Ramsey shadowing and recurrence

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Table of contents

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- The system (X, f) has **shadowing** if for every $\varepsilon > 0$ there is a $\delta > 0$ such that every δ-pseudo-orbit can be ε-shadowed by some $x \in X$.

A familiar definition

A *filter* $\mathcal{F}$ on ω is a set of subsets of ω satisfying:

1. **Nontriviality:** $\emptyset \notin \mathcal{F}$ and $\omega \in \mathcal{F}$.
2. **Upwards heredity:** if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.
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Every family $\mathcal{F}$ has a **dual**, namely
$\mathcal{F}^* = \{A \subseteq \omega \colon \forall B \in \mathcal{F}\ A \cap B \neq \emptyset\}$.

The Ramsey property

A family $\mathcal{F}$ has the *Ramsey property* if whenever $A \in \mathcal{F}$ and $A = B_1 \cup \cdots \cup B_n$, then there is some $i \leq n$ such that $B_i \in \mathcal{F}$.

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For example:

- Families with the Ramsey property include the infinite sets, the sets of nonzero density, any ultrafilter, the piecewise syndetic sets, the set of all sets containing arbitrarily long arithmetic sequences (van der Waerden), the IP sets (Hindman)

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- Families with the Ramsey property include the infinite sets, the sets of nonzero density, any ultrafilter, the piecewise syndetic sets, the set of all sets containing arbitrarily long arithmetic sequences (van der Waerden), the IP sets (Hindman)
- Families without the Ramsey property include the cofinite sets, the sets containing infinite arithmetic sequences, any nonmaximal filter, the thick sets, the syndetic sets, the dense sets with respect to the topology on $\mathbb{Q}$

The Ramsey shadowing property

- If X is a dynamical system, let us say that $x \in X$ shadows a sequence $\langle x_n \colon n \in \omega \rangle$ on A if $\{n \in \omega \colon d(f^n(x), x_n) < \varepsilon\} = A$.

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- X has the **Ramsey shadowing property** if, whenever $\mathcal{F}$ is a family with the Ramsey property, $\xi = \langle x_n \colon n \in \omega \rangle$ is a sequence in X, and $\varepsilon > 0$, there is some $x \in X$ that ε-shadows ξ on a set in $\mathcal{F}$.

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Theorem (Brian, Meddaugh, and Raines, 2014)

If X is chain transitive and has the shadowing property, then X has the Ramsey shadowing property.

The basic idea: global to local

- X has the **local Ramsey shadowing property** if, whenever $\mathcal{F}$ is a family with the Ramsey property and $x_0 \in X$, there is some $x \in X$ that shadows the constant sequence $\langle x_0, x_0, x_0, \dots \rangle$ on a set in $\mathcal{F}$.

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- Our main lemma is to show that local Ramsey shadowing is equivalent to Ramsey shadowing.
- Clearly Ramsey shadowing implies local Ramsey shadowing. For the converse, it is necessary to introduce two more (related) types of shadowing: ultrafilter shadowing and local ultrafilter shadowing.
- The proof goes like this:

$$LRS \Rightarrow LUS \Rightarrow US \Rightarrow RS$$

Glasner's theorem

A theorem of Glasner provides the first and third implications:

Theorem (Glasner, 1980)

1. *A family $\mathcal{F}$ has the Ramsey property iff $\mathcal{F}^*$ is a filter.*
2. *If $\mathcal{F}$ has the Ramsey property, there is an ultrafilter $p \subseteq \mathcal{F}$.*

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Corollary

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Proof.

Use part 2 of Glasner's Theorem for the reverse direction. For the forward direction, use the fact that every ultrafilter has the Ramsey property. $\square$

From local to global via ultrafilters

Theorem

Local ultrafilter shadowing implies ultrafilter shadowing.

Proof.

Let X be a dynamical system with local ultrafilter shadowing, let $\xi = \langle x_n \colon n \in \mathbb{N} \rangle$ be an arbitrary sequence in X, let p be an ultrafilter, and let $\varepsilon > 0$. Let $\{y_i \colon i \leq n\} \subseteq X$ be a finite set of points such that $X = \bigcup_{i \leq n} B_{\frac{\varepsilon}{2}}(y_i)$.

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Recurrence properties

- For any two subsets U and V of a dynamical system X, define $N(U,V) = \{n \in \mathbb{N} \colon f^n(U) \cap V \neq \emptyset\}$. Similarly, if $x \in X$ and $U \subseteq X$ define $N(x,U) = N(\{x\},U) = \{n \in \mathbb{N} \colon f^n(x) \in U\}$.

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- X has **strong uniform open set recurrence** if for every open $U \subseteq X$ there is some $x \in U$ such that $N(x,U)$ is syndetic.

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- X has **uniform open set recurrence** if for every open $U \subseteq X$, $N(U,U)$ is syndetic.
- X has **strong uniform open set recurrence** if for every open $U \subseteq X$ there is some $x \in U$ such that $N(x,U)$ is syndetic.
- X is **chain recurrent** if for every $x \in X$, every neighborhood U of X, and every $\delta > 0$, there is a finite δ-pseudo-orbit (of length 2 or more) that ends in U.

Ramsey shadowing as a recurrence property

Theorem

Strong uniform open set recurrence $\Rightarrow$
the Ramsey shadowing property $\Rightarrow$
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Ramsey shadowing as a recurrence property

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The proof is left as an exercise, with the following lemma as a hint:

Lemma

The following are equivalent for any $A \subseteq \mathbb{N}$*:*

1. A *is syndetic.*
2. *If* $\mathcal{F}$ *is any ultrafilter on* $\mathbb{N}$ *then there is some* $n \in \mathbb{N}$ *such that* $A - n \in \mathcal{F}$.
3. *If* $\mathcal{F}$ *is any family with the Ramsey property then there is some* $n \in \mathbb{N}$ *such that* $A - n \in \mathcal{F}$.

Equivalence with shadowing

Not unexpectedly, the situation becomes cleaner if shadowing is assumed:

Corollary

If X has the shadowing property, then the following are equivalent:

1. *strong uniform open set recurrence*
2. *the Ramsey shadowing property*
3. *uniform open set recurrence*
4. *chain recurrence*

This generalizes the aforementioned theorem of the author, Meddaugh, and Raines.

Tying up loose ends

Proposition

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Question

Is the Ramsey shadowing property equivalent to strong uniform open set recurrence (the existence of a dense set of minimal points)?

The End

Thank you for listening!

Any questions?