice that the eigenvalue constraint is not continuous.

The problem of maximizing the minimal eigenvalue Here we want to maximize the smallest eigenvalue of (8) subject to equilibrium conditions and constraints on the compliance and volume. Maximization of the smallest eigenfrequency is of paramount importance in many industrial application, e.g., in civil engineering.

$$\max_{x\in\mathbb{R}^m, u_\ell\in\mathbb{R}^n} \lambda_{\min}(x) \tag{P$_{\rm eig}$}$$

subject to

$$\left(\sum_{i=1}^m x_i K_i\right) u_\ell = f_\ell, \quad \ell = 1,\dots,n_\ell$$

$$f_\ell^T u_\ell \le \overline{\gamma}, \quad \ell = 1,\dots,n_\ell$$

$$\sum_{i=1}^m x_i \le \overline{V}$$

$$x_i \ge 0, \quad i = 1,\dots,m\,.$$

Here the objective function is $(x,u) \mapsto \lambda_{\min}(x)$, which is a possibly discontinuous function. This discontinuity is the reason that a standard perturbation approach widely used by practitioners for the solution of (P$_{\rm eig}$) may fail. If, with some small $\epsilon > 0$, the nonnegativity constraints are replaced by the constraints $x_i \ge \varepsilon$ for all i, and if x_ε^* denotes a solution of this perturbed problem (together with some u_ε^*), then x_ε^* may not converge to some solution x^* of the unperturbed problem (cf. Ex. 2.4 above).

We mention that each of the above three problems has already been considered in the literature with more or less small modifications, and that all problems find valuable interest in practical applications (cf. [15, 17, 11]). To the knowledge of the authors, however, a rigorous treatment of these problems in the situation of positive semidefinite matrices K and M (i.e., permitting $x_i = 0$ for some i, as needed in topology optimization) has not been considered, so far.

2.3 Interrelations of original formulations for $M_0 = 0$

In this section we study relations of the three problems (P$_{\rm vol}$), (P$_{\rm compl}$), and (P$_{\rm eig}$) when $M_0 = 0$. These relations are directly given by rescaling arguments but will also appear as special cases of problems with arbitrary M_0 treated in the next section. Note that in the following theorems we do not discuss the *existence* of solutions. Instead, we discuss their interrelations when existence is guaranteed. We start with an auxiliary result.

Lemma 2.6. *Let* $(x, u) \in \mathbb{R}^m \times \mathbb{R}^{n \cdot n_\ell}$, $x \geq 0$, *satisfy the equilibrium condition*

$$K(x)u_\ell = f_\ell \tag{15}$$

for some load vector f_ℓ. *Then* $f_\ell^T u_\ell > 0$ *and* $\sum_{i=1}^{m} x_i > 0$.

Proof. Because each of the matrices K_i is symmetric and positive semidefinite, it is clear that $f_\ell^T u_\ell = u_\ell^T K(x) u_\ell \geq 0$. Assume that $f_\ell^T u_\ell = 0$. Then $u_\ell^T K(x) u_\ell = 0$, and simple linear algebra shows that

$$K(x)u_\ell = 0_{\mathbb{R}^n} . \tag{16}$$

Eqn. (16), however, is a contradiction to the assumptions (15) and (1). If $\sum_{i=1}^{m} x_i = 0$ then $x = 0$, and the contradiction to (15) and (1) is obvious. □

Next we observe that the function $\lambda_{\min}(\,.\,)$ is independent of scaling of the structure, provided $M_0 = 0$. Recall that $\lambda_{\min}(x)$ is a well-defined non-negative number for any $x \in X$ (see Prop. 2.2(a)).

Lemma 2.7. *Let* $M_0 = 0$ *and* $x \geq 0$ *be any vector. Then*

$$\lambda_{\min}(\mu x) = \lambda_{\min}(x) \qquad \textit{for all } \mu > 0.$$

Proof. Because $K(\,\cdot\,)$ and $M(\,\cdot\,)$ are linear functions, the eigenvalue equation $K(\mu x)v = \lambda\, M(\mu x)v$ is equivalent to $K(x)v = \lambda\, M(x)v$ for all $\mu > 0$. □

We first show that each solution of $(\mathrm{P}_{\mathrm{vol}})$ immediately leads to a solution of $(\mathrm{P}_{\mathrm{compl}})$.

Theorem 2.8. *Let* $M_0 = 0$ *and* (x^*, u^*) *be a solution of* $(\mathrm{P}_{\mathrm{vol}})$.

(a) Then $\max_{1 \leq \ell \leq n_\ell} f_\ell^T u_\ell^* = \overline{\gamma}$.

(b) Put $\overline{V} := \sum_{i=1}^{m} x_i^*$ *in problem* $(\mathrm{P}_{\mathrm{compl}})$ *and copy the value of* $\overline{\lambda}$ *from problem* $(\mathrm{P}_{\mathrm{vol}})$. *Then* (x^*, u^*) *is optimal for* $(\mathrm{P}_{\mathrm{compl}})$ *with optimal objective function value* $\overline{\gamma}$.

Proof. For the proof of (a), denote

$$\gamma^* := \max_{1 \leq \ell \leq n_\ell} f_\ell^T u_\ell^* .$$

We must show that $\gamma^* = \overline{\gamma}$. Due to Lemma 2.6 we have

$$\gamma^* > 0 \qquad \text{and} \qquad V^* := \sum x_i^* > 0.$$

Consider the couple

$$(\tilde{x}^*, \tilde{u}^*) := \Big(\frac{\gamma^*}{\overline{\gamma}}\, x^*,\ \frac{\overline{\gamma}}{\gamma^*}\, u^* \Big) \,;$$

by the definition of γ^* we obtain

$$f_\ell^T \tilde{u}_\ell^* = \frac{\overline{\gamma}}{\gamma^*} f_\ell^T u_\ell^* \le \frac{\overline{\gamma}}{\gamma^*}\, \gamma^* = \overline{\gamma} \qquad \text{for all } \ell = 1, \dots, n_\ell, \tag{17}$$

and, obviously,

$$\left(\sum_{i=1}^m \tilde{x}_i^* K_i\right) \tilde{u}_\ell^* = \frac{\gamma^* \overline{\gamma}}{\overline{\gamma}\gamma^*} \left(\sum_{i=1}^m x_i^* K_i\right) u_\ell^* = f_\ell \qquad \text{for all } \ell = 1, \dots, n_\ell.$$

This, together with Lemma 2.7, shows that $(\tilde{x}^*, \tilde{u}^*)$ is feasible for ($\mathrm{P}_{\mathrm{vol}}$). Hence optimality of (x^*, u^*) in ($\mathrm{P}_{\mathrm{vol}}$) yields

$$V^* \le \sum_{i=1}^m \tilde{x}_i^* = \frac{\gamma^*}{\overline{\gamma}} \sum_{i=1}^m x_i^* = \frac{\gamma^*}{\overline{\gamma}} V^*.$$

Because $V^* > 0$, this means

$$\overline{\gamma} \le \gamma^*.$$

Eqn. (17), however, shows that $\gamma^* \le \overline{\gamma}$. All in all, we arrive at $\gamma^* = \overline{\gamma}$, as stated in (a).

Now we prove (b). Due to the choice of $\overline{V}$ it is clear that (x^*, u^*) is feasible for problem ($\mathrm{P}_{\mathrm{compl}}$). Moreover, (a) shows that the corresponding objective function value is $\overline{\gamma}$. Let (x, u) be an arbitrary feasible point of ($\mathrm{P}_{\mathrm{compl}}$). Lemma 2.6 shows that the value $\gamma := \max\limits_{1 \le \ell \le n_\ell} f_\ell^T u_\ell$ is positive, and hence the couple

$$(\tilde{x}, \tilde{u}) := \left(\frac{\gamma}{\overline{\gamma}}\, x, \frac{\overline{\gamma}}{\gamma}\, u\right)$$

is well-defined. As in (a), we conclude that $(\tilde{x}, \tilde{u})$ is feasible for ($\mathrm{P}_{\mathrm{vol}}$). Optimality of (x^*, u^*) in ($\mathrm{P}_{\mathrm{vol}}$) gives

$$\sum_{i=1}^m x_i^* \le \sum_{i=1}^m \tilde{x}_i = \frac{\gamma}{\overline{\gamma}} \sum_{i=1}^m x_i. \tag{18}$$

Now, $\sum\limits_{i=1}^m x_i^* = \overline{V}$ by the definition of $\overline{V}$, and we have $\sum\limits_{i=1}^m x_i \le \overline{V}$ by the feasibility of (x, u) for ($\mathrm{P}_{\mathrm{compl}}$). Hence (18) becomes $\overline{V} \le \frac{\gamma}{\overline{\gamma}} \overline{V}$ which in turn means that $\overline{\gamma} \le \gamma$. Thus we have shown (use (a)) that

$$\max_{1 \le \ell \le n_\ell} f_\ell^T u_\ell^* \le \max_{1 \le \ell \le n_\ell} f_\ell^T u_\ell,$$

i.e., optimality of (x^*, u^*) for problem ($\mathrm{P}_{\mathrm{compl}}$). $\square$

The first assertion of the theorem shows that, when $M_0 = 0$, the compliance constraint in ($\mathrm{P}_{\mathrm{vol}}$) is always active for at least one load case. Later we will demonstrate this theorem by means of a numerical example (see Ex. 4.1).

A completely analogous theorem to Thm. 2.8 can be stated when problems ($\mathrm{P}_{\mathrm{vol}}$) and ($\mathrm{P}_{\mathrm{compl}}$) are interchanged. The proof uses the same arguments and is thus omitted.

Theorem 2.9. *Let $M_0 = 0$ and let (x^*, u^*) be a solution of* ($\mathrm{P}_{\mathrm{compl}}$).

(a) Then $\sum\limits_{i=1}^m x_i^* = \overline{V}$.

(b) Put $\overline{\gamma} := \max\limits_{1 \le \ell \le n_\ell} f_\ell^T u_\ell^*$ in problem ($\mathrm{P}_{\mathrm{vol}}$) *and copy $\overline{\lambda}$ from* ($\mathrm{P}_{\mathrm{compl}}$). *Then (x^*, u^*) is optimal for* ($\mathrm{P}_{\mathrm{vol}}$) *with optimal objective function value $\overline{V}$.*

The interrelations of ($\mathrm{P}_{\mathrm{vol}}$) (resp., of ($\mathrm{P}_{\mathrm{compl}}$)) and ($\mathrm{P}_{\mathrm{eig}}$) are a bit more cumbersome because the objective function ($\mathrm{P}_{\mathrm{eig}}$) is invariant with respect to scaling, as shown in Lemma 2.7. As a first and simple result, we obtain the following proposition (where all sums run over $i = 1, \dots, m$).

Proposition 2.10. *Let $M_0 = 0$, and let (x^*, u^*) be a solution of problem* ($\mathrm{P}_{\mathrm{eig}}$).

(a) Then for each

$$\mu \in \Big[\frac{\sum x_i^*}{\overline{V}}; \frac{\overline{\gamma}}{\max\limits_{1\le\ell\le n_\ell} f_\ell^T u_\ell^*}\Big] \tag{19}$$

the couple $(\frac{1}{\mu}x^*, \mu u^*)$ is also a solution of ($\mathrm{P}_{\mathrm{eig}}$).

