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Optimization Problems Aith Equilibrium Constraints And Their Numerical Solution

by

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Abstract. We consider a class of optimization problems with a generalized equation among the constraints. This class covers several problem types like MPEC (Mathematical Programs with Equilibrium Constraints) and MPCC (Mathematical Programs with Complementarity Constraints). We briefly review techniques used for numerical solution of these problems: penalty methods, nonlinear programming (NLP) techniques and Implicit Programming approach (ImP). We further present a new theoretical framework for the ImP technique that is particularly useful in case of difficult equilibria. Finally, three numerical examples are presented: an MPEC that can be solved by ImP but can hardly be formulated as a nonlinear program, an MPCC that cannot be solved by ImP and finally an MPEC solvable by both, ImP and NLP techniques. In the last example we compare the efficiency of the two approaches.

1. Introduction

As early as in 1934 mathematicians were confronted with an optimization problem, the feasible set of which was a subset of the minimizers to another, lower-level optimization problem. This was the famous Stackelberg game ([55]). Stackelberg games play now an important role in economy, design of mechanical structures and other application areas; see [11]. If the lower-level problem happens to be a convex program, it can be replaced (under a constraint qualification) by a variational inequality or a generalized equation (GE) ([47]). But in this way one can also describe equilibria that are not related to optimization problems. This is, for instance, the noncooperative (Nash) equilibrium or Wardrop's user equilibrium. The obtained problem is thus called Mathematical Program with Equilibrium Constraints, or MPEC, a terminology introduced in [22] and nowadays widely used in the literature. In the course of time, the number and the complexity of equilibrium problems increased. What is more, a new class of problems appeared with a different meaning of variables (the control variable is not present as opposed to standard MPECs).

In the next section we propose a simple classification and a partially new terminology which takes these phenomena into account. It is based on the description of equilibria via suitable GEs; this description can cover a majority of cases considered in the

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literature so far (it does not cover, e.g., bilevel programs with nonconvex optimization problems on the lower level).

At present, plenty of rather different techniques are available to the numerical solution of MPECs, many of them tailored to problems with a special structure [3, 10, 20, 26, 31]. When aiming to solve a broader class of problems, one can recognize at least three main approaches ([33, 42, 11]): (i) penalty approach, where the equilibrium constraint is either converted to a (smooth) equation or augmented to the objective via a suitable error bound; (ii) implicit programming approach (ImP), where one treats the equilibrium constraint via a generalized Implicit Function Theorem; (iii) piecewise programming approach, based on the combinatorial structure inherent in many equilibria.

All the above techniques can be coupled with various standard optimization methods, the result of which are concrete methods and algorithms. For instance, the piecewise programming approach connected with the standard sequential quadratic programming (SQP) method gives rise to the PSQP method; see [33]. Recently, in case of equilibria governed by complementarity problems, elaborated SQP codes have been successfully applied directly to the nonlinear programming (NLP) formulation of the respective MPEC. And it seems that also other NLP methods could lead, in the same way, to effective MPEC solvers. So one can speak of an NLP approach. In Section 3 we briefly discuss the penalty approach and pay more attention to the ImP and NLP approaches.

A combination of ImP with a bundle method of nonsmooth optimization enables us to handle MPECs with very complex equilibria. The computation of subgradient information, required by the bundle method, becomes then a difficult task that is not always easily tractable by the technique from [42]. Hence, in Section 4 we propose an alternative technique, based on the generalized differential calculus of B. Mordukhovich.

Section 5 contains three examples coming from mechanics. They illustrate the application areas and the effectiveness of both, the ImP technique combined with a classic bundle code [54] and the NLP approach, here tested with several SQP and one interior point code. It seems that these two approaches, combined with proper algorithms, create an efficient and versatile tool to the numerical solution of a broad class of optimization problems with equilibrium constraints.

The following notation is used: x^i is the ith component of a vector $x \in I\!R^n$, $\overline{I\!R}$ is the extended real line and $I\!R^n_+$ denotes the nonnegative orthant of $I\!R^n$. E is the unit matrix. For an $[m \times n]$ matrix A and an index set $I \subset \{1, 2, \ldots, m\}$, A_I is the submatrix of A with rows specified by I. Furthermore, δ_Λ is the indicatrix of a set Λ, $\mathrm{epi}\, f$ is the epigraph of a function f and $\overline{\partial} f(x)$ denotes Clarke's subdifferential of f at x. If f is vector-valued, $\overline{\partial} f(x)$ denotes Clarke's generalized Jacobian of f at x. For a multifunction $Q[I\!R^n \rightsquigarrow I\!R^m]$, $\mathrm{Gph}\, Q = \{(x, y) \in I\!R^n \times I\!R^m \mid y \in Q(x)\}$ and $\mathrm{Ker}\, Q = \{x \in I\!R^n \,|\, 0 \in Q(x)\}$.

Finally, we define basic objects from the generalized differential calculus of B. Mordukhovich, used extensively in Section 4. Consider a closed set $\Pi \subset I\!R^p$.

Definition 1. *(i) The* Fréchet normal cone *to Π at $\overline{a}$, denoted $\widehat{N}_\Pi(\overline{a})$, is given by*

$$\widehat{N}_\Pi(\overline{a}) = \{v \in I\!R^p \mid \limsup_{a \xrightarrow{\Pi} \overline{a}} \frac{\langle v, a - \overline{a}\rangle}{\|a - \overline{a}\|}\} \leq 0\,.$$

(ii) The limiting normal cone *to Π at $\overline{a}$, denoted $N_\Pi(\overline{a})$, is given by*

$$N_\Pi(\overline{a}) = \limsup_{a \xrightarrow{\Pi} \overline{a}} \widehat{N}_\Pi(a)\,,$$

where "$\limsup$*" is the upper limit of multifunctions in the sense of Kuratowski-Painlevé; see [2].*

Definition 2. *Let $\varphi[I\!R^p \to \overline{I\!R}]$ be an arbitrary extended real-valued function and $a \in \operatorname{dom}\varphi$. The set*

$$\partial\varphi(a) := \{a^* \in I\!R^p | (a^*, -1) \in N_{\operatorname{epi}\varphi}(a, \varphi(a))\}$$

is called the limiting subdifferential *of φ at a.*

Definition 3. *Let $\Phi[I\!R^p \rightsquigarrow I\!R^q]$ be an arbitrary multifunction with a closed graph and $(a,b) \in \operatorname{Gph}\Phi$.*
(i) The multifunction $\widehat{D}^*\Phi(a,b)\,[I\!R^q \rightsquigarrow I\!R^p]$ defined by

$$\widehat{D}^*\Phi(a,b)\,(b^*) := \{a^* \in I\!R^p | (a^*, -b^*) \in \widehat{N}_{\operatorname{Gph}\Phi}(a,b)\}, \quad b^* \in I\!R^q$$

is called the regular coderivative *of Φ at (a,b).*
(ii) The multifunction $D^*\Phi(a,b)\,[I\!R^q \rightsquigarrow I\!R^p]$ defined by

$$D^*\Phi(a,b)\,(b^*) := \{a^* \in I\!R^p | (a^*, -b^*) \in N_{\operatorname{Gph}\Phi}(a,b)\}, \quad b^* \in I\!R^q$$

is called the coderivative *of Φ at (a,b).*

2. Problem classification

The general optimization problem considered in this paper takes the form

$$\begin{aligned} &\min_z f(z) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(MPGE)}\\ &\text{subject to}\\ &\qquad 0 \in F(z) + Q(z)\\ &\qquad z \in \Omega\,, \end{aligned}$$

where $z \in I\!R^s$, $f[I\!R^s \to I\!R]$ and $F[I\!R^s \to I\!R^d]$ are assumed to be continuously differentiable, $Q[I\!R^s \rightsquigarrow I\!R^d]$ has a closed graph, and $\Omega \subset I\!R^s$ comprises all other constraints imposed on z. We will call this problem *Mathematical Program with a Generalized Equation Constraint* (MPGE).

The constraint given by the generalized equation

$$0 \in F(z) + Q(z)\,, \qquad (1)$$

typically describes some (mechanical, economic) equilibrium and we will often refer to it as to *equilibrium problem*.

MPGE contains two distinct classes with a completely different background. In the first one, the variable z splits into two vectors, the *control* or *design* variable and the *state* variable. We want to find a feasible control-state pair that is optimal from a certain point of view. These problems are typical in shape optimization, control of technological processes, and in certain economic models [33, 42, 11]. In the second case, we only want to identify a specific solution of the equilibrium problem; there is no control variable here. We can meet this kind of problems in economic and mechanical models; see Section 5.2.

These two cases can be formally distinguished by the dimensions s and d.

Case I: $s > d$, in particular, $s = n + m$ and $d = m$. Here z can be written as $z = (x, y), x \in I\!R^n, y \in I\!R^m$, where x and y play the roles of control and state variables. This is the case that is known as *Mathematical Program with Equilibrium Constraints* or *MPEC*; see [33, 42]. In the rest of the paper, we will frequently refer to this problem; let us thus specify it together with its dimensions as follows:

$$\min_{x,y} f(x,y) \qquad \text{(MPEC)}$$

subject to

$$0 \in F(x,y) + Q(x,y)$$
$$(x,y) \in \Omega$$

where

$$x \in I\!R^n,\ y \in I\!R^m,\ F[I\!R^n \times I\!R^m \to I\!R^m],\ Q[I\!R^n \times I\!R^m \rightsquigarrow I\!R^m]\,.$$

Case II: $s = d$. In this case we have just one "natural" variable z and try to identify a specific solution of the generalized equation (1). We will call this problem *Square Mathematical Program with Generalized Equation Constraint*, shortly *SMPGE*, and z the *decision* variable. Again, let us write down this particular form of MPGE together with its dimensions:

$$\min_{z} f(z) \qquad \text{(SMPGE)}$$

subject to

$$0 \in F(z) + Q(z)$$
$$z \in \Omega$$

where

$$z \in I\!R^s,\ F[I\!R^s \to I\!R^s],\ Q[I\!R^s \rightsquigarrow I\!R^s]\,.$$

Besides the above two cases, MPGE also covers other problems, for instance, those in which the range space of F and Q has a smaller dimension than s, but z cannot be partitioned into a control-state pair.

