Asymptotic behavior of solutions to the generalized Becker-Döring equations for general initial data

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These results were obtained under the supervision and help of Stéphane Mischler. I wish to thank him for his explanations and suggestions. The author has been supported by a HYKE grant.

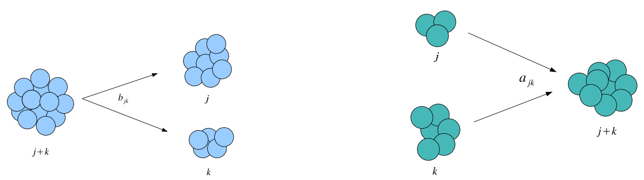
Main result

In [3] (to appear) we prove that the well-known asymptotic behavior of solutions to the generalized Becker-Döring system takes place for general initial data, extending the previous knowledge that placed some restrictions on it.

The coagulation-fragmentation equations

The coagulation-fragmentation equations describe the evolution of a large number of clusters which can stick together or break. Here we deal with the discrete version.

$c_j \equiv$ number density of clusters of size j



$b_{jk} \equiv$ rate of occurrence of reaction $j+k \to j,k$

$a_{jk} \equiv$ rate of occurrence of reaction $j \to j+k$

$$\frac{d}{dt}c_j = \frac{1}{2}\sum_{k=1}^{j-1} a_{k,j-k}c_k c_{j-k} \qquad \text{Coagulation gain}$$

$$-\sum_{k=1}^{\infty} a_{jk}c_j c_k \qquad \text{Coagulation loss}$$

$$+\sum_{k=j+1}^{\infty} b_{j,k-j}c_k \qquad \text{Fragmentation gain}$$

$$-\frac{1}{2}\sum_{k=1}^{j-1} b_{k,j-k}c_j \qquad \text{Fragmentation loss}$$

The generalized Becker-Döring system is the special case where a_{jk} and b_{jk} are zero whenever $\min\{j,k\} > N$ for some N. For $N=1$ the system is the Becker-Döring system.

Asymptotic Behavior

The study of the long-time behavior of solutions to these equations is expected to be a model of physical processes such as phase transition. Under certain general conditions which include a detailed balance we can ensure the existence of equilibrium states. In these conditions, there is a critical mass $\rho_s \in]0,\infty[$ such that any solution that initially has mass $\rho_0 \leq \rho_s$ will converge for large times, in a certain strong sense, to an equilibrium solution with mass ρ_0. On the other hand, any solution with mass above ρ_s converges (in a weak sense) to the only equilibrium with mass ρ_s; this weak convergence can then be interpreted as a phase transition in the physical process modelled by the equation.

Convergence in this weak sense means that a fixed part of the total mass of particles is found to be forming larger and larger clusters as time passes and the mean size of clusters goes to infinity. The physical interpretation of this, depending on the context, can be a change of phase or the apparition of crystals, for example.

Below critical mass $\rightarrow$ $\begin{cases}\text{Trend to equilibrium}\\ \text{Strong convergence}\end{cases}$

Over critical mass $\rightarrow$ $\begin{cases}\text{Large clusters created}\\ \text{Weak convergence}\end{cases}$

Previous results

Becker-Döring system	Ball, Carr, Penrose [1, 2] (1986-88)
Generalized Becker-Döring (rapidly decaying initial data)	Carr, da Costa [4] (1994)
Generalized Becker-Döring (small initial data)	da Costa [5] (1998)

Sketch of the proof

Our proof is a generalization of a method used in unpublished notes by Ph. Laurençot and S. Mischler [6], inspired by the proof of uniqueness of solutions to the Becker-Döring equation in [7].

It is known that, under common assumptions, *there is always* at least weak convergence to a certain equilibrium state; the problem reduces to show that for an initial density under the critical one solutions converge *strongly* to the equilibrium *with the same density*. To prove this, it is enough to show that the tails of the solutions are small enough, so that strong convergence holds. The following estimate, roughly stated here, is the key of our proof:

Main estimate

If $c = \{c_j\}_{j\geq 1}$ is a solution to the generalized Becker-Döring equations with density below the critical one, then there is some sequence r_i (which tends to zero as $i \to \infty$) such that the tails of the solution have mass below r_i; this is,

$$\sum_{k=i}^{\infty} k c_k(t) \leq r_i$$

for all times t after some time t_0.

The proof of this consists mainly of an estimate obtained by differentiating the quantity $H_i := (G_i - r_i)_+$ (the positive part of $G_i - r_i$), proving with a differential inequality that it must remain zero for all times starting from a certain t_0.

References

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