(b) In particular,

$$\Big(\frac{\overline{V}}{\sum x_i^*}\, x^*,\ \frac{\sum x_i^*}{\overline{V}}\, u^*\Big)$$

is also a solution of ($\mathrm{P}_{\mathrm{eig}}$) *where the volume constraint is attained as an equality.*

(c) Analogously to (b),

$$\Big(\frac{\max\limits_{1\le\ell\le n_\ell} f_\ell^T u_\ell^*}{\overline{\gamma}}\, x^*,\ \frac{\overline{\gamma}}{\max\limits_{1\le\ell\le n_\ell} f_\ell^T u_\ell^*}\, u^*\Big)$$

is also a solution of ($\mathrm{P}_{\mathrm{eig}}$) *where the compliance constraint is attained as an equality for at least one load case ℓ.*

Proof. First, feasibility of (x^*, u^*) in ($\mathrm{P}_{\mathrm{eig}}$) and Lemma 2.6 yield

$$0 < \frac{\sum x_i^*}{\overline{V}} \le 1 \le \frac{\overline{\gamma}}{\max\limits_{1\le\ell\le n_\ell} f_\ell^T u_\ell^*},$$

and hence the interval in (19) is well-defined and non-empty. Moreover, it is straightforward to see that

$$\sum \frac{1}{\mu} x_i^* \le \overline{V} \qquad \text{and} \qquad f_\ell^T u_\ell^* \le \overline{\gamma} \quad \text{for all } \ell = 1, \dots, n_\ell$$

hold if and only if μ satisfies (19). Thus for each μ from (19), the point $(\frac{1}{\mu}x^*, \mu u^*)$ is feasible in problem ($\mathrm{P}_{\mathrm{eig}}$). Hence Lemma 2.7 shows that it is even an optimal solution. Assertions (b) and (c) are straightforward consequences of (a). ☐

This proposition relies on the fact that, for $M_0 = 0$, $\lambda_{\min}(\cdot)$ is invariant with respect to scaling of the structure. Hence, if either the volume constraint or the compliance constraints are inactive at the optimum, the optimal structure can be scaled without changing the value of the objective function $\lambda_{\min}(\cdot)$. This shows that (for $M_0 = 0$) problem ($\mathrm{P}_{\mathrm{eig}}$) rather looks for an optimal "shape" of the structure independently of the appropriate scaling. Later in Section 4 will see a numerical example illustrating Prop. 2.10 (see Ex. 4.3).

2.4 Interrelations of original formulations for arbitrary M_0

In this section, we do not make any restrictions on M_0 apart from the general requirements already mentioned, i.e., that M_0 is symmetric and positive semidefinite. In the following, when relating two different optimization problems, the matrix M_0 is considered to be the same in both problems.

We start with a general result on the relation of optimization problems where the objective function of one problem acts as a constraint of the other one and vice versa. Through this result we will then be able to state all interrelationships of the formulations ($\mathrm{P}_{\mathrm{vol}}$), ($\mathrm{P}_{\mathrm{compl}}$), and ($\mathrm{P}_{\mathrm{eig}}$).

Theorem 2.11. *Let $Y \subseteqq \mathbb{R}^k$ be non-empty, and let the functions $f_1, f_2 : Y \longrightarrow \mathbb{R}$ be given. For $\overline{f}_1, \overline{f}_2 \in \mathbb{R}$ define the two optimization problems*

$$\min_{y \in Y} \{ f_1(y) \mid f_2(y) \leq \overline{f}_2 \} \tag{$\mathrm{P}_1[\overline{f}_2]$}$$

and

$$\min_{y \in Y} \{ f_2(y) \mid f_1(y) \leq \overline{f}_1 \}. \tag{$\mathrm{P}_2[\overline{f}_1]$}$$

Let $\overline{f}_2$ be fixed and the set Y_1^* of solutions to problem $(\mathrm{P}_1[\overline{f}_2])$ be non-empty. The optimal function value is denoted by

$$f_1^* := f_1(y^*) \qquad \text{for all } y^* \in Y_1^*.$$

Put

$$f_2^* := \inf\{ f_2(y^*) \mid y^* \in Y_1^* \}, \tag{20}$$

and let the infimum be attained at some $\hat{y}^* \in Y_1^*$. Consider problem $(\mathrm{P}_2[\overline{f}_1])$ with $\overline{f}_1 := f_1^*$.

Then $\hat{y}^*$ is optimal for problem $(\mathrm{P}_2[\overline{f}_1])$ with optimal objective function value f_2^*.

Proof. Optimality, and hence feasibility, of $\hat{y}^*$ for $(\mathrm{P}_1[\overline{f}_2])$ shows that this point is also feasible for $(\mathrm{P}_2[\overline{f}_1])$ due to the definition of $\overline{f}_1 := f_1^*$. By the choice of $\hat{y}^*$, the value of the objective function of $\hat{y}^*$ in $(\mathrm{P}_2[\overline{f}_1])$ is f_2^*. Now, let y be an arbitrary feasible point of $(\mathrm{P}_2[\overline{f}_1])$ with

$$f_2(y) \leq f_2^*. \tag{21}$$

We must prove that $f_2(y) \geq f_2^*$.

First, the choice of $\hat{y}^*$ shows that

$$f_2^* = f_2(\hat{y}^*) \leq \overline{f}_2.$$

Hence, using (21), we see that

$$f_2(y) \leq \overline{f}_2.$$

Thus, due to feasibility of y in $(\mathrm{P}_2[\overline{f}_1])$, it is clear that (x, u) is also feasible for $(\mathrm{P}_1[\overline{f}_2])$. The definition of $\overline{f}_1$ and the optimality of $\hat{y}^*$ for $(\mathrm{P}_1[\overline{f}_2])$ show that

$$\overline{f}_1 = f_1^* = f_1(\hat{y}^*) \leq f_1(y). \tag{22}$$

The feasibility of (x, u) for $(\mathrm{P}_2[\overline{f}_1])$, however, shows that

$$f_1(y) \leq \overline{f}_1$$

which together with (22) and with the definition $\overline{f}_1 := f_1^*$ proves

$$f_1(y) = f_1^*.$$

We conclude that y is optimal for $(\mathrm{P}_1[\overline{f}_2])$, i.e., $y \in Y_1^*$. Hence, by the definition of f_2^*,

$$f_2(y) \geq f_2^*,$$

and the proof is complete. $\square$

Now we collect certain tools which are needed to show that the infimum in (20) is attained in all situations. For this, we define the function

$$\begin{array}{rcl} c:\ \{x \in \mathbb{R}^m \mid x \geq 0\} & \longrightarrow & \mathbb{R} \cup \{+\infty\}, \\ x & \mapsto & \sup\limits_{1 \leq \ell \leq n_\ell} \sup\limits_{u_\ell \in \mathbb{R}^n} \left\{ 2 f_\ell^T u_\ell - u_\ell^T \Big(\sum\limits_{i=1}^m x_i K_i \Big) u_\ell \right\}. \end{array}$$

Obviously, the function c denotes the maximum (over all load cases) of the negative minimum potential energies of the structure x.

Proposition 2.12 (Properties of the function c)**.**

(a) Let $x \geq 0$. Then $c(x) < +\infty$ if and only if there exist "displacement vectors" $u_1, \ldots, u_{n_\ell} \in \mathbb{R}^n$ such that

$$K(x) u_\ell = f_\ell \qquad \textit{for all } \ell = 1, \ldots, n_\ell. \tag{23}$$

(b) Let $x \geq 0$. If $c(x) < +\infty$ then

$$c(x) = \max_{1 \leq \ell \leq n_\ell} f_\ell^T u_\ell$$

for all $u_1, \ldots, u_{n_\ell}$ which satisfy (23).

(c) The function $c(\cdot)$ is finite and continuous on the set $\{x \in \mathbb{R}^m \mid x > 0\}$ and lower semi-continuous (l.s.c.) on $\{x \mid x \geq 0\}$, i.e.,

$$\liminf_{\substack{x \to \bar{x} \\ x \geq 0}} c(x) \geq c(\bar{x}), \qquad \bar{x} \geq 0.$$

Proof. All assertions were proved in [1]. Assertions (a) and (b), however, are easily deduced from the necessary and sufficient conditions of the inner sup-problems over u_ℓ and from the fact that a convex quadratic function is unbounded if and only if it does not possess a stationary point. Concerning (c), we mention that the finiteness of c on $\{x \mid x > 0\}$ is based on assumption (5), and that c possesses much stronger continuity properties than just being l.s.c. on $\{x \mid x \geq 0\}$ (see [1]). $\square$

For simplification of notation, we define

$$\mathrm{vol}(x) := \sum_{i=1}^m x_i$$

for $x \in \mathbb{R}^m$, $x \geq 0$. Moreover, we define

$$\mathcal{S}^*_{\mathrm{vol}}, \mathcal{S}^*_{\mathrm{compl}}, \mathcal{S}^*_{\mathrm{eig}} \subset \{x \in \mathbb{R}^m \mid x \geq 0\} \times \mathbb{R}^{n \cdot n_\ell}$$

as the solution sets of the problems ($\mathrm{P_{vol}}$), ($\mathrm{P_{compl}}$), and ($\mathrm{P_{eig}}$), respectively. Notice that these sets may well be empty.

Our first result based on Thm. 2.11 relates problem ($\mathrm{P_{vol}}$) with the problems ($\mathrm{P_{compl}}$) and ($\mathrm{P_{eig}}$), respectively.

Theorem 2.13. *Let $\mathcal{S}^*_{\mathrm{vol}}$ be non-empty. Denote the optimal function value of problem* ($\mathrm{P_{vol}}$) *by V^*, i.e.,*

$$V^* := \sum_{i=1}^m x_i^* \qquad \textit{for all } (x^*, u^*) \in \mathcal{S}^*_{\mathrm{vol}}.$$

Put

$$\gamma^* := \inf\left\{ \max_{1\le\ell\le n_\ell} f_\ell^T u_\ell^* \,\middle|\, (x^*, u^*) \in \mathcal{S}^*_{\text{vol}} \right\}, \tag{24}$$

and

$$\lambda^* := \sup\left\{ \lambda_{\min}(x^*) \,\middle|\, (x^*, u^*) \in \mathcal{S}^*_{\text{vol}} \right\}. \tag{25}$$

Then the following assertions hold:

(a) *The infimum in (24) is attained at some $(\hat{x}^*, \hat{u}^*) \in \mathcal{S}^*_{\text{vol}}$. Moreover, with $\overline{V} := V^*$, and with $\overline{\lambda}$ copied from problem $(\mathrm{P}_{\text{vol}})$, the point $(\hat{x}^*, \hat{u}^*)$ is optimal for problem $(\mathrm{P}_{\text{compl}})$ with optimal objective function value γ^*.*

(b) *The supremum in (25) is attained at some $(\tilde{x}^*, \tilde{u}^*) \in \mathcal{S}^*_{\text{vol}}$. Moreover, with $\overline{V} := V^*$, and with $\overline{\gamma}$ copied from problem $(\mathrm{P}_{\text{vol}})$, the point $(\tilde{x}^*, \tilde{u}^*)$ is optimal for problem $(\mathrm{P}_{\text{eig}})$ with optimal objective function value λ^*.*

Proof. Consider the set

$$\mathcal{X}^*_{\text{vol}} := \{\, x^* \mid (x^*, u^*) \in \mathcal{S}^*_{\text{vol}} \}.$$

Using Prop. 2.12(a) and (b) it is easy to see that

$$\mathcal{X}^*_{\text{vol}} = \left\{ x \ge 0 \,\middle|\, \text{vol}(x) = V^*,\ c(x) \le \overline{\gamma},\ \lambda_{\min}(x) \ge \overline{\lambda} \right\}. \tag{26}$$

Because $x \ge 0$ and $\text{vol}(x) = V^*$ for all $x \in \mathcal{X}^*_{\text{vol}}$, the set $\mathcal{X}^*_{\text{vol}}$ is bounded. Moreover, because $\text{vol}(\cdot)$ is continuous, $\lambda_{\min}(\cdot)$ is u.s.c. (see Prop. 2.2(d)), and $c(\cdot)$ is l.s.c. (see Prop. 2.12(c)), the set $\mathcal{X}^*_{\text{vol}}$ is closed. All in all, $\mathcal{X}^*_{\text{vol}}$ is a compact set.