2.1. Special cases

Both problems, MPEC and SMPGE, can be further divided into subclasses, depending on the specific form of the generalized equation (equilibrium problem). Below we list some of the most important ones.

2.1.1. Complementarity problem Assume that in (1)

$$Q(z) = N_{\mathbb{R}^d_+}(z_I)\,,$$

where $z_I \in \mathbb{R}^d$ is a subvector of z. The GE (1) then corresponds to a nonlinear complementarity problem (NCP)

$$F(z) \geq 0,\ z_I \geq 0,\ \langle F(z),\, z_I\rangle = 0. \tag{2}$$

For MPEC we have $z = (x, y)$ with $z_I = y$ and

$$Q(x, y) = N_{\mathbb{R}^d_+}(y)\,.$$

The equilibrium problem then corresponds to an NCP in variable y depending on a control variable x:

$$F(x, y) \geq 0,\ y \geq 0,\ \langle F(x, y),\, y\rangle = 0.$$

In case of SMPGE, we simply have $z_I = z$.

In both cases (MPEC and SMPGE), we call the resulting problem *Mathematical Program with Complementarity Constraints* or *MPCC*. Since many numerical methods are tailored to its solution, we will pay special attention to this problem in Section 3.

2.1.2. Variational inequality This case will only be considered for MPECs. Let in (1)

$$Q(y) = \partial J(y)\,,$$

where $J[\mathbb{R}^m \to \overline{\mathbb{R}}]$ is lower semicontinuous (l.s.c.).

We further distinguish several subcases.

(i) Let $J = \delta_\Lambda$ with a closed convex set $\Lambda \subset \mathbb{R}^m$, i.e.,

$$\partial J(y) = N_\Lambda(y)\,.$$

In this case the GE (1) corresponds to an x-dependent variational inequality (VI) of the first kind:

$$\left.\begin{array}{l}\text{Find } y \in \Lambda \text{ such that}\\ \quad \langle F(x, y),\, v - y\rangle \geq 0 \quad \text{for all } v \in \Lambda.\end{array}\right\} \tag{3}$$

(ii) Let $J = J_1 + \delta_\Lambda$ with a convex continuous $J_1[\mathbb{R}^m \to \mathbb{R}]$ and a closed convex set $\Lambda \subset \mathbb{R}^m$. Then one has

$$\partial J(y) = \partial J_1(y) + N_\Lambda(y)\,.$$

Now the GE (1) corresponds to an x-dependent variational inequality of the second kind:

$$\left.\begin{array}{l}\text{Find } y \in \Lambda \text{ such that}\\ \quad \langle F(x, y),\, v - y\rangle + J_1(v) - J_1(y) \geq 0 \quad \text{for all } v \in \Lambda.\end{array}\right\} \tag{4}$$

(iii) Assume that J is generally nonconvex. The GE (1) is then related to an x-dependent *hemivariational inequality* (HVI):

$$\left.\begin{array}{l}\text{Find } y \in I\!R^m \text{ such that} \\ \quad \langle F(x,y),\, v-y\rangle + \sum\limits_{i=1}^{m} J^o(y; v-y) \geq 0 \quad \text{for all } v \in I\!R^m\end{array}\right\} \tag{5}$$

as introduced by Panagiotopoulos [44] (here $J^o(a;b)$ denotes Clarke's directional derivative of J at a in direction b [9]). The HVI (5) is actually equivalent to the GE

$$0 \in F(x,y) + \overline{\partial} J(y)\,.$$

2.1.3. Variational inequality with control dependent Λ Here we again consider only MPECs. Let in (1)

$$Q(x,y) = N_{\Lambda(x)}(y)\,. \tag{6}$$

Assume further that, for any x from a certain subset of $I\!R^n$, $\Lambda(x)$ is given by a system of equations and inequalities:

$$\Lambda(x) = \left\{\zeta \in I\!R^m \mid h^i(x,\zeta) = 0,\ i = 1,\dots,o_1,\ g^j(x,\zeta) \leq 0,\ j = 1,\dots,o_2\right\}, \tag{7}$$

with h^i, $i = 1,\dots,o_1$, affine in the second variable and twice continuously differentiable and g^i, $i = 1,\dots,o_2$, convex in the second variable and twice continuously differentiable. Suppose that for all admissible x the Slater constraint qualification is satisfied, i.e., there exists $\overline{\eta} \in I\!R^m$ such that $h^i(x,\overline{\eta}) = 0$, $i = 1,\dots,o_1$, and $g^j(x,\overline{\eta}) < 0$, $j = 1,\dots,o_2$. Then for each $\eta \in \Lambda(x)$ one has (see [2])

$$N_{\Lambda(x)}(\eta) = \left\{\sum_{i=1}^{o_1} \mu^i\, \nabla_\eta\, h^i(x,\eta) + \sum_{i=1}^{o_2} \lambda^i\, \nabla_\eta\, g^i(x,\eta) \mid \mu \in I\!R^{o_1}, \lambda \in I\!R^{o_2}_+, \right.$$
$$\left. \lambda^i\, g^i(x,\eta) = 0,\ i = 1,\dots,o_2 \right\},$$

and, with the map

$$\mathcal{L}(x,\eta,\mu,\lambda) := F(x,\eta) + \sum_{i=1}^{o_1} \mu^i\, \nabla_\eta\, h^i(x,\eta) + \sum_{i=1}^{o_2} \lambda^i\, \nabla_\eta\, g^i(x,\eta)\,,$$

we can write the GE (1) in the "expanded" form

$$0 \in \begin{bmatrix} \mathcal{L}(x,y,\mu,\lambda) \\ H(x,y) \\ -G(x,y) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ N_{I\!R^{o_2}_+}(\lambda) \end{bmatrix} \tag{8}$$

with $H(\eta) := [h^1(\eta),\dots,h^{o_1}(\eta)]^T$ and $G(\eta) := [g^1(\eta),\dots,g^{o_2}(\eta)]^T$.

The GE (8) has a substantially simpler multi-valued part in comparison with (6). The price is, however, also high: instead of the original state variable y, we now have to work with the triple (y,μ,λ). Nevertheless, sometimes this extension may be advantageous, as it enables us to work with objectives f that depend also on the "multipliers" μ,λ.

2.1.4. Equilibria given by fixed points, quasivariational inequalities To illustrate this complex class of equilibria, consider an MPEC with Q given by a composition

$$Q(y) = \varphi(y) \bullet \partial J(y)\,, \tag{9}$$

where $\varphi[I\!R^m \to I\!R^m]$ is a smooth map, $J[I\!R^m \to \overline{I\!R}]$ is again l.s.c. and convex, and "$\bullet$" denotes the Hadamard (componentwise) product of vectors.

Such a structure can be encountered, e.g., in the discretized two-dimensional contact problems with Coulomb friction; see [6]. The state of the ith node on the contact part of the boundary is described by the triple $(u^i_\tau, u^i_\nu, \lambda^i_\nu)$, where λ^i_ν is the normal stress and u^i_τ, u^i_ν is the tangential and the normal displacement, respectively. The associated part of Q attains the form

$$Q^i(u^i_\tau, u^i_\nu, \lambda^i_\nu) = \begin{bmatrix} \lambda^i_\nu \partial |u^i_\tau| \\ 0 \\ N_{I\!R_+}(\lambda^i_\nu) \end{bmatrix} \tag{10}$$

so that the appropriate Cartesian product of Q^i has exactly the form (9) with $y := (u_\tau, u_\nu, \lambda_\nu)$. The fixed point nature of this equilibrium is reflected by the coupling in (10): the normal stress λ^i_ν arises as a parameter in the term $\lambda^i_\nu \partial |u^i_\tau|$ which is related to the tangential slip of the ith node. In Section 5.1 we report about numerical solution of such an MPEC.

3. Solution methods

It is not the purpose of the paper to give an exhaustive overview of solution methods for MPECs and SMPGEs; we refer the reader to monographs [33, 42, 11] and the annotated bibliography [12]. Instead, we summarize three basic approaches to the solution of MPGEs and point out two methods that, currently, seem to be the most efficient ones.

The first two approaches try to reformulate the original problem in such a way that it (or its approximation) can be solved by standard NLP software. The techniques used here are typically based on penalization, relaxation, or smoothing of the original problem or its reformulation. In the following we will speak of *penalty approach*, whenever some penalty term will be added to the objective function f. All other techniques are called *NLP reformulations*.

The third technique, used for MPECs, is based on a generalized Implicit Function Theorem. The original problem is here rewritten as a standard, albeit nonsmooth and nonconvex, mathematical program.

3.1. Penalty approaches

One possible way how to deal with the equilibrium constraint is to replace the GE (1) by the equation $G(z) = 0$, where G is a suitable "gap" function. For the NCP (2) we may use, for instance, the so-called NCP functions

$$G_1(z) := \min\{F(z),\, z_I\} \qquad \text{(componentwise)}$$

or

$$G_2(z) := \begin{bmatrix} \Phi(F^1(z),\, z_I^1) \\ \vdots \\ \Phi(F^k(z),\, z_I^k) \end{bmatrix} \quad \text{with} \quad \Phi(a,b) := \sqrt{a^2+b^2} - (a+b);$$

see, e.g., [32]. Such a gap function can be then added to the objective f in form of a penalty. Some gap functions lead, under certain conditions, even to exact penalties. This is related to the theory of error bounds in mathematical programming ([34,57,41]). Let $\mathcal{E}$ be the set of (local) solutions to MPGE and $\hat{z} \in \mathcal{E}$. Further assume that to a gap function G there exist a neighbourhood $\mathcal{O}$ of $\hat{z}$ and a penalty parameter $\gamma > 0$ such that

$$\text{dist}_{\mathcal{E}}(z) \leq \gamma \|G(z)\| \quad \text{for all } z \in \mathcal{O} \cap \Omega\,.$$

Then we say that G generates a *(local) Lipschitz error bound* for the GE (1) near $\hat{z}$ with respect to Ω ([45]).

Consider the (MPGE) with equilibrium problem given by NCP (2). For $z \in I\!R^s$ define the index sets

$$\begin{aligned} L(z) &:= \{i \in \{1,\dots,d\} \mid z_I > 0\} \\ I_+(z) &:= \big\{i \in \{1,\dots,d\} \mid F^i(z) > 0\big\} \\ I_0(z) &:= \{1,\dots,d\} \setminus (L(z) \cup I_+(z))\,. \end{aligned}$$

It can be shown that the gap function G_1 generates a local Lipschitz error bound near the solution $\hat{z}$ with respect to Ω, if there are no nonzero vectors $u, v \in I\!R^d$ such that

$$\begin{aligned} &\text{for } i \in L \text{ one has } u^i = 0 \\ &\text{for } i \in I_+ \text{ one has } v^i = 0 \\ &\text{for } i \in I_0 \text{ one has either} \quad u^i v^i = 0 \quad \text{or} \quad u^i < 0 \quad \text{and} \quad v^i > 0 \end{aligned}$$

and

$$\begin{bmatrix} u \\ 0 \end{bmatrix} - (\nabla F(\hat{z}))^T\, v \in -N_\Omega(\hat{z})\,;$$

see [40, Thm. 3.1] and [43, Lemma 2.2]. This condition guarantees the Aubin property (see, e.g., [50]) of the map

$$r \mapsto \{z \in \Omega \mid r + G_1(z) = 0\}$$

around $(0, \hat{z})$ which, in turn, means that we compute $\hat{z}$ by solving a penalized problem

$$\begin{aligned} &\min_z f(z) + \gamma \|G_1(z)\| \\ &\text{subject to} \\ &\qquad z \in \Omega\,, \end{aligned}$$

whenever we start sufficiently close to $\hat{z}$ and $\gamma > 0$ is sufficiently large (but finite).

The exact penalization in MPECs is also studied in detail in [51] and [53], where the authors use results from the analysis of piecewise differentiable functions. In this approach the equilibrium constraint is written in the so-called normal equation form (see

[49]) and one works with a special "nonsmooth" Mangasarian–Fromowitz constraint qualification ([29]). This approach again leads to optimality conditions and a numerical method.

The Penalty Interior Point Algorithm (PIPA) of Luo, Pang and Ralph [33] is an SQP like iterative method that aims to stay strictly feasible with respect to the inequality constraints in the NCP (2). At each step, one solves a quadratic program to get a search direction. A line search in this direction is performed with the goal to stay strictly feasible and to sufficiently reduce the penalty function (for equilibrium problem given by NCP (2))

$$f(z) + \gamma\langle F(z),\, z_I\rangle$$

with a penalty parameter γ that is updated in every iteration. It was shown in [30], however, that PIPA can converge to a non-stationary point, for certain problems.

3.2. NLP reformulations

Many authors proposed various reformulations of (MPGE) as a smooth nonlinear problem and tried to solve this problem (or sequence of problems) by NLP methods like SQP. Since these techniques are mainly connected to MPCC, let us consider *only these problems* in the rest of this section.

Note first that MPCC of the type

$$\begin{aligned} &\min_z f(z) \qquad (11)\\ &\text{subject to}\\ &\quad h^i(z) = 0, \quad i = 1,\dots,o_1\\ &\quad g^i(z) \le 0, \quad i = 1,\dots,o_2\\ &\quad F(z) \ge 0, \quad z_I \ge 0\\ &\quad \langle F(z)\,,\, z_I\rangle = 0 \end{aligned}$$

is an NLP as such. (As before, h^i, $i = 1,\dots,o_1$, and g^i, $i = 1,\dots,o_2$, are assumed to be twice continuously differentiable.) Unfortunately, as shown in [8], Mangasarian–Fromowitz constraint qualification (MFCQ) for this problem is violated at all feasible points, and thus we may expect serious difficulties when trying to solve (11) directly by standard NLP algorithms.

Several techniques have been proposed to cope with this hurdle. The most obvious one is to replace the complementarity constraint in (11) by $\langle F(z)\,,\, z_I\rangle \le \tau$ with some $\tau > 0$, and solve a sequence of problems with $\tau \to 0$. This approach was analyzed in [51]; see also [16].

Another option is to replace the complementarity constraints by a nonsmooth equation (as in the previous section) and use a smoothing technique. Facchinei et al. [15] propose to use a smoothened min-function

$$G_\tau(z) := \sqrt{(F^i(z) - z_I^i)^2 + 4\tau} - F^i(z) - z_I^i = 0$$

with $\tau > 0$. Under some assumptions, they prove that the resulting NLPs possess nonempty and compact solution sets and the respective Karush-Kuhn-Tucker vectors exist. They further proposed and analysed several algorithms based on different choices of the sequence of smoothing parameters μ. An alternative technique, proposed by Jiang and Ralph [25], uses the smoothened Fischer-Burmeister function

$$G_\tau(z) := \sqrt{(F^i(z))^2 + (z_I^i)^2 + \tau} - F^i(z) - z_I^i = 0\,.$$

Again, one solves (inexactly) a sequence of NLPs with τ tending to zero. The authors further propose to treat τ as a variable and add an equation $e^\tau - 1 = 0$ to the NLP in order to ensure that, at the optimum, $\tau = 0$.

Finally, let us introduce a technique that, numerically, seems to be the most efficient one within this class. Consider again the MPCC problem (11) and replace in the complementarity constraint equality by inequality:

$$\begin{aligned} &\min_z f(z) \qquad (12)\\ &\text{subject to}\\ &\quad h^i(z) = 0, \quad i = 1, \dots, o_1\\ &\quad g^i(z) \le 0, \quad i = 1, \dots, o_2\\ &\quad F(z) \ge 0, \quad z_I \ge 0\\ &\quad \langle F(z)\,,\, z_I \rangle \le 0\,. \end{aligned}$$

Now solve this problem directly by an NLP code. Obviously, the new problem is equivalent to (11) and thus still does not satisfy MFCQ. However, numerical experiments performed independently by Anitescu [1] and Fletcher and Leyffer [18] showed that some SQP methods perform very well on many MPCC problems. Fletcher and Leyffer [18] report on results of three SQP codes, namely their own code FilterSQP, KNITRO [7] and SNOPT [21], on a collection of 137 MPCC problems. Two of these codes fail to find a solution only in 5 and 7 cases, respectively. Fletcher et al. [19] analyze under which conditions SQP methods applied to MPCC converge. The consequence of MFCQ not being satisfied is the unboundedness of the Lagrangian multipliers. It is shown in [19] that there exists a so-called basic multiplier vector; the multiplier set is then a ray whose base is this basic multiplier vector. And SQP methods are shown to converge quadratically to the basic multiplier, provided all QP subproblems remain consistent. But another consequence of the failure of MFCQ is the fact that any QP subproblem in SQP may be inconsistent. Anitescu [1] showed the importance of the *elastic mode*, implemented in some SQP codes (e.g., in SNOPT). This consists in modifying the NLP by relaxing the constraints and adding a penalty term to the objective function. SQP with elastic mode converges globally for problems (12) [1]. Another way how to cope with the QP inconsistency is to relax the linearization of the complementarity constraints, as adopted in FilterMPEC together with the so-called restoration mode [18].

In their paper, Fletcher and Leyffer [18] also report on results of the primal-dual interior point code LOQO [56]. Compared to the SQP codes, these results are not very satisfactory as LOQO fails in 20 instances. Benson et al. [5] analyze these failures and propose remedies. They show that by careful handling of the slack variables, by

proper initialization of the slacks, and by special treatment of problems with unbounded multipliers, they can get much better behaviour of LOQO than as reported in [18]. They further show (as also suggested in [18]) that the failure of LOQO is usually not due to failure of MFCQ but to other reasons, like unboundedness of the feasible set or rank-deficiency of the problem.

Let us list the main advantages and disadvantages of the NLP approach.

+ The approach is suitable for all MPCCs, i.e., also for those belonging to the SMPGE class. When solving MPEC–MPCC, no uniqueness of the equilibrium problem is needed (opposite to the ImP technique discussed below).
+ The NLP approach can directly handle state constraints (unlike the ImP technique).
+ There are several well-developed, robust and sophisticated NLP codes, both academic and commercial. Also, MPCC problems allow for formulations using tools like GAMS or AMPL. In this way, one can easily generate new problems and solve them using standard codes.

On the other hand,

– The fact that the NLP approach works on Cartesian product of the variables (for MPEC problems) prevents from using special solvers for the equilibrium problems (finite element solvers, multigrid, etc.). Both variables, control and state, are treated in the same way and any structure in the problem is ignored (in the current methods).
– The NLP approach is limited to MPCC problems and cannot be used for general MPECs.

In the rest of the paper, we will refer to the latter technique (direct solution of (12) by NLP software) as to *NLP technique* or *NLP approach*.