We first prove (a). Proposition 2.12(a) and (b) shows that

$$\gamma^* = \inf\{\, c(x) \mid x \in \mathcal{X}^*_{\text{vol}} \,\}, \tag{27}$$

and that the infimum in (24) is attained if and only if the infimum in (27) is attained. The latter, however, is straightforward because $c(\cdot)$ is a l.s.c. function, and $\mathcal{X}^*_{\text{vol}}$ is a compact set (each l.s.c. function attains its infimum on a compact set; see, e.g., [13, Thm. 2.13.1]). The rest of the assertion follows directly from Thm. 2.11 with the settings

$$\begin{array}{rcll}
Y & := & \{\, (x,u) \in \mathbb{R}^m \times \mathbb{R}^{n \cdot n_\ell} \mid & K(x)u_\ell = f_\ell, \quad \ell = 1,\ldots,n_\ell, \\
 & & & x_i \ge 0, \qquad\quad\ \ i = 1,\ldots,m, \\
 & & & \lambda_{\min}(x) \ge \overline{\lambda} \qquad\qquad\qquad\quad \}, \\
f_1(x,u) & := & \text{vol}(x), & \\
f_2(x,u) & := & c(x), & \\
\overline{f}_2 & := & \overline{\gamma}, & \\
\overline{f}_1 & := & V^*. &
\end{array}$$

The proof of (b) is analogous. We have to show that the supremum

$$\lambda^* = \sup\{\, \lambda_{\min}(x) \mid x \in \mathcal{X}^*_{\text{vol}} \,\}$$

is attained at some $\tilde{x}^*$. This is the case because $\lambda_{\min}(\cdot)$ is u.s.c. (see Prop. 2.2(d)) and $\mathcal{X}^*_{\text{vol}}$ is compact (see above). Notice that $c(\tilde{x}^*) \le \overline{\gamma} < +\infty$ (see (26)), and hence corresponding vectors

$\tilde{u}_1^*, \ldots, \tilde{u}_{e_\ell}^*$ exist by Prop. 2.12(a) and (b) such that $(\tilde{x}^*, \tilde{u}^*)$ is feasible (and optimal) for ($\mathrm{P}_{\mathrm{vol}}$)). The rest of assertion (b) follows directly from Thm. 2.11 with the settings

$$
\begin{array}{rcll}
Y & := & \{\, (x,u) \in \mathbb{R}^m \times \mathbb{R}^{n \cdot n_\ell} \mid & K(x)u_\ell = f_\ell, \quad \ell = 1,\ldots,n_\ell, \\
& & & x_i \geq 0, \qquad\quad\;\; i = 1,\ldots,m, \\
& & & f_\ell^T u_\ell \leq \overline{\gamma} \qquad\quad \ell = 1,\ldots,n_\ell \quad \}, \\
f_1(x,u) & := & \mathrm{vol}(x), & \\
f_2(x,u) & := & -\lambda_{\min}(x), & \\
\overline{f}_2 & := & -\overline{\lambda}, & \\
\overline{f}_1 & := & V^*. &
\end{array}
$$

□

Theorem 2.13(a) reflects the fact that at some solution (x^*, u^*) of ($\mathrm{P}_{\mathrm{vol}}$) none of the compliance constraints may be satisfied with equality, and hence the "post-optimization" in (24) is needed to select a proper solution of ($\mathrm{P}_{\mathrm{vol}}$) to obtain a solution of ($\mathrm{P}_{\mathrm{compl}}$). Theorem 2.13(a) also shows that—with the appropriate settings of $\overline{V}$ and $\overline{\lambda}$—there is always a structure x^* which is optimal for *both* problems at the same time (provided there exists a solution at all). Analogous comments, of course, can be made for Thm. 2.13(b) concerning solutions of ($\mathrm{P}_{\mathrm{eig}}$). A numerical example illustrating Thm. 2.13 is given in Section 4 (Ex. 4.4).

Theorem 2.13 substantially simplifies in the following special situation.

Corollary 2.14. *Let the set $\mathcal{X}_{\mathrm{vol}}^* = \{x^* \mid (x^*, u^*) \in \mathcal{S}_{\mathrm{vol}}^*\}$ be a singleton. Then the following assertions hold:*

(a) Put $\overline{V} := \mathrm{vol}(x^*)$ in problem ($\mathrm{P}_{\mathrm{compl}}$) and copy the value $\overline{\lambda}$ from problem ($\mathrm{P}_{\mathrm{vol}}$). Then (x^*, u^*) is optimal for problem ($\mathrm{P}_{\mathrm{compl}}$) with optimal objective function value $\max\limits_{1 \leq \ell \leq n_\ell} f_\ell^T u_\ell^*$.

(b) Put $\overline{V} := \mathrm{vol}(x^*)$ in problem ($\mathrm{P}_{\mathrm{eig}}$) and copy the value $\overline{\gamma}$ from problem ($\mathrm{P}_{\mathrm{vol}}$). Then (x^*, u^*) is optimal for problem ($\mathrm{P}_{\mathrm{eig}}$) with optimal objective function value $\lambda_{\min}(x^*)$.

Proof. If $\mathcal{X}_{\mathrm{vol}}^* = \{x^*\}$ then the infimum in (24) is attained at any $(x^*, u^*) \in \mathcal{S}_{\mathrm{vol}}^*$ because for each $u^*, \tilde{u}^*$ with $K(x^*)u_\ell^* = K(x^*)\tilde{u}_\ell^* = f_\ell$ for all ℓ the compliance values

$$f_\ell^T u_\ell^* = \tilde{u}_\ell^{*T} K(x^*) u_\ell^* = \tilde{u}_\ell^{*T} f_\ell = f_\ell^T \tilde{u}_\ell^*, \qquad \ell = 1,\ldots,n_\ell$$

are constant. Because $\mathcal{X}_{\mathrm{vol}}^*$ is the singleton x^*, and because $\lambda_{\min}(\cdot)$ does not depend on u^*, it is trivial to see that the supremum in (25) is attained at each $(x^*, u^*) \in \mathcal{S}_{\mathrm{vol}}^*$. Now apply Thm. 2.13. □

Remark 2.15. Theorem 2.13(a) generalizes Thm. 2.8(b) of the previous chapter. If $M_0 = 0$ in Thm. 2.13(a) then Thm. 2.8(a) shows that

$$\max_{1 \leq \ell \leq n_\ell} f_\ell^T u_\ell^* = \overline{\gamma} \qquad \text{for all } (x^*, u^*) \in \mathcal{S}_{\mathrm{vol}}^*.$$

Hence $\gamma^* = \overline{\gamma}$, and the infimum in (24) is attained at each solution $(x^*, u^*) \in \mathcal{S}_{\mathrm{vol}}^*$. Similar comment cannot be made for Thm. 2.13(b). The setting $M_0 = 0$ does not guarantee that for each solution (x^*, u^*) of ($\mathrm{P}_{\mathrm{vol}}$) the eigenvalue constraint is attained as an equality. This will also be demonstrated by Example 4.4 below. The background lies in the invariance of $\lambda_{\min}(\cdot)$ w.r.t. scaling of the structure; see Lemma 2.7. ◊

Analogously to Thm. 2.13, we may derive solutions of problems ($\mathrm{P}_{\mathrm{vol}}$) and ($\mathrm{P}_{\mathrm{eig}}$), respectively, from solutions of problem ($\mathrm{P}_{\mathrm{compl}}$).

Theorem 2.16. *Let $\mathcal{S}^*_{\mathrm{compl}}$ be non-empty. Denote the optimal function value of problem* ($\mathrm{P}_{\mathrm{compl}}$) *by γ^*, i.e.,*

$$\gamma^* := \max_{1\le \ell \le n_\ell} f_\ell^T u_\ell^* \qquad \textit{for all } (x^*, u^*) \in \mathcal{S}^*_{\mathrm{compl}}.$$

Put

$$V^* := \inf \Big\{ \sum_{i=1}^m x_i^* \,\Big|\, (x^*, u^*) \in \mathcal{S}^*_{\mathrm{compl}} \Big\}, \tag{28}$$

and

$$\lambda^* := \sup \Big\{ \lambda_{\min}(x^*) \,\Big|\, (x^*, u^*) \in \mathcal{S}^*_{\mathrm{compl}} \Big\}. \tag{29}$$

Then the following assertions hold:

(a) The infimum in (28) is attained at some $(\hat{x}^*, \hat{u}^*) \in \mathcal{S}^*_{\mathrm{compl}}$. Moreover, with $\overline{\gamma} := \gamma^*$, and with $\overline{\lambda}$ copied from problem ($\mathrm{P}_{\mathrm{compl}}$)*, the point $(\hat{x}^*, \hat{u}^*)$ is optimal for problem* ($\mathrm{P}_{\mathrm{vol}}$) *with optimal objective function value V^*.*

(b) The supremum in (29) is attained at some $(\tilde{x}^*, \tilde{u}^*) \in \mathcal{S}^*_{\mathrm{compl}}$. Moreover, with $\overline{\gamma} := \gamma^*$, and with $\overline{V}$ copied from problem ($\mathrm{P}_{\mathrm{compl}}$)*, the point $(\tilde{x}^*, \tilde{u}^*)$ is optimal for problem* ($\mathrm{P}_{\mathrm{eig}}$) *with optimal objective function value λ^*.*

Proof. We modify the proof of Thm. 2.13. Consider the set

$$\mathcal{X}^*_{\mathrm{compl}} := \{ x^* \mid (x^*, u^*) \in \mathcal{S}^*_{\mathrm{compl}} \}.$$

In view of Prop. 2.12(a) and (b) it is easy to see that

$$\mathcal{X}^*_{\mathrm{compl}} = \Big\{ x \ge 0 \,\Big|\, \mathrm{vol}(x) \le \overline{V},\ c(x) = \gamma^*,\ \lambda_{\min}(x) \ge \overline{\lambda} \Big\}. \tag{30}$$

Because γ^* is the optimal objective function value, there is no $x \ge 0$ such that $\mathrm{vol}(x) \le \overline{V}$, $c(x) < \gamma^*$, and $\lambda_{\min}(x) \ge \overline{\lambda}$. Hence the set $\mathcal{X}^*_{\mathrm{compl}}$ remains unchanged if we change the equality sign in "$c(x) = \gamma^*$" to an inequality sign:

$$\mathcal{X}^*_{\mathrm{compl}} = \Big\{ x \ge 0 \,\Big|\, \mathrm{vol}(x) \le \overline{V},\ c(x) \le \gamma^*,\ \lambda_{\min}(x) \ge \overline{\lambda} \Big\}. \tag{31}$$

Because $x \ge 0$ and $\mathrm{vol}(x) \le \overline{V}$ for all $x \in \mathcal{X}^*_{\mathrm{compl}}$, the set $\mathcal{X}^*_{\mathrm{compl}}$ is bounded. Moreover, each of the functions $\mathrm{vol}(\cdot)$, $-\lambda_{\min}(\cdot)$, and $c(\cdot)$ is l.s.c. (see Props. 2.2(d) and 2.12(c)). Hence the description (31) shows that $\mathcal{X}^*_{\mathrm{compl}}$ is a closed set, and thus $\mathcal{X}^*_{\mathrm{compl}}$ is compact (notice that the *level line* of a l.s.c. function $f(\cdot)$ for some value α, i.e., the set $\{y \mid f(y) = \alpha\}$, needs not be closed, but the *level set* $\{y \mid f(y) \le \alpha\}$ is always closed).

First we prove (a). Obviously, the infimum in (28) is attained because $\mathcal{X}^*_{\mathrm{compl}}$ is a compact set and $\mathrm{vol}(\cdot)$ is continuous. Now apply Thm. 2.11 with the settings

$$\begin{array}{rcl}
Y & := & \{ (x,u) \in \mathbb{R}^m \times \mathbb{R}^{n \cdot n_\ell} \mid \begin{array}{ll} K(x)u_\ell = f_\ell, & \ell = 1, \ldots, n_\ell, \\ x_i \ge 0, & i = 1, \ldots, m, \\ \lambda_{\min}(x) \ge \overline{\lambda} & \end{array} \}, \\
f_1(x,u) & := & \max\limits_{1 \le \ell \le n_\ell} f_\ell^T u_\ell, \\
f_2(x,u) & := & \mathrm{vol}(x), \\
\overline{f}_2 & := & \overline{V}, \\
\overline{f}_1 & := & \gamma^*.
\end{array}$$

The proof of (b) is analogous to that of Thm. 2.13(b). $\square$

The following corollary parallels Cor. 2.14. Its proof is even simpler because neither $\mathrm{vol}(\cdot)$ nor $\lambda_{\min}(\cdot)$ in (28) and (29), respectively, depend on u^*.