3.3. Implicit programming approach (ImP)

This approach is naturally connected with the MPEC problem class. Let $\Omega = \omega \times I\!R^m$, where ω is a set of feasible controls. The basic idea is to define a *solution map* of the equilibrium problem: a multifunction $S[I\!R^n \rightsquigarrow I\!R^m]$ that assigns each control variable $x \in \omega$ a set $S(x)$ of solutions to the equilibrium problem (viewed as a problem in variable y with a parameter x). Assume that for each pair $(\overline{x}, \overline{y}) \in (\omega \times I\!R^m) \cap \mathrm{Gph}\, S$ there exist neighborhoods $\mathcal{U}$ of $\overline{x}$, $\mathcal{V}$ of $\overline{y}$ and a function $\sigma[\mathcal{U} \to I\!R^m]$ such that $\overline{y} = \sigma(\overline{x})$ and $S(x) \cap \mathcal{V} = \{\sigma(x)\}$ for all $x \in \mathcal{U}$. In such a case, we can write (MPEC) as the following optimization problem:

$$\begin{aligned} &\min_x \Theta(x) := f(x, \sigma(x)) \\ &\text{subject to} \\ &\qquad x \in \omega\,. \end{aligned} \tag{13}$$

Obviously, this approach cannot be used for SMPGE problems, as it is based on the existence of a control variable. If y is subjected to an additional constraint

$$y \in \kappa \subset I\!R^m\,,$$

we cannot handle it directly by ImP. Instead, this constraint is usually treated via a penalty term added to the objective Θ. This is related to the so-called calmness property ([50]) of the constraint system

$$x \in \omega, \qquad \sigma(x) \in \kappa$$

and will not be tackled in the sequel. Instead, we will concentrate on a numerical method for solving (13) that proved to be rather efficient in a number of MPECs.

In fact, a number of minimization techniques can be used to solve (13) numerically; the choice depends mainly on the properties of the composite objective Θ. Each of these techniques is connected with some additional assumption on the problem data (f, S, ω). For instance, the method of Han et al. [46] proposed in [33] to the solution of (13) requires f to be continuously differentiable and σ locally Lipschitz and directionally differentiable.

The *bundle method* proposed in [42] requires Θ to be locally Lipschitz and weakly semismooth [54], which has to be translated into conditions imposed on f and σ. Additionally, ω must be given by affine equalities and inequalities. The weak semismoothness of Θ is ensured, provided f is continuously differentiable and σ is a PC^1-mapping [52]. This situation occurred in most problems that were analyzed and numerically solved in [42]. If σ does not happen to be PC^1, the weak semismoothness of Θ must be verified by analyzing the particular MPEC in question.

A typical bundle code [24] needs at each iterate x_k

– the function value $\Theta(x_k)$

and

– one element (subgradient) of Clarke's subdifferential $\overline{\partial}\Theta(x_k)$.

The computation of the function value is straightforward: for a given control x^k, one solves the equilibrium problem and plugs the solution into f. The critical point of the approach is the computation of the subgradient and we will discuss it thoroughly in the next section.

As in the NLP case, we list the main advantages and disadvantages of the ImP approach.

+ Clearly, from the computational viewpoint, the biggest advantage is that we separate the control and the state variable. Very often, MPEC models a real-world problem in which the two variables play a completely different role. They may also differ in the dimension: while the dimension of the state variable may be very high (typical for problems coming from finite element discretization), there may be only few (5–50) control variables. Computationally the most demanding part is then the solution of the state (and adjoint) problem for a given control. But since the variables are separated, one can use special codes developed for efficient solution of the state problem (like black-box finite element solvers based on multigrid or domain-decomposition methods).
+ It turns out that problem (13) if often very well structured for the use of bundle-type methods. These methods are particularly efficient for

- ◦ difficult nondifferentiable problems when we can only compute one arbitrary subgradient. This is our case: the nonsmoothness comes from the nonsmooth dependence of the solution of the equilibrium problem on the control variable and it may be extremely time consuming to compute the full subdifferential;
- ◦ problems with relatively few variables, say, up to 100. Again, this is often the case; see the above point.

On the minus side we have the following points:

- – The local single-valuedness and the Lipschitz continuity of S is certainly a restrictive assumption.
- – When the number of control variables becomes high, the existing codes of nondifferentiable optimization may become inefficient. For very difficult problems, the codes may not be robust enough, even for smaller dimensions.
- – The ImP approach cannot handle state constraints directly. Typically, they are treated by penalty terms added to the objective function in (13).
- – The approach is naturally limited to MPEC problems and cannot be used for solving SMPGEs with no control variables.
- – In case of complicated equilibria, the computation of Clarke's subgradients of Θ may be a difficult task. The next section is devoted to this particular problem.

4. Computation of subgradients for ImP

So far, in all papers dealing with ImP in connection with bundle methods, the subgradients needed by the method were computed via generalized Jacobians of the respective selection σ of S. In fact, one exactly follows the definition of the generalized Jacobian ([9]). When dealing with complex equilibria, like the three-dimensional contact problem with Coulomb friction or a HVI, this technique is not very handy. As a plausible alternative, one can employ the generalized differential calculus of B. Mordukhovich which is richer and deeper developed than its Clarke counterpart. This fact represents a significant advantage, particularly in cases when Q is not just the normal cone mapping of a simple set (Paragraphs 2.1.1, 2.1.3) but has a more complicated structure. Using the Mordukhovich calculus, one can also easily derive conditions ensuring that the computed vectors are indeed Clarke's subgradients (see conditions MF1, MF2 in [42]). In this section we will briefly explain how this alternative computation of subgradients can be performed.

Consider first an equilibrium governed by the GE

$$0 \in F(x, y) + Q(x, y), \tag{14}$$

with the dimensions as specified in the Section 2. Recall that F is continuously differentiable and Q has a closed graph. Assume that $(\overline{x}, \overline{y})$ is a *reference pair*, feasible with respect to (14), and put $\overline{z} := -F(\overline{x}, \overline{y})$.

The next statement concerns the properties of the respective solution map required by ImP.

Theorem 1. *Consider the solution map $S[\mathbb{R}^n \rightsquigarrow \mathbb{R}^m]$ given by (14) and assume that it is locally single-valued around $(\overline{x},\overline{y})$. Additionally suppose that the qualification condition*

$$\begin{pmatrix} x^* \\ 0 \end{pmatrix} \in (\nabla F(\overline{x},\overline{y}))^T\ v + D^*\, Q(\overline{x},\overline{y},\overline{z})\,(v) \Longrightarrow \begin{cases} x^* = 0 \\ v = 0 \end{cases} \tag{15}$$

is fulfilled. Then the (unique) selection σ of S, passing $(\overline{x},\overline{y})$, is Lipschitz on a neighborhood of $\overline{x}$.

Proof. Condition (15) implies the Aubin property of S around $(\overline{x},\overline{y})$; see [37, Thm. 4.3]. Since S is even locally single-valued around $(\overline{x},\overline{y})$, the result follows.

As f is continuously differentiable then, under assumptions of Theorem 1, one has

$$\partial\Theta(\overline{x}) = \nabla_x f(\overline{x},\overline{y}) + (\overline{\partial}\sigma(\overline{x}))^T\, \nabla_y f(\overline{x},\overline{y}) \tag{16}$$

due to the Chain Rule I of [9]. Since $D^*S(\overline{x})(y^*) \subset (\overline{\partial}\sigma(\overline{x}))^T y^*$ for all $y^* \in \mathbb{R}^m$ ([35]), each vector from the set $D^*S(\overline{x})(\nabla_y f(\overline{x},\overline{y}))$ leads to a vector from $\overline{\partial}\Theta(\overline{x})$.

Theorem 2. *(i) Let $(\overline{x}^*,\overline{v}) \in \mathbb{R}^n \times \mathbb{R}^m$ be a solution of the GE*

$$\begin{bmatrix} x^* \\ -\nabla_y f(\overline{x},\overline{y}) \end{bmatrix} \in (\nabla F(\overline{x},\overline{y}))^T v + \widehat{D}^* Q(\overline{x},\overline{y},\overline{z})\,(v). \tag{17}$$

Then one has

$$\overline{x}^* \in D^*S(\overline{x})(\nabla_y f(\overline{x},\overline{y}))\,.$$

(ii) Let the assumptions of Theorem 1 be satisfied. Then to each $x^* \in D^*S(\overline{x})(\nabla_y f(\overline{x},\overline{y}))$ there exists a vector $v \in \mathbb{R}^m$ such that

$$\begin{bmatrix} x^* \\ -\nabla_y f(\overline{x},\overline{y}) \end{bmatrix} \in (\nabla F(\overline{x},\overline{y}))^T v + D^* Q(\overline{x},\overline{y},\overline{z})\,(v). \tag{18}$$

Proof. The GE (14) can be rewritten to the form

$$\Phi(x,y) \in \operatorname{Gph} Q \qquad \text{with} \quad \Phi(x,y) = \begin{bmatrix} x \\ y \\ -F(x,y) \end{bmatrix}$$

so that, by Definition 3,

$$D^*S(\overline{x})(y^*) = \{x^* \in \mathbb{R}^n \mid (x^*,-y^*) \in N_{\Phi^{-1}(\operatorname{Gph} Q)}(\overline{x},\overline{y})\}\,.$$

Since $N_{\Phi^{-1}(\operatorname{Gph} Q)}(x,y) = \partial(\delta_{\operatorname{Gph} Q}(\Phi(x,y)))$, we have to do with a composition of a proper l.s.c. function and a smooth map. The statements (i) and (ii) follow now from [50, Thm. 10.6].

By comparing (17) and (18), one observes that

$$D^*S(\overline{x})(\nabla_y f(\overline{x},\overline{y})) = \{\overline{x}^* \in I\!R^n \mid \exists \overline{v} \in I\!R^m \text{ such that } (\overline{x}^*,\overline{v}) \text{ solves the GE (18)}\} \tag{19}$$

whenever $\mathrm{Gph}\, Q$ is regular at $(\overline{x},\overline{y},\overline{z})$. The GE (18) is called *adjoint generalized equation* (AGE).

In some cases $\mathrm{Gph}\ Q$ is not regular at $(\overline{x},\overline{y})$ and the GE (17) is difficult to solve. Then one can try to enforce the equality in (19) by some finer, problem-dependent assumption. Nevertheless, for a large class of equilibria a standard bundle method will deliver an acceptable solution even in the case when it is supplied by vectors $\nabla_x f(\overline{x},\overline{y}) + x^*$ with x^* from the right-hand side of (19), $\mathrm{Gph}\, Q$ being regular or not. Such a solution, however, is not necessarily an approximation of a Clarke stationary point, but of a point which is stationary in a weaker sense; see [11].

Another hurdle is associated with the solution of the AGE (18). Our goal is to choose such a selection of the multifunction $D^*Q(\overline{x},\overline{y},\overline{z})$ that will reduce (18) to an easily solvable system of equations. By means of a concrete equilibrium, we will now illustrate how this goal can be achieved.

Example 1. Consider the GE (14) with

$$Q(x,y) = N_{I\!R^m_+}(y - \Phi(x,y)), \tag{20}$$

where $\Phi[I\!R^n \times I\!R^m \to I\!R^m]$ is continuously differentiable. This GE describes a control-dependent Implicit Complementarity Problem (ICP). In [42] this GE was converted to an equivalent "expanded" form

$$0 \in \begin{bmatrix} F(x,y) - \lambda \\ y - \Phi(x,y) \end{bmatrix} + \begin{bmatrix} 0 \\ N_{I\!R^m_+}(\lambda) \end{bmatrix} \tag{21}$$

with an additional state variable $\lambda \in I\!R^m_+$. This reformulation enables us to apply the standard approach using generalized Jacobians. The essential requirement, discussed at the beginning of this section, can then be ensured via Robinson's strong regularity [42, Theorem 5.10]. The approach proposed here can be, however, directly applied to the original (non-expanded) GE.

Let us define the following index sets associated with the reference pair $(\overline{x},\overline{y})$:

$$\begin{aligned} L(\overline{x},\overline{y}) &:= \left\{ i \in \{1,2,\ldots,m\} \mid \overline{y}^i - \Phi^i(\overline{x},\overline{y}) > 0 \right\} \\ I_+(\overline{x},\overline{y}) &:= \{ i \in \{1,2,\ldots,m\} \mid -\overline{z} > 0 \} \\ I_0(\overline{x},\overline{y}) &:= \{1,2,\ldots,m\} \setminus \left(L(\overline{x},\overline{y}) \cup I_+(\overline{x},\overline{y}) \right). \end{aligned}$$

Lemma 1. *Assume that the matrix* $(-\nabla_x \Phi(\overline{x},\overline{y}),\ E - \nabla_y \Phi(\overline{x},\overline{y}))$ *has full row rank and* v *is a vector from* $I\!R^m$ *such that* $v^i = 0$ *for* $i \in I_+(\overline{x},\overline{y})$. *Then, for the multifunction*

(20), one has

$$D^*\,Q(\overline{x},\overline{y},\overline{z})\,(v)=\left\{\begin{bmatrix}-(\nabla_x\Phi(\overline{x},\overline{y}))^Tu\\ u-(\nabla_y\Phi(\overline{x},\overline{y}))^Tu\end{bmatrix}\mid u^i=0 \text{ for } i\in L(\overline{x},\overline{y}),\right.$$
$$u^i\in I\!R \text{ for } i\in I_+(\overline{x},\overline{y})$$
$$\left.\text{and, for } i\in I_0(\overline{x},\overline{y}),\, u^i\in\begin{Bmatrix} I\!R & \text{if } v^i=0\\ I\!R_- & \text{if } v^i<0\\ 0 & \text{otherwise}\end{Bmatrix}\right\}. \quad (22)$$

Proof. Define the function $\Psi[I\!R^n\times I\!R^m\times I\!R^m\to I\!R^m]$ by

$$\Psi(x,y,u):=\begin{pmatrix}y-\Phi(x,y)\\ u\end{pmatrix}.$$

The one has

$$\mathrm{Gph}\,Q=\Psi^{-1}(\mathrm{Gph}\,N_{I\!R^m_+})$$

and, like in the proof of Theorem 2,

$$N_{\mathrm{Gph}\,Q}(\overline{x},\overline{y},\overline{z})=\partial\delta_{\Psi^{-1}(\mathrm{Gph}\,N_{I\!R^m_+})}(\overline{x},\overline{y},\overline{z})=\partial\left(\delta_{\mathrm{Gph}\,N_{I\!R^m_+}}(\Psi(\overline{x},\overline{y},\overline{z}))\right).$$

By virtue of [50, Ex. 10.7]

$$\partial\left(\delta_{\mathrm{Gph}\,N_{I\!R^m_+}}(\Psi(\overline{x},\overline{y},\overline{z}))\right)=\begin{bmatrix}-(\nabla_x\Phi(\overline{x},\overline{y}))^T & 0\\ E-(\nabla_y\Phi(\overline{x},\overline{y}))^T & 0\\ 0 & E\end{bmatrix}N_{\mathrm{Gph}\,N_{I\!R^m_+}}(\overline{y}-\Phi(\overline{x},\overline{y}),\overline{z})$$

due to our surjectivity assumption. Consequently, by Definition 2,

$$D^*\,Q(\overline{x},\overline{y},\overline{z})\,(v)=\left\{\begin{bmatrix}-(\nabla_x\Phi(\overline{x},\overline{y}))^Tu\\ u-(\nabla_y\Phi(\overline{x},\overline{y}))^Tu\end{bmatrix}\mid u\in D^*N_{I\!R^m_+}(\overline{y}-\Phi(\overline{x},\overline{y}),\overline{z})(v)\right\}.$$

As shown in [43, Lemma 2.2], the coderivative $D^*N_{R^m_+}(\overline{y}-\Phi(\overline{x},\overline{y}),\overline{z})\,(v)$ is nonempty, provided $v^i=0$ for $i\in I_+(\overline{x},\overline{y})$. Moreover, for such vectors v, one has

$$D^*N_{I\!R^m_+}(\overline{y}-\Phi(\overline{y},\overline{y}),\overline{z})\,(v)=\bigtimes_{i=1}^m D^*N_{I\!R_+}(\overline{y}^i-\Phi^i(\overline{x},\overline{y}),\overline{z}^i)\,(v^i)$$

and

$$D^*N_{I\!R_+}(\overline{y}^i-\Phi^i(\overline{x},\overline{y}),\overline{z}^i)\,(v^i)=\begin{cases}0 & \text{for } i\in L(\overline{x},\overline{y})\cup\{j\in I_0(\overline{x},\overline{y})\mid v^j>0\}\\ I\!R & \text{for } i\in I_+(\overline{x},\overline{y})\cup\{j\in I_0(\overline{x},\overline{y})\mid v^j=0\}\\ I\!R_- & \text{for } i\in\{j\in I_0(\overline{x},\overline{y})\mid v^j<0\}.\end{cases}$$

The result has been established.

In this example, the AGE (18) reduces to a particularly simple form when we put $u^i=0$ for $i\in I_0(\overline{x},\overline{y})$. It can be solved in the following two steps:

1. Solve the linear system

$$\left[(\nabla_y F_{L\cup I_0}(\overline{x},\overline{y}))^T \qquad E_{I_+}^T - (\nabla_y \Phi_{I_+}(\overline{x},\overline{y}))^T\right] w = -\nabla_y f(\overline{x},\overline{y})$$
$$\text{in variable } w \in I\!R^m. \quad (23)$$

2. Put

$$x^* = \left[\,(\nabla_x F_{L\cup I_0}(\overline{x},\overline{y}))^T \qquad -(\nabla_x \Phi_{I_+})(\overline{x},\overline{y})^T\,\right] w. \quad (24)$$

The nonsingularity of the matrix on the left-hand side of (23) follows immediately from the qualification condition (15). Under this condition, system (23) has a unique solution and the respective vector x^*, computed in the 2nd step, belongs to the set on the right-hand side of (19).

Formulas (23), (24) coincide with those derived in [42] via generalized Jacobians under Robinson's strong regularity condition (see below) imposed on the expanded GE (21) at $(\overline{x},\overline{y},\overline{\lambda})$, $\overline{\lambda} = -\overline{z}$.

Consider now an equilibrium governed by the GE

$$0 \in F(x,y) + Q(y), \quad (25)$$

where Q maps this time $I\!R^m$ into subsets of $I\!R^m$. Again, we suppose that F is continuously differentiable, Q has a closed graph, $(\overline{x},\overline{y})$ is a reference pair and $\overline{z} = -F(\overline{x},\overline{y})$. In this case, the situation is slightly easier and most formulas can be significantly simplified.

For GEs of the type (25), the existence of a locally unique Lipschitz selection σ of S, passing the reference point $(\overline{x},\overline{y})$, is usually ensured via Robinson's strong regularity condition, see [48] and [13].

Definition 4. *We say that the GE (25) fulfils the* Strong Regularity Condition *(SRC) at* $(\overline{x},\overline{y})$*, provided the multifunction*

$$\Sigma(\xi) := \{y \in I\!R^m \mid \xi \in F(\overline{x},\overline{y}) + \nabla_y F(\overline{x},\overline{y})\,(y-\overline{y}) + Q(y)\}$$

is locally single-valued and Lipschitz around $(0,\overline{y})$.