Corollary 2.17. *Let the set $\mathcal{X}^*_{\text{compl}} = \{x^* \mid (x^*, u^*) \in \mathcal{S}^*_{\text{compl}}\}$ be a singleton. Then the following assertions hold:*

(a) Put $\overline{\gamma} := \max_{1\le\ell\le n_\ell} f_\ell^T u_\ell^*$ in problem $(\mathrm{P_{vol}})$ and copy the value $\overline{\lambda}$ from problem $(\mathrm{P_{compl}})$. Then (x^*, u^*) is optimal for problem $(\mathrm{P_{vol}})$ with optimal objective function value $\mathrm{vol}(x^*)$.

(b) Put $\overline{\gamma} := \max_{1\le\ell\le n_\ell} f_\ell^T u_\ell^*$ in problem $(\mathrm{P_{eig}})$ and copy the value $\overline{V}$ from problem $(\mathrm{P_{compl}})$. Then (x^*, u^*) is optimal for problem $(\mathrm{P_{eig}})$ with optimal objective function value $\lambda_{\min}(x^*)$.

Remark 2.18. Similarly as in Remark 2.15, Thm. 2.16(a) generalizes Thm. 2.9(b) of the previous section. If $M_0 = 0$ in Thm. 2.16(a) then Thm. 2.9(a) shows that

$$\mathrm{vol}(x^*) = \overline{V} \qquad \text{for all } (x^*, u^*) \in \mathcal{S}^*_{\text{compl}}.$$

Hence $V^* = \overline{V}$, and the infimum in (28) is attained at each solution $(x^*, u^*) \in \mathcal{S}^*_{\text{compl}}$. $\Diamond$

Finally, we may derive solutions of problems $(\mathrm{P_{vol}})$ and $(\mathrm{P_{compl}})$ from solutions of $(\mathrm{P_{eig}})$.

Theorem 2.19. *Let $\mathcal{S}^*_{\text{eig}}$ be non-empty. Denote the optimal function value of problem $(\mathrm{P_{eig}})$ by λ^*, i.e.,*

$$\lambda^* := \lambda_{\min}(x^*) \qquad \textit{for all } (x^*, u^*) \in \mathcal{S}^*_{\text{eig}}.$$

Put

$$V^* := \inf\Big\{ \sum_{i=1}^m x_i^* \,\Big|\, (x^*, u^*) \in \mathcal{S}^*_{\text{eig}} \Big\}, \tag{32}$$

and

$$\gamma^* := \inf\Big\{ \max_{1\le\ell\le n_\ell} f_\ell^T u_\ell^* \,\Big|\, (x^*, u^*) \in \mathcal{S}^*_{\text{eig}} \Big\}. \tag{33}$$

Then the following assertions hold:

(a) The infimum in (32) is attained at some $(\hat{x}^*, \hat{u}^*) \in \mathcal{S}^*_{\text{eig}}$. Moreover, with $\overline{\lambda} := \lambda^*$, and with $\overline{\gamma}$ copied from problem $(\mathrm{P_{eig}})$, the point $(\hat{x}^*, \hat{u}^*)$ is optimal for problem $(\mathrm{P_{vol}})$ with optimal objective function value V^*.

(b) The infimum in (33) is attained at some $(\tilde{x}^*, \tilde{u}^*) \in \mathcal{S}^*_{\text{eig}}$. Moreover, with $\overline{\lambda} := \lambda^*$, and with $\overline{V}$ copied from problem $(\mathrm{P_{eig}})$, the point $(\tilde{x}^*, \tilde{u}^*)$ is optimal for problem $(\mathrm{P_{compl}})$ with optimal objective function value γ^*.

Proof. The proof of this Theorem is analogous to that of Thm. 2.13 with the role of the functions $\mathrm{vol}(\cdot)$ and $\lambda_{\min}(\cdot)$ interchanged. $\square$

For illustration of this theorem, we refer to Example 4.3. The proof of the following corollary is analogous to that of Cor. 2.14.

Corollary 2.20. *Let the set $\mathcal{X}^*_{\text{eig}} = \{x^* \mid (x^*, u^*) \in \mathcal{S}^*_{\text{eig}}\}$ be a singleton. Then the following assertions hold:*

(a) Put $\overline{\lambda} := \lambda_{\min}(x^*)$ in problem $(\mathrm{P_{vol}})$ and copy the value $\overline{\gamma}$ from problem $(\mathrm{P_{eig}})$. Then (x^*, u^*) is optimal for problem $(\mathrm{P_{vol}})$ with optimal objective function value $\mathrm{vol}(x^*)$.

(b) Put $\overline{\lambda} := \lambda_{\min}(x^*)$ *in problem* $(\mathrm{P}_{\mathrm{compl}})$ *and copy the value* $\overline{V}$ *from problem* $(\mathrm{P}_{\mathrm{eig}})$*. Then* (x^*, u^*) *is optimal for problem* $(\mathrm{P}_{\mathrm{compl}})$ *with optimal objective function value* $\max\limits_{1\le\ell\le n_\ell} f_\ell^T u_\ell^*$.

To conclude this theoretical study of relations of the three original problem formulations we would like to give a few comments on their practical use. Obviously, a direct implementation of one of the Theorems 2.13, 2.16, and 2.19 for numerical purposes is difficult because one would need to know the set of *all* solutions to one of the problems, or one should be able to solve the inf- or sup-problems on the optimal set. There are ways to do this, as has been recently shown in [8]. However, as we will see in Section 3, there is no need to proceed from a solution of one (nonlinear!) problem to the solution of some other problem, because global solutions of some of the original problems can be calculated through equivalent (quasi)convex problem formulations.

2.5 Brief discussion on the variation of M_0

In this section we want to briefly prove what is widely known among practicioners: what happens when the non-structural mass is changed or even removed. For example, if volume minimization is considered then a bigger non-structural mass will generally increase the optimal volume. Similarly, if maximization of the minimal eigenvalue is considered, the removal of the non-structural mass will generally lead to a smaller minimal eigenvalue. Hence, in this section, we briefly consider the variation of M_0 and use the extended notation (see Prop. 2.2(c))

$$\lambda_{\min}(x, M_0) := \sup\{\lambda \mid K(x) - \lambda(M(x) + M_0) \succeq 0\}. \tag{34}$$

Lemma 2.21. *Let* $x \ge 0$*, and let* $\widetilde{M}_0, M_0 \in \mathbb{R}^{n\times n}$ *be symmetric with* $\widetilde{M}_0 \succeq M_0 \succeq 0$*. Then* $\lambda_{\min}(x, \widetilde{M}_0) \le \lambda_{\min}(x, M_0)$*.*

Proof. Put $\tilde{\lambda} := \lambda_{\min}(x, \widetilde{M}_0)$. Then

$$\begin{aligned} 0 \ &\preceq\ K(x) - \tilde{\lambda}(M(x) + \widetilde{M}_0) \ =\ K(x) - \tilde{\lambda}M(x) - \tilde{\lambda}\widetilde{M}_0 \\ &\preceq\ K(x) - \tilde{\lambda}M(x) - \tilde{\lambda}M_0 \ =\ K(x) - \tilde{\lambda}(M(x) + M_0). \end{aligned}$$

Hence,

$$\tilde{\lambda} \le \sup\{\lambda \mid K(x) - \lambda(M(x) + M_0) \succeq 0\} = \lambda_{\min}(x, M_0).$$

□

As a simple conclusion concerning the optimal objective function values of our three problems we obtain

Proposition 2.22. *Consider two problems of the type* $(\mathrm{P}_{\mathrm{vol}})$ *(or* $(\mathrm{P}_{\mathrm{compl}})$ *or* $(\mathrm{P}_{\mathrm{eig}})$*), with the same constraint bounds* $\overline{\gamma}$ *and* $\overline{\lambda}$ *(resp.* $\overline{V}$ *and* $\overline{\lambda}$*, resp.* $\overline{V}$ *and* $\overline{\gamma}$*) but with different non-structural mass matrices* $M_0, \widetilde{M}_0$ *where* $\widetilde{M}_0 \succeq M_0$*. Let both problems possess a solution, and denote the optimal objective function values by* $V^*, \widetilde{V}^*$ *(resp.* $\gamma^*, \widetilde{\gamma}^*$*, resp.* $\lambda^*, \widetilde{\lambda}^*$*). Then* $V^* \ge \widetilde{V}^*$ *(resp.* $\gamma^* \ge \widetilde{\gamma}^*$*, resp.* $\lambda^* \le \widetilde{\lambda}^*$*).*

Proof. Consider the pair of minimum volume problems. Notice that each feasible point (x, u) of problem $(\mathrm{P}_{\mathrm{vol}})$ with non-structural mass $\widetilde{M}_0$ is also feasible for the problem with non-structural mass M_0 due to Lemma 2.21(a). Hence, $\tilde{V}^* \le V^*$.

The proof for the pair of min-max compliance problems is analogous. For the pair of max-min eigenvalue problems it is even simpler, because the set of feasible points is the same for both problems, and Lemma 2.21(a) applies directly on the objective function values. (Notice that for this type of problems, we are *max*imizing, and thus we have "$\le$" in the assertion.) □

More detailed results than in the above proposition can hardly be obtained, apart from the effect of simple joint scalings of the bounds $\overline{V}, \overline{\gamma}, \overline{\lambda}$ and M_0. Because the total mass matrix in the problem is $(M(x) + M_0)$, a pure change of only M_0 always has nonlinear impact in the problem, and hence, is difficult to describe. As a consequence, the optimal topology changes as well with a change of M_0. Such a numerical example is presented Section 4 (see Ex. 4.6).

3 SDP reformulations

All the original formulations are nonconvex, some even discontinuous. Furthermore, all of them implicitly include the computation of the smallest eigenvalue of (8). Below we give reformulations of the problems ($\mathrm{P}_{\mathrm{vol}}$), ($\mathrm{P}_{\mathrm{compl}}$), ($\mathrm{P}_{\mathrm{eig}}$) to problems that are much easier to analyze and to solve numerically. All these reformulations have been known. The third one, however, has never been used for the numerical treatment, up to our knowledge. We will further use a unified approach to these reformulation that offers a clear look at their mutual relations.

We start with an auxiliary result.

Proposition 3.1. *Let $x \in \mathbb{R}^m$, $x \geq 0$, and $\gamma \in \mathbb{R}$ be fixed, and fix an index $\ell \in \{1, \ldots, n_\ell\}$. Then there exists $u_\ell \in \mathbb{R}^n$ satisfying*

$$K(x)u_\ell = f_\ell \qquad \textit{and} \qquad f_\ell^T u_\ell \leq \gamma$$

if and only if

$$\begin{pmatrix} \gamma & -f_\ell^T \\ -f_\ell & K(x) \end{pmatrix} \succeq 0\,.$$

Proof. Note that $K(x)$ may be singular in our case, so that we cannot directly use the Schur complement theorem. We first write the matrix inequality equivalently as

$$\alpha^2\gamma - 2\alpha f_\ell^T v + v^T K(x) v \geq 0 \quad \forall \alpha \in \mathbb{R}, \forall v \in \mathbb{R}^n\,. \tag{35}$$

"$\Rightarrow$" As $K(x) \succeq 0$, we know that u_ℓ minimizes the quadratic functional $v \mapsto v^T K(x) v - 2 f_\ell^T v$ with the minimal value $-f_\ell^T u_\ell$. Thus

$$v^T K(x) v - 2 f_\ell^T v \geq -f_\ell^T u_\ell \geq -\gamma \quad \forall v \in \mathbb{R}^n\,.$$

Using the substitution $v = \sigma w$, $\sigma \in \mathbb{R}$, we can write this as

$$(\sigma w)^T K(x)(\sigma w) - 2 f_\ell^T (\sigma w) \geq -\gamma \quad \forall \sigma \in \mathbb{R}, \forall w \in \mathbb{R}^n\,,$$

hence

$$w^T K(x) w - \frac{1}{\sigma} 2 f_\ell^T w \geq -\frac{1}{\sigma^2}\gamma \quad \forall \sigma \in \mathbb{R} \setminus \{0\}, \forall w \in \mathbb{R}^n$$

which is just (35) with $\alpha = \frac{1}{\sigma}$.
"$\Leftarrow$" Put $\alpha = \frac{1}{2}$; then we get from (35)

$$\frac{1}{4}\gamma + v^T(K(x)v - f_\ell) \geq 0 \quad \forall v \in \mathbb{R}^n$$

and so

$$K(x)v = f_\ell\,.$$

Inserting this into (35) with $\alpha = 1$, we have $\gamma + v^T(f_\ell - 2f_\ell) \geq 0$, that is, $\gamma \geq f_\ell^T v$, and we are done. ◻

With this proposition, we immediately get the following reformulations of our three original problems.