The other possibility consists in a suitable modification of Theorem 1.

Theorem 3. *Consider the solution map S given by (25) and assume that it is locally single-valued around $(\overline{x},\overline{y})$. Additionally, suppose that the qualification condition*

$$0 \in (\nabla_y F(\overline{x},\overline{y}))^T v + D^*Q(\overline{y},\overline{z})\,(v) \quad \Longrightarrow \quad v = 0 \quad (26)$$

is fulfilled. Then the assertion of Theorem 1 remains true.

Proof. The map S can be equivalently written in the form

$$S(x) = \left\{ y \in I\!R^m \mid \begin{pmatrix} y \\ -F(x,y) \end{pmatrix} \in \mathrm{Gph}\ Q \right\}, \quad (27)$$

so that [35, Theorem 6.10] can be applied. By virtue of this result, condition (26) ensures that for all $y^* \in I\!R^m$

$$D^*S(\overline{x},\overline{y})\,(y^*) \subset \left\{(\nabla_x F(\overline{x},\overline{y}))^T v \mid 0 \in y^* + (\nabla_y F(\overline{x},\overline{y}))^T v + D^*Q(\overline{y},\overline{z})(v)\right\}.$$

Moreover, again by (26), $D^*S(\overline{x},\overline{y})\,(0) = \{0\}$, which is equivalent to the Aubin property of S around $(\overline{x},\overline{y})$ ([36]). The statement thus follows from the assumed local single-valuedness of S around $(\overline{x},\overline{y})$.

The relation between the strong regularity and the assumptions of Theorem 3 is clarified in the next statement.

Theorem 4. *Let the GE (25) fulfil (SRC) at $(\overline{x},\overline{y})$. Then the assumptions of Theorem 3 are satisfied.*

Proof. The local single-valuedness of S around $(\overline{x},\overline{y})$ follows directly from [13]. Thus it remains to prove only the implication (26). Since Σ is locally single-valued and Lipschitz around $(0,\overline{y})$, one has ([36])

$$D^*\Sigma(0,\overline{y})\,(0) = \{0\}$$

or, equivalently,

$$\operatorname{Ker} D^*\Sigma^{-1}(\overline{y},0) = \{0\}.$$

For the coderivative of Σ^{-1}, however, it holds that

$$D^*\Sigma^{-1}(\overline{y},0)\,(v) = (\nabla_y F(\overline{x},\overline{y}))^T v + D^*Q(\overline{y},\overline{z})\,(v)$$

for all $v \in I\!R^m$ ([35, Cor. 4.4]). The result has been established.

There are equilibria in which the essential assumption for the application of ImP is ensured by a completely different argumentation. This is, for instance, the case of discretized contact problems with Coulomb friction; see [6] and Section 5.1. Such equilibria amount to fixed-point problems to which, under certain assumptions, the Banach Fixed-Point Theorem can be applied. In this way, one can prove the required properties of S but the qualification condition (26) need not be satisfied, in general.

Theorem 5. *Consider the GE (25) around the reference pair $(\overline{x},\overline{y})$ and assume that all nonzero solutions to the GE in (26) do not belong to $\operatorname{Ker}(\nabla_x F(\overline{x},\overline{y}))^T$. Let $\overline{v}$ be a solution of the GE*

$$0 \in \nabla_y f(\overline{x},\overline{y}) + (\nabla_y F(\overline{x},\overline{y}))^T v + D^*Q(\overline{y},\overline{z})\,(v). \tag{28}$$

Then, for all $y^* \in I\!R^m$, one has

$$D^*S(\overline{x},\overline{y})\,(y^*) \subset \left\{(\nabla_x F(\overline{x},\overline{y}))^T\overline{v} \mid \overline{v} \text{ is a solution of (28)}\right\}. \tag{29}$$

This inclusion becomes equality, provided either $\nabla_x F(\overline{x},\overline{y})$ is surjective or $\operatorname{Gph} Q$ is regular at $(\overline{y},\overline{z})$.

Proof. As in the proof of Theorem 3, we rewrite the GE (25) in the form (27) and apply [35, Theorem 6.10]. From this result, one easily derives the above assumption as well as relations (28), (29). It also implies the equality in (29) under the regularity of $\operatorname{Gph} Q$ at $(\overline{y},\overline{z})$. To show that the surjectivity of $\nabla_x F(\overline{x},\overline{y})$ also implies this equality, one can employ the argumentation used in the proof of Lemma 1.

Remark 1. Since (28) plays the same role as (18), we will call it AGE, too.

The assumption in Theorem 5 is trivially satisfied when either the qualification condition (26) holds true or the matrix $\nabla_x F(\overline{x},\overline{y})$ is surjective. In this way, we have no problems with this assumption whenever the required behaviour of S is ensured via the strong regularity or via Theorem 3.

In many situations $Q(y) = \bigtimes_{i=1}^{m} Q^i(y^i)$. In this case, the surjectivity and/or regularity conditions from Theorem 5 (ensuring equality in (29)) can be applied to each Q^i separately and combined together. In such a way, one arrives at finer conditions, under which

$$\nabla_x f(\overline{x},\overline{y}) + (\nabla_x F(\overline{x},\overline{y}))^T \overline{v} \in \partial\Theta(\overline{x})$$

with some solution $\overline{v}$ of (28).

The last issue concerns the solution of (28). If the GE (25) models equilibria discussed in [42] and (SRC) is fulfilled, then it is always possible to choose a selection of $D^*Q(\overline{y},\overline{z})$ in such a way that (28) reduces to a uniquely solvable linear system. This goal can be achieved, however, also in case of more complicated equilibria. To illustrate the procedure, consider equilibria governed by the GE (25), where

$$F(x,y) = A(x)\,y + \ell(x) \quad \text{and} \quad Q(y) = \partial J(y). \tag{30}$$

Here the maps $A[I\!R^n \to I\!R^{m\times m}]$, $\ell[I\!R^n \to I\!R^m]$ are continuously differentiable and $J[I\!R^m \to \overline{I\!R}]$ is a proper convex and l.s.c. function. Furthermore, suppose that $(\overline{x},\overline{y})$ is a reference pair and $A(\overline{x})$ is positive definite. Then it is easy to show that the respective GE fulfils (SRC) at $(\overline{x},\overline{y})$ and hence also condition (26) is satisfied. The computation of the coderivative of the multi-valued part in (30) is generally rather difficult. Nevertheless, besides the simple case

$$J(y) = \delta_{I\!R^m_+}(y),$$

corresponding to control-dependent linear complementarity problems, there are other classes of functions for which the coderivative of their subdifferential mapping was computed. In [14] the authors consider the case

$$J(y) = \delta_{\Lambda}(y),$$

where Λ is a convex polyhedron. In [38] another situation is investigated: d is a given natural number and

$$J(y) = \sum_{i=1}^{d} j^i\left(\langle c_i, y\rangle + b^i\right),$$

where $c_i \in I\!R^m$, $b^i \in I\!R$ and $j^i[I\!R \to \overline{I\!R}]$ is a sum of a piecewise C^2 function and the indicatrix of a closed interval, $i = 1,2,\ldots,d$. Assume that the $[m \times d]$ matrix

$$C^T := \left[c_1 \cdots c_d\right]$$

has full column rank and denote by $\overline{u}$ the (unique) vector satisfying the relations

$$\begin{aligned}
&\overline{z} = C^T\overline{u} \quad (\overline{z} = -A(\overline{x})\,\overline{y} - \ell(\overline{x}))\,,\\
&\overline{u}^i \in \partial j^i\left(\langle c_i,\overline{y}\rangle + b^i\right), \quad i = 1,2,\ldots,d.
\end{aligned}$$

Then, for all $v \in I\!R^m$, one has ([38, Theorem 3.4])

$$D^*Q(\overline{y},\overline{z})\,(v) = C^T \bigtimes_{i=1}^{d} D^*\partial j^i \left(\langle c_i,\overline{y}\rangle + b^i,\overline{u}^i\right)\left(\langle c_i, v\rangle\right). \tag{31}$$

Example 2. Consider an academic equilibrium given by the GE

$$0 \in \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} + \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \partial J(y), \tag{32}$$

where $J(y) = j^1(y^1 + 0.5) + j^2(y^1 + y^2 + 1)$ with

$$j^1(\xi) = 0.25 \max\left\{e^{\xi}, e^{-\xi}\right\}, \quad j^2(\xi) = \delta_{I\!R_-}(\xi),$$

around the reference pair $\overline{x}^1 = 0.25$, $\overline{x}^2 = 0.5$, $\overline{y}^1 = \overline{y}^2 = -0.5$. The GE (32) represents the necessary and sufficient optimality conditions of the convex nonsmooth control-dependent program

$$\min \left[\frac{1}{2}(y^1)^2 + \frac{1}{2}(y^2)^2 + x^1y^1 + x^2y^2 + \max\left\{e^{y^1+0.5}, e^{-y^1-0.5}\right\}\right]$$

subject to

$$y^1 + y^2 \le -1.$$

One easily verifies that all above assumptions are fulfilled and the AGE (28) attains the form

$$0 \in \nabla_y f(\hat{x},\hat{y}) + \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{33}$$

with $\xi \in D^*\partial j^1(0,\,0.25)\,(v^1)$ and $\eta \in D^*N_{I\!R_-}(0,0)\,(v^1+v^2)$. By Definitions 2 and 3, $D^*\partial j^1(0,\,0.25)\,(0) = D^*N_{I\!R_-}(0,0)\,(0) = I\!R$. One of the solutions to (33) has thus a particularly simple form: $\overline{v} = 0$ ($\eta = -\nabla_{y^2} f(\hat{x},\hat{y}),\ \xi = -\nabla_{y^1} f(\hat{x},\hat{y}) - \eta$).