The minimum volume problem In this problem, $\overline{\gamma}$ and $\overline{\lambda}$ are given, and we minimize the upper bound V on the volume.

$$\min_{x\in\mathbb{R}^m, V\in\mathbb{R}} V \tag{$\mathrm{P}_{\mathrm{vol}}^{\mathrm{SDP}}$}$$
subject to
$$\begin{pmatrix} \overline{\gamma} & -f_\ell^T \\ -f_\ell & K(x) \end{pmatrix} \succeq 0, \quad \ell = 1,\dots,n_\ell$$
$$\sum_{i=1}^m x_i \leq V$$
$$x_i \geq 0, \quad i = 1,\dots,m$$
$$K(x) - \overline{\lambda}(M(x) + M_0) \succeq 0\,.$$

We mention that this problem has first been formulated and studied in [14].

The minimum compliance problem Here $\overline{V}$ and $\overline{\lambda}$ are given, and we minimize the upper bound γ on the compliance.

$$\min_{x\in\mathbb{R}^m, \gamma\in\mathbb{R}} \gamma \tag{$\mathrm{P}_{\mathrm{compl}}^{\mathrm{SDP}}$}$$
subject to
$$\begin{pmatrix} \gamma & -f_\ell^T \\ -f_\ell & K(x) \end{pmatrix} \succeq 0, \quad \ell = 1,\dots,n_\ell$$
$$\sum_{i=1}^m x_i \leq \overline{V}$$
$$x_i \geq 0, \quad i = 1,\dots,m$$
$$K(x) - \overline{\lambda}(M(x) + M_0) \succeq 0\,.$$

The problem of maximizing the minimal eigenvalue Now $\overline{\gamma}$ and $\overline{V}$ are given, and λ is the variable. For the sake of a common problem structure in all three formulations, we *minimize* and put a minus in front of the objective function.

$$\min_{x\in\mathbb{R}^m, \lambda\in\mathbb{R}} -\lambda \tag{$\mathrm{P}_{\mathrm{eig}}^{\mathrm{SDP}}$}$$
subject to
$$\begin{pmatrix} \overline{\gamma} & -f_\ell^T \\ -f_\ell & K(x) \end{pmatrix} \succeq 0, \quad \ell = 1,\dots,n_\ell$$
$$\sum_{i=1}^m x_i \leq \overline{V}$$
$$x_i \geq 0, \quad i = 1,\dots,m$$
$$K(x) - \lambda(M(x) + M_0) \succeq 0\,.$$

The proof of the following proposition is immediate, and thus is skipped.

Proposition 3.2. *(a) If (x^*, u^*) is a global minimizer of $(\mathrm{P}_{\mathrm{vol}})$ then (x^*, V^*) is a global minimizer of $(\mathrm{P}_{\mathrm{vol}}^{\mathrm{SDP}})$ where $V^* := \sum x_i^*$, and the optimal values of both problems coincide.*

(b) If (x^*, V^*) is a global minimizer of $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}})$ *then there exists u^* such that (x^*, u^*) is a global minimizer of* $(\mathrm{P}_{\mathrm{vol}})$*, and the optimal values of both problems coincide.*

Analogous statements hold for the pairs of problems $(\mathrm{P}_{\mathrm{compl}})$-$(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{compl}})$ *and* $(\mathrm{P}_{\mathrm{eig}})$-$(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$*, respectively, where in the latter case, the optimal function values coincide up to a sign.*

Note that problems $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}})$ and $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{compl}})$ are linear SDPs, while $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$ is an SDP problem with a bilinear matrix inequality (BMI) constraint, i.e., it is generally nonconvex. We should emphasize that, due to the SDP reformulation, the originally discontinuous problems became continuous; a fact of big practical value.

Theorem 3.3. *Each local minimizer of problem* $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}})$ *is also a global minimizer. Analogous statement holds for problem* $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{compl}})$*.*

Proof. Problems $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}})$ and $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{compl}})$ are linear SDPs, i.e., convex problems, and the assertions follow. □

Needless to say that this theorem is of paramount interest from the practical point of view.

Clearly, a statement similar to Thm. 3.3 does not hold for the problem $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$; see Example 2.4 where the function $\lambda_{\min}(\cdot)$ is constant for $x_2 > 0$ and has thus infinitely many local minima which are, however, greater than the global minimum attained at $x_2 = 0$.

We remark, however, that problem $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$ hides a *quasi*convex structure. To see this, use Def. 2.1 to write problem $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$ in the form

$$\min\{\, -\lambda_{\min}(x) \mid x \in \mathcal{F} \,\} \tag{36}$$

with the feasible set

$$\mathcal{F} := \left\{ x \in \mathbb{R}^m \,\middle|\, x \geq 0;\ \begin{pmatrix} \overline{\gamma} & -f_\ell^T \\ -f_\ell & K(x) \end{pmatrix} \succeq 0,\ \ell = 1, \ldots, n_\ell;\ \sum_{i=1}^m x_i \leq \overline{V} \right\}.$$

Then Prop. 2.2(f) and the fact that the cone of positive semidefinite matrices is convex show that we minimize here a quasiconvex function over a convex feasible set $\mathcal{F}$. This fact might be useful, e.g., for the application of cutting plane algorithms from global optimization. Unfortunately, the function $-\lambda_{\min}(\cdot)$ lacks to be strictly quasiconvex as already explained in Example 2.4.

Formulation (36) of problem $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$ immediately clarifies the existence of solutions:

Theorem 3.4. *Problem* $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$ *(or, equivalently, problem* $(\mathrm{P}_{\mathrm{eig}})$*) possesses a solution if and only if it possesses feasible points.*

Proof. Consider problem $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$ in the form (36). Since the cone of positive semidefinite matrices is closed, the set $\mathcal{F}$ is compact. Moreover, $0 \notin \mathcal{F}$ due to assumption (1), and hence $(-\lambda_{\min})$ is l.s.c. on $\mathcal{F}$ by Prop. 2.2(d). Each l.s.c. function attains its infimum on a non-empty compact set (see, e.g., [13, Thm. 2.13.1]). □

Instead of using methods from global optimization for the calculation of a global minimizer of problem $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$, we may use the close relation to the convex problems $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}})$ and $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{compl}})$. In the following we propose a practical framework for finding the global solution of $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$ based on the solutions of a sequence of problems which are of the type $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}})$. Analogous considerations can be done with problems of the type $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{compl}})$.

For fixed $\lambda \geq 0$ and fixed $\delta \geq 0$ consider the following linear SDP:

$$\min_{x\in\mathbb{R}^m, V\in\mathbb{R}} V \qquad (\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}}(\lambda,\delta))$$

subject to

$$\begin{aligned} &\begin{pmatrix} \overline{\gamma} & -f_\ell^T \\ -f_\ell & K(x) \end{pmatrix} \succeq 0, \quad \ell = 1,\dots,n_\ell \\ &\sum_{i=1}^m x_i \leq V \\ &V \leq \overline{V} \\ &x_i \geq 0, \quad i = 1,\dots,m \\ &K(x) - (\lambda+\delta)(M(x) + M_0) \succeq 0\,. \end{aligned}$$

Notice that this problem is just problem ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}}$) with the choice $\overline{\lambda} := \lambda + \delta$, and with the supplementary linear constraint $V \leq \overline{V}$. In the following, the feasible set of this problem is denoted by

$$\mathcal{F}(\lambda,\delta),$$

for simplicity. Notice that ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}}(\lambda,\delta)$) is a linear SDP, i.e., a convex optimization problem for which a global minimizer can be calculated, provided $\mathcal{F}(\lambda,\delta) \neq \varnothing$. Moreover, since ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}}(\lambda,\delta)$) is a convex SDP, modern solution procedures are able to recognize whether $\mathcal{F}(\lambda,\delta) = \varnothing$.

The following proposition gives a tool for the estimation of the (globally) optimal objective function value of problem ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$).

Proposition 3.5. *Let $(\tilde{x},\lambda)$ be feasible for ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$), and let $(-\lambda^{**})$ denote the (globally) optimal function value of problem ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$). Moreover, let $\delta > 0$ be arbitrary, and consider problem ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}}(\lambda,\delta)$) with the parameters $\overline{\gamma}$ and $\overline{V}$ copied from ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$). Then the following assertions hold:*

(a) If $\mathcal{F}(\lambda,\delta) \neq \varnothing$ then for each $(x,V) \in \mathcal{F}(\lambda,\delta)$ the point $(x,\lambda+\delta)$ is feasible for ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$), i.e.,

$$-\lambda^{**} \leq -(\lambda+\delta) < -\lambda. \tag{37}$$

(b) If $\mathcal{F}(\lambda,\delta) = \varnothing$ then

$$-(\lambda+\delta) < -\lambda^{**} \leq -\lambda. \tag{38}$$

Proof. For the proof of (a), let $(x,V) \in \mathcal{F}(\lambda,\delta)$ be arbitrary. It is straightforward to see that $(x,\lambda+\delta)$ is feasible for ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$), and hence its objective function value $(-(\lambda+\delta))$ satisfies (37).

To prove (b), first notice that the second inequality in (38) is a simple consequence of (x,λ) being feasible for ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$). The first inequality in (38) is now proved by contradiction. Assume that $-(\lambda+\delta) \geq -\lambda^{**}$, i.e., there exists $x \in \mathbb{R}^n$ such that $(x,\lambda+\delta)$ is feasible for ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$). Put $V := \mathrm{vol}(x)$, and consider problem ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}}(\lambda,\delta)$). Because $(x,\lambda+\delta)$ is feasible for ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$), we see that the point (x,V) satisfies the LMIs, the two volume constraints $V = \sum x_i \leq \overline{V}$, and the non-negativity constraints in ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}}(\lambda,\delta)$). Moreover, feasibility of $(x,\lambda+\delta)$ for ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$) also yields that $K(x) - (\lambda+\delta)(M(x)+M_0) \succeq 0$. All in all, we obtain that $(x,V) \in \mathcal{F}(\lambda,\delta)$ which contradicts the assumption. □

As an immediate consequence of Prop. 3.5 we get the following assertion.

Corollary 3.6. *Let (x, λ) be feasible for* $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$*. Then (x, λ) is a global minimizer of* $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$ *if and only if $\mathcal{F}(\lambda, \delta) = \varnothing$ for all $\delta > 0$.*

The practical value of Prop. 3.5 lies in the possibility to improve upper and lower bounds for λ^{**} which can be numerically calculated through solutions (or only feasible points) of convex linear SDPs. As a pre-processing step, we first calculate initial lower and upper bounds λ_0^L, λ_0^U on λ^{**}. For this, first calculate a feasible point (x, λ) of $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$ and choose arbitrary $\bar{\delta} > 0$. Then find the smallest $k \in \mathbb{N}$ such that $\mathcal{F}(\lambda, 2^k\bar{\delta}) = \varnothing$ by solving $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}}(\lambda, 2^k\bar{\delta}))$ repeatedly. Set

$$\lambda_0^L := \lambda + 2^{k-1}\bar{\delta} \qquad \text{and} \qquad \lambda_0^U := \lambda + 2^k\bar{\delta} \ .$$

Then Prop. 3.5 shows that

$$0 \leq \lambda_0^L \leq \lambda^{**} < \lambda_0^U \ . \tag{39}$$

With these bounds it is easy to construct a bisection type algorithm which in each step reduces the gap $(\lambda_k^U - \lambda_k^L)$ by a factor of (at least) $\frac{1}{2}$.

Algorithm 3.1. Choose an accuracy $\eta > 0$, a feasible point $(\hat{x}, \hat{\lambda})$ for $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$. Put $(x_0, \lambda_0) := (\hat{x}, \hat{\lambda})$, $\delta_0 := \frac{1}{2}(\lambda_0^U - \lambda_0^L)$, and $k := 0$. Go to Step 2.

1. Calculate a feasible point [or even a local minimizer] (x_k, λ_k) of $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$ with the additional constraint "$\lambda \geq \lambda_k^L$".
2. If $\lambda_k > \lambda_k^L$ then update λ_k^L by $\lambda_k^L := \lambda_k$.
3. If $\lambda_k^U - \lambda_k^L \leq \eta$ then EXIT with the result $(x^*, \lambda^*) := (x_k, \lambda_k)$.
4. Put $\delta_k := \frac{1}{2}(\lambda_k^U - \lambda_k^L)$, and consider problem $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}}(\lambda_k, \delta_k))$.
   If $\mathcal{F}(\lambda_k, \delta_k) \neq \varnothing$ then:

   4A. Put $\lambda_{k+1}^L := \lambda_k^L + \delta_k$, $k := k + 1$, and go to Step 1.

   Otherwise, if $\mathcal{F}(\lambda_k, 2^k\bar{\delta}) = \varnothing$, then:

   4B Put $\lambda_{k+1}^U := \lambda_k^U - \delta_l$, $k := k + 1$, and go to Step 1.

Proposition 3.7. *Let* $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$ *possess a solution (x^{**}, λ^{**}) (cf. Thm. 3.4). Then the following assertions hold.*

(a) Algorithm 3.1 is well-defined, and after each iteration we have

$$\lambda_k^L \leq \lambda^{**} < \lambda_k^U \qquad \textit{and} \qquad \lambda_k^U - \lambda_k^L \leq 2^{-k}(\lambda_0^U - \lambda_0^L).$$

(b) Algorithm 3.1 terminates after a finite number K of iterations, and

$$K \leq \left\lceil \frac{\ln(\lambda_0^U - \lambda_0^L) - \ln(\eta)}{\ln(2)} \right\rceil .$$

At termination, the result (x^*, λ^*) is feasible for $(\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}})$ *with*

$$\lambda^{**} - \lambda^* \leq \eta \ .$$

The proof of this proposition is a straightforward exercise.

Notice that the additional constraint "$\lambda \geq \lambda_k^L$" in Step 1 does not cause any trouble but guarantees that $(\lambda_k)_k$ is monotonically increasing. Moreover, the calculation of global minimizers (in Step 4A), resp. local minimizers (in Step 1), instead of just feasible points should significantly speed up the algorithm. In this case the update of λ_k^U in Step 4B, resp. of λ_k^L in Step 2, may lead to a much bigger reduction of the gap $\lambda_k^U - \lambda_k^U$. Obviously, Step 1 must be carried out in each iteration. Notice also, that λ_k^L is increased in Step 4A, while it remains untouched in Steps 4B. Denote by K' the number of iterations in which Steps 4A have been performed. Moreover, if Steps 4A has been performed in iteration $k-1$, let (x_k, λ_k) in Step 1 be a local optimizer. Then, consequently,

$$K' \leq \left| \left\{ \lambda \,|\, (x,\lambda) \text{ is a local optimizer of } (\mathrm{P}_{\mathrm{eig}}^{\mathrm{SDP}}) \right\} \right|,$$

i.e. K' is limited by the number of levels of the objective function which are attained at a local optimizer. We believe that this cardinality is very small in applications. As an illustration consider Ex. 2.4 where $K' = 2$.

For the numerical treatment of the SDP problems ($\mathrm{P}_{\mathrm{vol}}^{\mathrm{SDP}}$), ($\mathrm{P}_{\mathrm{compl}}^{\mathrm{SDP}}$), ($\mathrm{P}_{\mathrm{eig}}^{\mathrm{SDP}}$) one must resort to methods of semidefinite programming. Such methods, and corresponding codes, are nowadays available for linear SDPs. The limiting factor of these codes is, however, the problem size which, compared to general nonlinear programs, is restricted to problems of medium size. The problem ($\mathrm{P}_{\mathrm{eig}}^{\mathrm{SDP}}$) even requires a method which can deal with bilinear matrix inequalities. We will use such a method to solve examples in the next section. It should be noted, however, that algorithms and codes for SDPs with bilinear matrix inequalities are on the edge of current research and are not yet standard.

4 Numerical Examples

In this chapter we present numerical examples which, on the one hand, will illustrate some of the theoretical results above and, on the other hand, demonstrate the practical use of the SDP problem formulations.

The code we have used for the treatment of the SDP formulations is PENBMI, version 2.0 [10]. This code implements the generalized Augmented Lagrangian method, as described in [9, 19]. In particular, PENBMI can treat bilinear matrix inequalities as is necessary for problem ($\mathrm{P}_{\mathrm{eig}}^{\mathrm{SDP}}$) [7].

The examples were solved on a Pentium III-M 1GHz PC running Windows 2000. All problems were formulated and solved in MATLAB using the YALMIP parser [12] to PENBMI.

Example 4.1. This example illustrates Thms. 2.8, 2.9, and 2.19 with $M_0 = 0$. Consider a 3-by-3 truss with all nodes connected by potential bars. The nodes on the left-hand side are fixed in both directions, a horizontal force $(-1, 0)$ is applied at the right-middle node; see Figure 2-left. No nonstructural mass is considered, i.e., $M_0 = 0$. We consider the minimum volume problem ($\mathrm{P}_{\mathrm{vol}}^{\mathrm{SDP}}$) with $\overline{\gamma} = 1$ and $\overline{\lambda} = 5.0 \cdot 10^{-2}$. PENBMI calculated the (global) optimal solution (x^*, V^*) of this convex problem: the optimal design x^* is shown in Figure 2-right, while $V^* = 1.20229$. Prop. 3.2(b) shows that there exists u^* such that (x^*, u^*) is optimal for problem ($\mathrm{P}_{\mathrm{vol}}$).

Now consider the minimum compliance problem ($\mathrm{P}_{\mathrm{compl}}^{\mathrm{SDP}}$) with $\overline{V} = 1.20229$ and $\overline{\lambda} = 5.0 \cdot 10^{-2}$. As expected by Prop. 3.2(b) and Thm. 2.9, we obtain the solution (x^*, γ^*) with the same structure x^* as before (Fig. 2-right), and with $\gamma^* = 1$.

Finally, when solving the problem of maximizing the minimum eigenvalue ($\mathrm{P}_{\mathrm{eig}}^{\mathrm{SDP}}$) with $\overline{V} = 1.20229$ and $\overline{\gamma} = 1$, we again obtain x^* from before, and $\lambda^* = 5.0 \cdot 10^{-2}$. This shows that the value V^* in (32) and the value γ^* in (33) are attained for x^* because otherwise this would yield a

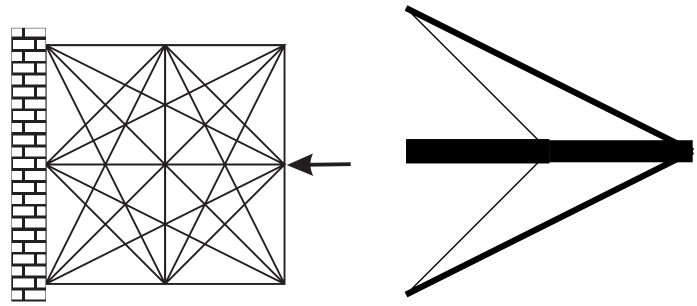

Figure 2: Three-by-three truss (Ex. 4.1): initial layout and optimal topology

contradiction to Thm. 2.19. The authors believe that in this simple example the solution structure x^* is the unique solution, and thus Cor. 2.20 may be applied. $\Diamond$

Example 4.2. In this example, as in Ex. 4.1 above, we again obtain the same optimal structure for all three problem formulations. Here, however, $M_0 \neq 0$, and thus these coincidences are somewhat unexpected.

We consider the same ground structure, boundary conditions, and external load as in the previous example. In addition, we assign a nonstructural mass of size 10 at the loaded node, i.e., $M_0 \neq 0$; see Figure 3-left. Consider the minimum weight problem ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}}$) with $\overline{\gamma} = 1$ and $\overline{\lambda} = 5.0 \cdot 10^{-2}$. Figure 3-right shows the optimal design x^*. The corresponding optimal weight is $V^* = 7.10157$.

Now consider the minimum compliance problem ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{compl}}$) with $\overline{V} = 7.1015$ and $\overline{\lambda} = 5.0 \cdot 10^{-2}$. We obtain the solution (x^*, γ^*) with the same structure x^* as before (Fig. 3-right), and with $\gamma^* = 1$.

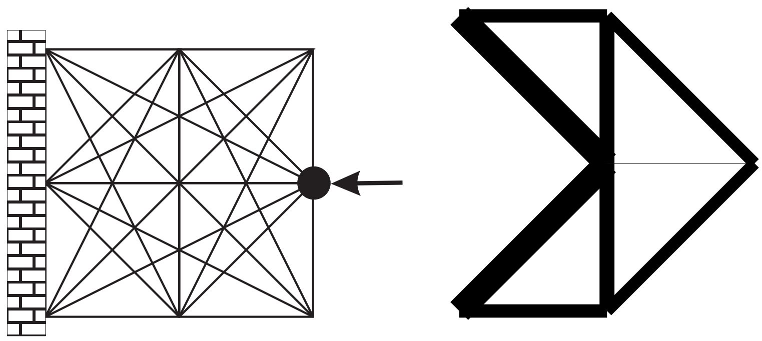

Figure 3: Three-by-three truss with nonstructural mass (Ex. 4.2): initial layout and optimal topology

Finally, when solving the problem of maximizing the minimum eigenvalue ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$) with $\overline{V} = 7.1015$ and $\overline{\gamma} = 1$, we again obtain x^* from above, and $\lambda^* = 5.0 \cdot 10^{-2}$. Again, we believe that the solution x^* is unique in each of the three problems. If this is the case, then the equivalence of the results holds by Corollaries 2.14, 2.17, and 2.20. $\Diamond$

Example 4.3. This academic example illustrates the possible nonuniqueness of solution to the problem ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$). Consider a 2×3 ground-structure with boundary conditions and load as depicted

Table 1: Results of Example 4.3 for different data

M_0	$\overline{V}$	γ^*	λ^*
0	1	1	-0.70711
0	10	0.1	-0.70711
10	1	1	-0.08761
10	10	0.1	-0.41421

in Figure 4-left. Put $M_0 = 0$, $\overline{\gamma} = 10$, and $\overline{V} = 10$. The computed optimal structure x^* is presented in Figure 4-right; the optimal objective function value of ($\mathrm{P}_{\mathrm{eig}}^{\mathrm{SDP}}$) is $-\lambda^* = -0.70711$, i.e., $\lambda_{\min}(x^*) = 0.70711$. While the volume constraint is active at x^*, the compliance constraint is inactive (more precisely, after calculating some u^* corresponding to x^*, we have $\gamma^* := f^T u^* = 0.1 < \overline{\gamma} = 10$). Proposition 2.10 suggests that if we scale the solution x^* by a certain factor μ, we will still get a solution to our problem. For instance, if we solve the same problem but with $\overline{V} = 1.0$, then we will obtain a solution with the same λ^* and with $\gamma^* = 1.0$, i.e., still within the $\overline{\gamma}$ limits. Table 1 summarizes these numbers. It also presents the results for the case when $M_0 = 10$ (and then Prop. 2.10 does not apply). In this case, the optimal solution is no longer scalable. $\diamondsuit$