5. Three examples

Here we present three numerical examples that should demonstrate the usability and efficiency of the NLP and ImP techniques. The first example (shape optimization of an elastic bodies in unilateral contact under Coulomb friction) represents an MPEC that is solvable by the ImP technique but its reformulation to an MPCC requires a rather complicated and nontransparent transformation ([42]).

The second example (unique reformulation of truss topology design problem) is an SMPGE; that means, we cannot apply ImP here. We will solve it by the NLP technique and compare various NLP solvers.

The third example (shape optimization of a membrane with a compliant obstacle) is an MPEC–MPCC with a uniquely solvable equilibrium problem. For its solution, we can apply both the ImP and the NLP technique.

5.1. Shape optimization in contact problems with Coulomb friction [6]

In Paragraph 2.1.4 we gave a brief introduction to this problem. The full details of the modeling and finite element analysis can be found in [6]. The respective GE is of the type discussed in Paragraph 2.1.4 and attains the form

$$\left.\begin{aligned} 0 &\in A_{\tau\tau}(x)u_\tau + A_{\tau\nu}(x)u_\nu - \ell_\tau(x) + Q(u_\tau, \lambda_\nu) \\ 0 &= A_{\nu\tau}(x)u_\tau + A_{\nu\nu}(x)u_\nu - \ell_\nu(x) - \lambda_\nu \\ 0 &\in u_\nu + x + N_{I\!R^p_+}(\lambda_\nu), \end{aligned}\right\} \tag{34}$$

with a control variable $x \in I\!R^n$ specifying the shape of the contact boundary Γ and state variables $u = (u_\tau, u_\nu) \in I\!R^{2p}$ (the normal and tangential components of the displacements on the contact boundary) and $\lambda_\nu \in I\!R^p$ (the normal component of the contact stress vector); see Fig. 1. The multivalued part in the first line of (34) is given by

$$Q(u_\tau, \lambda_\nu) = \lambda_\nu \bullet \partial j(u_\tau), \quad j(u_\tau) = \mathcal{F} \sum_{i=1}^{p} |u_\tau^i| \,.$$

The contact boundary Γ is modeled by a Bezier curve of order n; the design variable x is a vector of its control points. The end points of the Bezier curve are identical with the first and last control point.

The shape optimization problem is defined as follows:

$$\begin{aligned} &\min_{x,u,\lambda_\nu} f(x, u, \lambda_\nu) \\ &\text{subject to} \\ &\qquad (u, \lambda_\nu) \text{ solves the GE (34)} \\ &\qquad x \in \omega \end{aligned}$$

with

$$\omega = \{x \in I\!R^n \mid 0 \le x^i \le C_0,\ i = 0, 1, \ldots, n; \\ |x^{i+1} - x^i| \le \frac{C_1}{n},\ i = 0, 1, \ldots, n-1; \quad \sum_{i=0}^{n} x^i = C_2(n+1)\} \,,$$

where C_0, C_1, C_2 are given positive constants. The equality constraint in the definition of ω has a physical meaning of preserving the body volume.

We will solve this MPEC by the ImP technique, whereas the nonsmooth optimization problem will be solved by the BT code [54]. In every BT iteration we have to solve the equilibrium problem, i.e., the Signorini problem with Coulomb friction formulated as a fixed-point problem. For that, we use the splitting variant of the fixed-point method introduced in [23]. This is basically the method of successive approximations where, at each step, we solve the contact problem with a given friction. The iterative process then updates the coefficient of the given friction. The problem with the given friction is solved using the so-called reciprocal variational formulation that leads to a quadratic

programming problem with simple box constraints. For its solution we use a so-called splitting technique, a version of the Gauss–Seidel algorithm.

Next, we will present results of a numerical example. The shape of the unloaded elastic body $\mathcal{O}(x), x \in \omega$, is defined through a Bezier curve $\mathcal{B}_x$ as follows:

$$\mathcal{O}(x) = \{(\xi_1, \xi_2) \in I\!R^2 \mid \xi_1 \in (0, a),\ \mathcal{B}_x(x_1) < \xi_2 < b\};$$

see Figure 1. This figure also shows the distribution of external loads P on the boundary Γ_P. Further, Γ_u is the part of the boundary with prescribed Dirichlet condition.

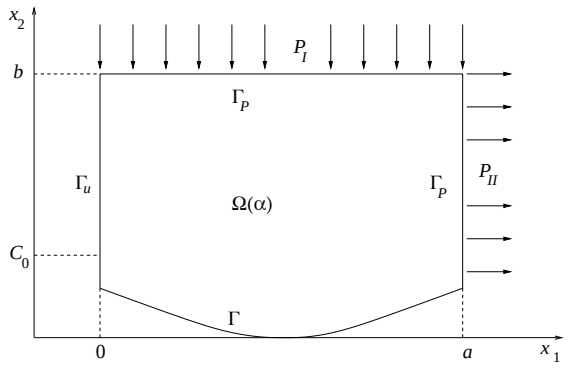

Fig. 1. The elastic body and applied loads.

Example 3. We try to identify the contact normal stress distribution with a given function $\overline{\lambda}_\nu$. The shape optimization problem can be written as

$$\begin{aligned} &\min \ \|\overline{\lambda}_\nu - \lambda_\nu\|_2^2 \\ &\text{subject to} \quad x \in \omega\,. \end{aligned}$$

The example was solved by the ImP technique in connection with the BT code. We discretized $\mathcal{O}(x)$ by a regular 29×9 mesh, i.e, we had 261 nodes and 522 unknowns in the state problem. The dimension of the control vector x, generating the Bezier curve and defining $\mathcal{O}(x)$, was set to 20.

Let us demonstrate the convergence behavior of the BT algorithm when minimizing the nonsmooth composite objective function Θ. Figure 2 shows the decrease of the function value in the example. We needed 123 BT iterations and 126 function and subgradient evaluations to reach the prescribed accuracy. This figure also presents the development of the BT stopping criterium during the iteration process.

5.2. Unique reformulation of truss topology design problem

A truss is an assemblage of pin-jointed uniform straight bars. The bars are subjected to only axial tension and compression when the truss is loaded at the joints. The truss is

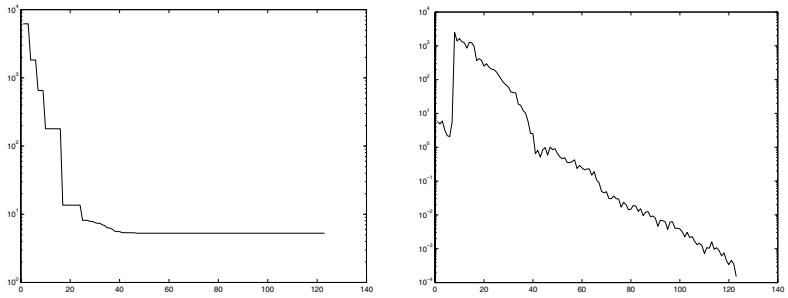

Fig. 2. Behavior of the BT algorithm in Example 3. Decrease of the cost functional (left) and of the value of the stopping criterium (right), both in logarithmic scale.

characterized by the set of nodes, bars and nodal forces. Some of the nodes are assumed to be fixed, that is, some components of the displacement vector are forced to be zero; let us denote by n the number of free displacement components. The nodal positions, forces and displacements are assembled in vectors $x \in I\!R^n$, $f \in I\!R^n$ and $u \in I\!R^n$, respectively. The bar volumes are denoted by t_i, $i = 1, \ldots, m$, where m is the number of bars.

Consider first the *truss analysis problem*, i.e., the problem of finding displacements for a given truss (a given vector t). This is a convex quadratic problem of minimizing the potential energy subject to possible unilateral contact constraints:

$$\begin{aligned} &\min_{u \in I\!R^n} \frac{1}{2} u^T A u + f^T u \\ &\text{subject to} \\ &\qquad Cu \geq 0\,. \end{aligned} \tag{35}$$

Here $A = A(t) \in I\!R^{n \times n}$ is the (symmetric and positive semidefinite) stiffness matrix of the truss and $C \in I\!R^{p \times n}$ contains the unilateral contact information. This problem does not have a unique solution, in general. The stiffness matrix may even have many zero eigenvalues when t is obtained as a result of topology optimization. There is, however, a unique "physical" solution of problem (35). This solution can be identified as the minimizer of another quadratic functional $u^T G u$ over all solutions of (35). Here $G \in I\!R^{n \times n}$ is a symmetric and positive definite matrix; its construction is explained in [27].

Formulating the truss analysis problem (35) as complementarity problem, we can write the *unique* truss analysis problem as the SMPGE

$$\begin{aligned} &\min_{u \in I\!R^n, \lambda \in I\!R^p} u^T G u \\ &\text{subject to} \\ &\qquad Au + f - C^T \lambda = 0 \\ &\qquad\qquad\qquad Cu \geq 0 \\ &\qquad\qquad\qquad \lambda \geq 0 \\ &\qquad\qquad\qquad \lambda^T C u \leq 0\,. \end{aligned} \tag{36}$$

In a similar way, we can define a unique reformulation of the *truss topology design problem*. In the traditional formulation of the problem, one minimizes the so-called compliance (maximizes the stiffness) of the truss subject to equilibrium conditions:

$$\min_{t\in I\!R^m, u\in I\!R^n} -f^T u \tag{37}$$

subject to

$$\left(\sum_{i=1}^m t_i A_i\right) u + f = 0$$
$$\sum_{i=1}^m t_i \le 1$$
$$t_i \ge 0, \quad i = 1,\dots,m\,;$$

here $A_i \in I\!R^{n\times n}$ are symmetric and positive semidefinite stiffness matrices of single bars. It is well-known that this (nonconvex) problem is extremely difficult to solve by standard NLP codes. It was shown by Ben-Tal and Bendsøe [4] that (37) can be equivalently formulated as a convex quadratically constrained quadratic program

$$\min_{\alpha\in I\!R, u\in I\!R^n} \alpha - f^T u \tag{38}$$

subject to

$$\frac{1}{2} u^T A_i u - \alpha \le 0 \quad i = 1,\dots,m\,.$$

Problems (37), (38) do not have a unique solution, in general. Using the same technique as for the truss analysis problem, we define a *unique* truss design problem as an SMPGE and use for its solution the NLP approach.

The NLP formulation of the unique truss design problem reads as:

$$\min_{t\in I\!R^m, u\in I\!R^n, \alpha\in I\!R} u^T G u \tag{39}$$

subject to

$$\left(\sum_{i=1}^m t_i A_i\right) u + f = 0$$
$$\sum_{i=1}^m t_i \le 1$$
$$\alpha - \frac{1}{2} u^T A_i u \ge 0, \quad i = 1,\dots,m$$
$$t_i \ge 0, \quad i = 1,\dots,m$$
$$\sum_{i=1}^m t_i (\alpha - \frac{1}{2} u^T A_i u) \le 0\,.$$

To demonstrate the capability of the NLP approach, we generated three examples of the unique truss analysis problem of increasing dimension and solved them by NLP

codes available on the NEOS[1] server. Figure 3 shows one such truss—it is the result of the truss topology optimization problem and we can see that many bars have zero volume. Also, there are many nodes lying on straight bars, a source of possible nonuniqueness. The problems were solved using formulation (36).

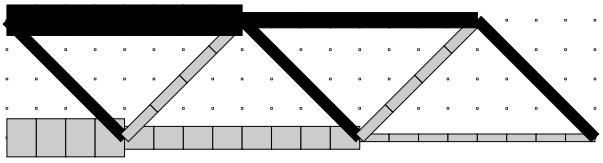

Fig. 3. Problem tr21x5. The truss is fixed at the left-hand side nodes and subjected to a vertical load at the right-lower node.

The following table shows results of the most successful NEOS codes, namely, LOQO [56], Filter [17], SNOPT [21], MINOS [39], and of the code PENNON [28]. We must remark that "one iteration" has different meaning in different codes, and thus

Table 1. Results for NLP reformulation of the unique truss analysis problem. Given are the numbers of iterations for each code. "F" stands for failure. "var" and "constr" are the numbers of variables and constraints, respectively.

problem	var	constr	LOQO	Filter	SNOPT	MINOS	PENNON
tr_11x3	47	34	15	10	63	53	13
tr_21x5	103	72	34	9	137	110	23
tr_41x9	569	534	F	F	F	F	F

the table basically presents the ability of the code to solve the particular problem.

The unique truss optimization problem (39) resembles problem (37). As (37) was a difficult NLP problem, we cannot expect (39) to be any simpler and this is clearly seen from our numerical results. It turned out that the examples from previous table are all too large, so we generated several smaller ones and solved them again by NLP codes available on the NEOS server. These examples include additional linear inequality constraints on the displacement of certain nodes (modeling rigid obstacle).

The only codes capable to solve at least some of these examples were SNOPT and MINOS, all the other codes failed. The following table shows the results[2]. These results indicate that SMPGE problems with difficult equilibria are the more difficult when solved by the NLP approach.

5.3. Design of membrane with compliant obstacle [42]

We first define the equilibrium problem—a membrane with a compliant obstacle. Let $\mathcal{O}(x)$ be a domain in $I\!R^2$ with a Lipschitz boundary described by a design variable x.

[1] `http://www-neos.mcs.anl.gov/neos/`

[2] All problems from this section are available in AMPL format on the author's webpage `http://www2.am.uni-erlangen.de/~kocvara/mpec/`.

Table 2. Results for NLP reformulation of the unique truss design problem. Given are the numbers of iterations for each code. "F" stands for failure. "var" and "constr" are the numbers of variables and constraints, respectively.

problem	var	constr	SNOPT	MINOS
tro_3x3	31	33	259	368
tro_4x4	64	66	593	F
tro_5x5	109	111	1122	F
tro_6x2	46	48	763	F
tro_11x3	151	153	F	F

Assume for the moment that x is kept fixed. The domain is occupied by a membrane subjected to a force ℓ. The state variable u represents the respective deflection. Again, we give directly a formulation of the problem discretized by the finite element method. Let $A(x)$ and $\ell(x)$ be the stiffness matrix and the force vector in $\mathcal{O}(x)$. The loaded membrane cannot penetrate a compliant obstacle; this condition is expressed by the inequality

$$u \geq G(x,u) := k(A(x)u - \ell(x)) + \chi(x)\,,$$

where χ describes the original shape of the obstacle and k is a coefficient of compliance. The equilibrium problem for a membrane with a compliant obstacle is an ICP in variable u:

$$\begin{aligned} &A(x)u - \ell(x) \geq 0, \quad u - G(x,u) \geq 0 \\ &\langle A(x)u - \ell(x),\, u - G(x,u)\rangle = 0. \end{aligned} \tag{40}$$

In the optimum design problem, our objective is to minimize the membrane surface under the state constraint that in a given subset $\mathcal{O}_0 \subset \mathcal{O}(x)$ of the membrane stays in contact with the obstacle ([42]). The design variables are the positions of the nodes on a part of the boundary of $\mathcal{O}(x)$. Let $\mathcal{D}_0$ be an index set including numbers of the nodes from $\mathcal{O}_0$. The optimum design problem reads as

$$\begin{aligned} &\min f(x,u) := \operatorname{meas} \mathcal{O}(x) \\ &\text{subject to} \\ &\qquad u \text{ solves the ICP (40)} \\ &\qquad (u - G(x,u))^i = 0 \quad \text{for } i \in \mathcal{D}_0 \\ &\qquad x \in \omega. \end{aligned} \tag{41}$$

The set of admissible design variables ω is defined in such a way that the boundary of $\mathcal{O}(x)$ can only move within given bounds.

In ImP, we cannot handle the state constraint in (41) explicitly; instead, we modify the objective function by an exact penalty with the penalty parameter γ (h is a discretization parameter):

$$\begin{aligned} &\min f_\gamma(x,u) := \operatorname{meas} \mathcal{O}(x) + \gamma h^2 \sum_{i \in \mathcal{D}_0} (u - G(x,u))^i \\ &\text{subject to} \\ &\qquad u \text{ solves the ICP (40)} \\ &\qquad x \in \omega. \end{aligned} \tag{42}$$

Example 4. We chose two example that are included in the MacMPEC[3] collection, in order to compare the ImP and NLP techniques. The examples differ by the choice of the obstacle functions $\chi_1(\xi_1,\xi_2) = -0.04(\xi_1^2 + (\xi_2^2 - 0.25)^2)$ and $\chi_2(\xi_1,\xi_2) = -0.05\xi_1$. The load is constant over the membrane.

We first solved these problems using the ImP technique. The equilibrium problems were solved by the nonsmooth Newton method applied to the following nonsmooth equation, equivalent to ICP (40):

$$\min\{A(x)u - \ell(x), u - G(x,u)\} = 0 \qquad \text{(componentwise)}.$$

The nonsymmetric linear systems were solved by the minimum residuum method without preconditioning. There are other alternative methods for solving of this type of ICP that may be equally effective. For the computation of subgradients of the composite function Θ we used formulas (23), (24) from Example 1.

Five discretizations were used with $h \in \{\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}\}$; the value of the penalty parameter was $r = 10^6$ in all cases. Table 3 summarizes the results. Here we use the same problem names as in MacMPEC. The numbers of variables and constraints are given from the ImP viewpoint. We observe a typical behaviour of a bundle algorithm in

Table 3. Results of the ImP approach using BT code. Given are the numbers of BT iterations and CPU time on Pentium 3 PC (1 GHz) running Windows 2000.

problem	var	constr	iter	CPU(sec)	f^*
pack-comp1-8	9	18	4	0.27	0.60000000
pack-comp1-16	17	34	36	0.84	0.61695165
pack-comp1-32	33	66	75	8.91	0.65297999
pack-comp1-64	65	130	93	105.98	0.68197856
pack-comp1-128	129	258	77	2262.30	0.69398364
pack-comp2-8	9	18	187	0.64	0.67311716
pack-comp2-16	17	34	76	1.38	0.72713674
pack-comp2-32	33	66	71	8.54	0.78260579
pack-comp2-64	65	130	164	180.49	0.82122227
pack-comp2-128	129	258	91	2515.19	0.83365644

MPEC: the number of iterations is virtually independent on the size of the problem.

In Table 4 we present results of the NLP approach for $h \in \{\frac{1}{8}, \frac{1}{16}, \frac{1}{32}\}$, whereas the NLP problems are solved by FilterMPEC and LOQO. These results were published in [18] and [56]. The CPU times were obtained on different computers and should be taken just as indications of the codes behaviour.

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[3] `http://www.mcs.anl.org/~sleyffer/macmpec/`

Table 4. Results of the NLP approach using FilterMPEC and LOQO.

			FilterMPEC		LOQO		
problem	var	constr	iter	CPU	iter	CPU	f^*
pack-comp1-8	156	180	6	0.07	40	0.71	0.60000
pack-comp1-16	692	736	8	1.66	31	2.78	0.61695
pack-comp1-32	2916	3068	49	1813.46	66	44.59	0.65298
pack-comp2-8	156	180	9	0.10	39	0.80	0.67312
pack-comp2-16	692	736	33	7.80	28	2.65	0.72714
pack-comp2-32	2916	3068	36	1027.19	54	33.55	0.78260

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