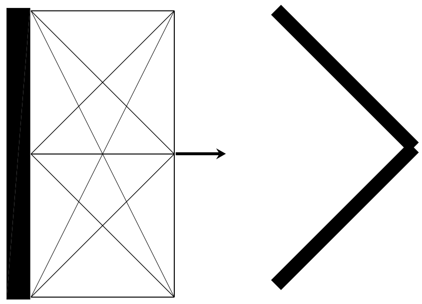

Figure 4: Example demonstrating possible nonuniqueness of solution of the ($\mathrm{P}_{\mathrm{eig}}^{\mathrm{SDP}}$) problem

Example 4.4. Here we demonstrate the possible nonuniqueness of solutions to the minimum volume problem ($\mathrm{P}_{\mathrm{vol}}$) (or ($\mathrm{P}_{\mathrm{vol}}^{\mathrm{SDP}}$)), and illustrate Thm. 2.13(b) in more detail. Consider the same ground-structure and boundary conditions as in Ex. 4.1. The load vector consists of a single vertical force $(0, 1)$ applied at the bottom-right node. Let further $\overline{\gamma} := 0.5$, and consider the single load min-volume problem without vibration constraint

$$\min_{x \in \mathbb{R}^m,\, u \in \mathbb{R}^n} \sum_{i=1}^m x_i \tag{40}$$

subject to

$$K(x)u = f,$$
$$f^T u \le \overline{\gamma},$$
$$x_i \ge 0, \quad i = 1, \ldots, m.$$

This problem can be formulated as a linear program [2] and thus the set $\mathcal{X}^*_{(40)}$ of solution structures of (40) is given by the set of all convex combinations of the most-left and most-right structure in Figure 6, i.e., by the set

$$\mathcal{X}^*_{(40)} = \{(1-\mu)x^{1*} + \mu x^{2*} \mid \mu \in [0, 1]\}$$

where x^{1*} denotes the most-left and x^{2*} the most-right structure in Fig. 6. We have $\mathrm{vol}(x^*) = 18$ and $c(x^*) = 1$ for all $x^* \in \mathcal{X}^*_{(40)}$. Figure 5 shows the dependence of the minimum vibration eigenvalue on the parameter μ of this convex combination, i.e., a plot of the function

$$\mu \mapsto \lambda_{\min}((1-\mu)x^{1*} + \mu x^{2*})$$

over the interval $[0,1]$. The points 1–5 in the plot correspond to the structures in Figure 6, left to right. We observe that $\lambda_{\min}$ is maximized at $\mu \approx 0.0536$, i.e., at structure number 3. Let us now add the vibration constraint to problem (40); thus we arrive at problem ($\mathrm{P}_{\mathrm{vol}}$). For example, put $\overline{\lambda} := 0.037$ which is the value of $\lambda_{\min}$ for structure number 2 in Figure 6. Then it is clear that any structure between truss number 2 and number 5 is a solution to problem ($\mathrm{P}_{\mathrm{vol}}$), and the vibration constraint will be inactive for the structures strictly in between. Moreover, truss number 3 is the structure $\hat{x}^*$ where the supremum in eqn. (25) in Thm. 2.13 is attained, i.e., truss number 3 is optimal for problem ($\mathrm{P}_{\mathrm{eig}}$) with the settings $\overline{V} := 18$ and $\overline{\gamma} := 1$ (according to Thm. 2.13(b)). $\diamondsuit$

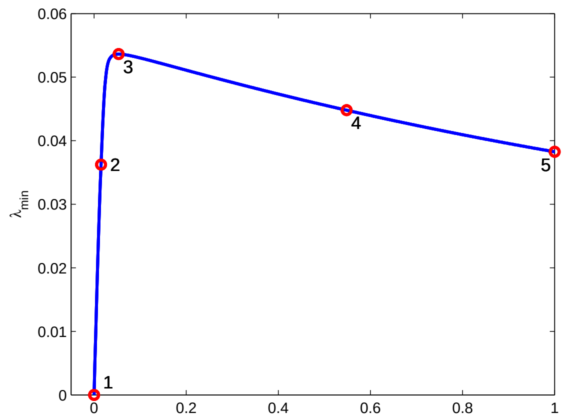

Figure 5: Example 4.4—graph of $\lambda_{\min}$ on interval between two structures of the same volume and compliance

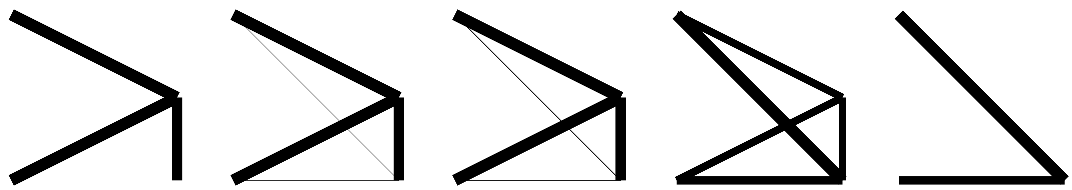

Figure 6: Example 4.4—structures corresponding to points 1–5 on the graph in Figure 5

Example 4.5. This example shows that not only can the minimum eigenvalue function be discontinuous (see Ex. 2.4) but it may also behave in a non-Lipschitz way. This is slightly unexpected, given the well-known fact that the eigenvalues of the *standard* symmetric eigenvalue problem are Lipschitz.

Consider again the 3×3 ground-structure from Ex. 4.1 with all nodes connected. A horizontal force is applied at the central node. Figure 7 shows the behavior of the objective function $\lambda_{\min}(\cdot)$

of the problem ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$) with $x \geq \varepsilon > 0$; denote the solution of this problem by x_ε. The left-hand figure shows the plot of the function $\lambda_{\min}(x_\varepsilon)$ for $1.5 \cdot 10^{-7} \leq \varepsilon \leq 2 \cdot 10^{-3}$; the function looks all but Lipschitz (for smaller values of ε we were unable to compute the function value due to round-off errors). To see its behavior more clearly, we plot in the right-hand figure the derivative (computed by finite differences) in the interval $[1.5 \cdot 10^{-7}, 1.6 \cdot 10^{-5}]$; this figure confirms the non-Lipschitz behavior. When we solve the minimum eigenvalue problem ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$) with $x \geq 0$, we obtain the optimum value $\lambda^* = -0.7071068$. Obviously, the picture is not a proof of a non-Lipschitz behavior, but it is very indicative. The optimal trusses for $\varepsilon = 2 \cdot 10^{-3}$ and for the problem with $x \geq 0$ are shown in Figure 8 (left and right, respectively). In the first case, only bars that are not equal to the lower bound are presented. In both cases, the compliance constraint was inactive. $\diamondsuit$

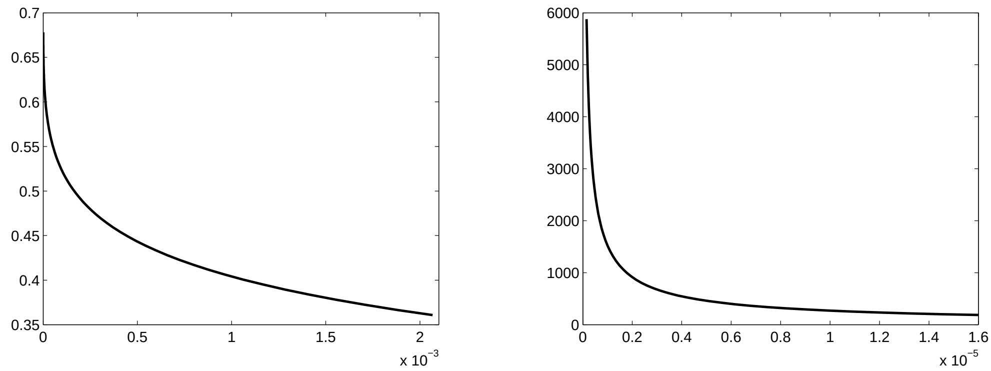

Figure 7: Example 4.5 demonstrating apparent non-Lipschitz behavior of the minimum eigenvalue function close to the boundary of the feasible region. The graph of the function (left) and its derivative (right) are shown.

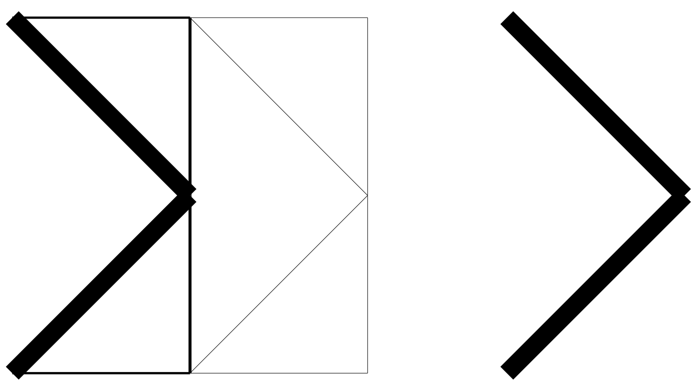

Figure 8: Example 4.5—optimal structures for $x_i \geq 2 \cdot 10^{-3}$ (left) and $x_i \geq 0$ (right).

Example 4.6. This example demonstrates that the change in M_0 may lead to a change of the topology of the optimal structure as has been suggested in the discussion after Prop. 2.22. We take the same ground-structure, boundary conditions and loads as in Example 4.2. Consider the minimum volume problem ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}}$) with three different values of M_0, namely, 0, 10 and 100. The bounds on compliance and minimum eigenvalue are $\overline{\gamma} = 20$ and $\overline{\lambda} = 1.0 \cdot 10^{-3}$. The optimal values of V^* are, respectively, 0.05012, 0.07284, and 0.63386. In the latter case (M_0=100), the compliance constraint was inactive. The respective optimal structures are presented in Figure 9.$\Diamond$

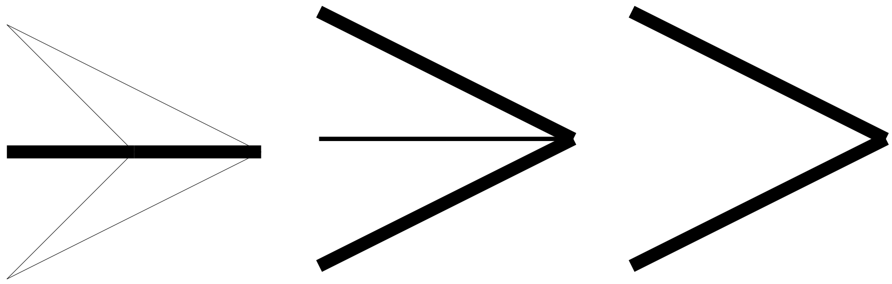

Figure 9: Example 4.6 demonstrating the dependence of the optimal structure on nonstructural mass changes; optimal results for $M_0 = 0, 10, 100$ are depicted left-to-right.

Example 4.7. With practical applications in mind, we also present an example of larger ground structure with multiple loads. Consider a 7×3 nodal grid with the ground-structure, boundary conditions and loads as depicted in Figure 10 top-left. Each of the load arrows indicates an independent load case. The result of the standard minimum volume multiple-load problem (with no vibration constraints) with $\overline{\gamma} = 10$ is shown in Figure 10 top-right—obviously resulting in two independent horizontal bars, one for each load. The volume of this structure is $V^* = 5.0$. Figure 10 bottom-left shows the result for the multiple load problem with a bound $\overline{\lambda} = 1.0 \cdot 10^{-3}$ on the minimum eigenvalue with the optimal volume $V^* = 7.8309$. For a comparison, we also show a result of the single load problem (both forces considered as a single load) with $\overline{\gamma} = 20$ and $\overline{\lambda} = 1.0 \cdot 10^{-3}$; the optimal structure with $V^* = 7.6166$ is presented in Figure 10 bottom-right. All solutions were obtained by PENBMI in less than 10 seconds. $\Diamond$

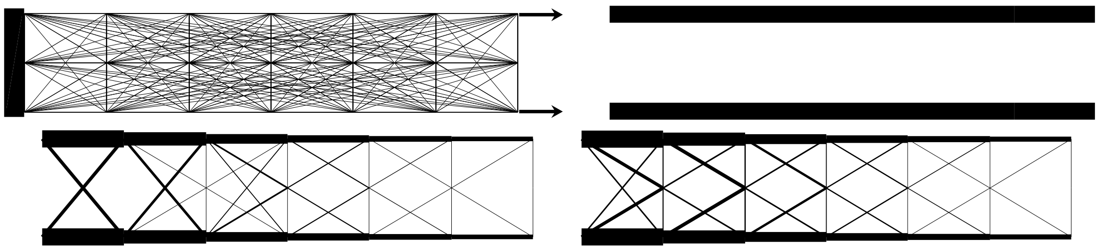

Figure 10: A medium size multiple-load example (Ex. 4.7): initial layout (top-left); optimal topology without (top-right) and with (bottom-left) vibration constraints; single-load optimal result with vibration constraints (bottom-left)

Example 4.8. We consider the same problem scenario as in Example 4.2 but with a 7x7 full ground-structure with 1176 potential bars; see Figure 11-left. Again we solve the minimum weight problem ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}}$) with $\overline{\gamma} = 1$ and $\overline{\lambda} = 5.0 \cdot 10^{-2}$ (and a nonstructural mass of size 10 at the loaded node). Figure 11-right shows the calculated optimal design x^*. The optimal weight is $V^* = 3.59874$, i.e., just one half of the optimal weight of the 3x3 ground-structure from before in Ex. 4.2. To solve the minimum volume problem by PENBMI, we needed 5 min 16 sec. To solve the other two formulations, ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{compl}}$) and ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$), the code needed 11 min 41 sec and 20 min 15 sec, respectively. As expected, formulation ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$) is computationally the most demanding one due to the presence of bilinear matrix inequalities. $\Diamond$

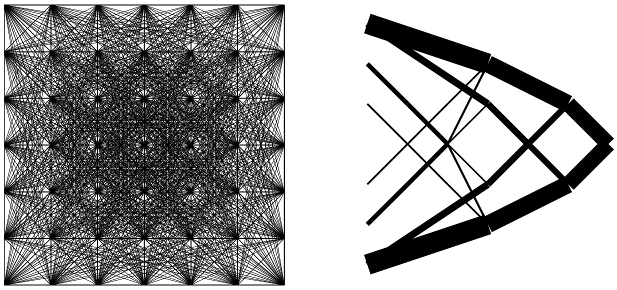

Figure 11: Example 4.8—a medium-size problem, initial layout and optimal topology

Example 4.9. Here we consider a medium-size example with an 11×5 ground-structure, having 100 degrees of freedom and 1485 potential bars. The bounds on compliance and on the eigenvalue were $\overline{\gamma} = 20$ and $\overline{\lambda} = 5.0 \cdot 10^{-4}$. A horizontal force $(-10, 0)$ is applied at the right-middle node; see Figure 12-left. No nonstructural mass is considered. The minimum volume problem was solved by PENBNI in 33 min 37 sec, and resulted in the optimal structure shown in Figure 12-right with $V^* = 1542.65$. According to Thm. 2.8 this structure is also optimal for the min-compliance problem ($\mathrm{P}_{\mathrm{compl}}$) with $\overline{V} := 1542.65$ and $\overline{\lambda}$ as above. $\diamondsuit$

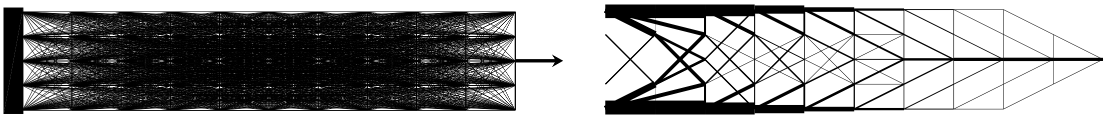

Figure 12: Example 4.9—a medium-size problem, initial layout and optimal topology

5 An Extension: the multiple-mass problem

Here we propose an extension to each of the three original problem formulations. Assume that we have n_k matrices $M_0^{(k)}$, $k = 1, \ldots, n_k$, corresponding to n_k different nonstructural masses that can be applied independently. The corresponding eigenvalue constraint extending the constraint "$\lambda_{\min}(x) \geq \overline{\lambda}$" in problem ($\mathrm{P}_{\mathrm{vol}}$) or in problem ($\mathrm{P}_{\mathrm{compl}}$) would then be stated as

$$\lambda_{\min}(x, M_0^{(k)}) \geq \overline{\lambda} \qquad \text{for all } k = 1, \ldots, n_k$$

where we have used the notation (34) from Sec. 2.5 for different nonstructural mass matrices. Similarly, the objective function $\lambda_{\min}(\cdot)$ in problem ($\mathrm{P}_{\mathrm{eig}}$) becomes

$$x \mapsto \min_{1 \leq k \leq n_k} \lambda_{\min}(x, M_0^{(k)})$$

(which is to be maximized). Generalizing the SDP problems from Sec. 3 we arrive at the following formulations possessing the same problem structure.

The minimum volume multiple-mass problem

$$\min_{x\in\mathbb{R}^m, V\in\mathbb{R}} V \tag{41}$$
subject to
$$\begin{pmatrix} \gamma & f_\ell^T \\ f_\ell & K(x) \end{pmatrix} \succeq 0, \quad \ell = 1,\dots,n_\ell$$
$$\sum_{i=1}^m x_i \le V$$
$$x_i \ge 0, \quad i = 1,\dots,m$$
$$K(x) - \overline{\lambda}(M(x) + M_0^{(k)}) \succeq 0, \quad k = 1,\dots,n_k\,.$$

The minimum compliance multiple-mass problem

$$\min_{x\in\mathbb{R}^m, \gamma\in\mathbb{R}} \gamma \tag{42}$$
subject to
$$\begin{pmatrix} \gamma & f_\ell^T \\ f_\ell & K(x) \end{pmatrix} \succeq 0, \quad \ell = 1,\dots,n_\ell$$
$$\sum_{i=1}^m x_i \le V$$
$$x_i \ge 0, \quad i = 1,\dots,m$$
$$K(x) - \overline{\lambda}(M(x) + M_0^{(k)}) \succeq 0, \quad k = 1,\dots,n_k\,.$$

The maximum lambda multiple-mass problem

$$\min_{x\in\mathbb{R}^m, \lambda\in\mathbb{R}} -\lambda \tag{43}$$
subject to
$$\begin{pmatrix} \gamma & f_\ell^T \\ f_\ell & K(x) \end{pmatrix} \succeq 0, \quad \ell = 1,\dots,n_\ell$$
$$\sum_{i=1}^m x_i \le V$$
$$x_i \ge 0, \quad i = 1,\dots,m$$
$$K(x) - \lambda(M(x) + M_0^{(k)}) \succeq 0, \quad k = 1,\dots,n_k\,.$$

Because the mathematical structure of these formulations is the same as that of the problems ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{vol}}$), ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{compl}}$), and ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$), we may use again the code PENBMI to numerically solve these problems. Let us look at a numerical example.

Example 5.1. Consider a 3-by-3 truss with all nodes connected by potential bars. The nodes on the left-hand side are fixed in both directions, two balls (nonstructural masses) are placed in the corners on the right-hand side; see Figure 13-left. Figure 13-middle shows the optimal design for formulation ($\mathrm{P}^{\mathrm{SDP}}_{\mathrm{eig}}$) when both masses are considered a “single” nonstructural mass. Figure 13-right presents the result of the multiple-mass formulation (43), where the two nonstructural masses are considered being independent from each other. The volume bound in both problems was $\overline{V} := 1$, and the resulting optimal eigenvalues were $\lambda^* = 4.758 \cdot 10^{-3}$ in the single-mass case and $\lambda^* = 7.365 \cdot 10^{-3}$ in the multiple-mass case. $\Diamond$

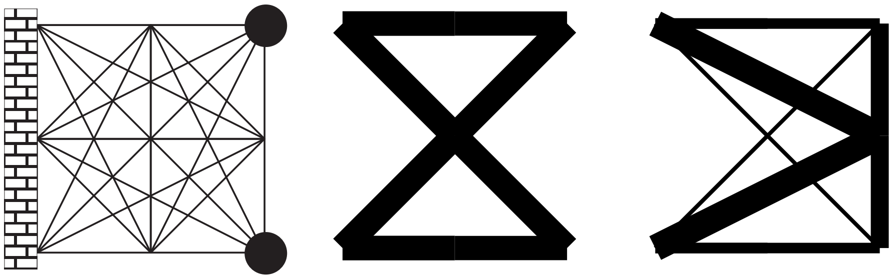

Figure 13: A multiple-mass problem (Ex. 5.1: initial layout (left), a “single-mass” result (middle) and a multiple-mass optimal structure (right)

Acknowledgment

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A Appendix

Let $x \in X$ be given and, for simplicity of notation, assume that $M_0 = 0$ (in general, we would assume that $\ker(M(x)) \subset \ker(M_0)$). We want to show that $M(x)$ can be partitioned into

$$M(x) = \begin{pmatrix} M_{\mathcal{A}\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}$$

where $M_{\mathcal{A}\mathcal{A}} \succ 0$.

Lemma A.1. *Let $Z_i \in \mathbb{R}^{n\times n}$, $Z_i \succ 0$, and $P_i \in \mathbb{R}^{k\times n}$, $k < n$, for $i = 1, \ldots, \mu$. Then, for any $z \neq 0$,*

$$\sum_{i=1}^{\mu} P_i Z_i P_i^T z = 0 \quad \Longrightarrow \quad \sum_{i=1}^{\mu} P_i P_i^T z = 0\,.$$

Proof. From the assumption we know that $\sum_{i=1}^{\mu} z^T P_i Z_i P_i^T z = 0$ and $z^T P_i Z_i P_i^T z \geq 0$ for each i (as $P_i Z_i P_i^T \succeq 0$). Thus $z^T P_i Z_i P_i^T z = 0$ for all i and therefore $\|Z^{1/2} P_i^T z\|_2^2 = 0$. As $Z^{1/2} \succ 0$, this immediately gives $P_i^T z = 0 \; \forall i$, and the lemma follows. □

Now let $\mathcal{I}$ includes the indices of all nonzero components x_i of x. Without loss of generality, let us assume that the nonzero components of x are equal to one, i.e., $x_i = 1$ for $i \in \mathcal{I}$. Hence $M = \sum_{i=1}^{m} x_i P_i \widehat{M}_i P_i^T = \sum_{i \in \mathcal{I}} P_i \widehat{M}_i P_i^T$. Define the projection

$$S = I_{n \times n} - \prod_{i \in \mathcal{I}} (I_{n \times n} - P_i P_i^T)\,;$$

clearly, S projects a vector $z \in \mathbb{R}^n$ to a subspace generated by Euclidean unit vectors associated with all degrees of freedom belonging to elements $i \in \mathcal{I}$, i.e., to the space $\operatorname{span}\{P_i P_i^T e,\ i \in \mathcal{I}\}$, where $e \in \mathbb{R}^n$ is the vector of all ones. From this definition, and from the construction of M, we immediately have that $M = SMS$. Without loss of generality, assume that S is of the form

$$S = \begin{pmatrix} I_{k \times k} & 0 \\ 0 & 0 \end{pmatrix}$$

where k is the rank of S. Hence M also has the form

$$M = \begin{pmatrix} \widetilde{M} & 0 \\ 0 & 0 \end{pmatrix}$$

with $\widetilde{M} \in \mathbb{R}^{k \times k}$.

Lemma A.2. *$\widetilde{M}$ is positive definite.*

Proof. Assume that $Mz = 0$ for some $z \neq 0$. We need to show that $\widetilde{M}\tilde{z} = 0$ only for $\tilde{z} = 0$, where $\tilde{z}$ includes the first k components of z. By definition,

$$Mz = \sum_{i \in \mathcal{I}} P_i \widehat{M}_i P_i^T z = 0\,.$$

From the above lemma, we have that

$$\sum_{i \in \mathcal{I}} P_i P_i^T z = 0\,.$$

Now, the matrix $\sum_{i \in \mathcal{I}} P_i P_i^T$ is of the same form as S and M, and its upper-left block consists of a (full) positive diagonal. Hence $\tilde{z} = 0$. □