COMPACTIFICATIONS AND REPRESENTABILITY OF GROUPS

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  (In reality we shall only treat a few specific cases.)

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Definition. Given a topological group $\mathcal{G}$ a *semigroup compactification* (X, ψ) of G is defined as a pair, where X is a semigroup with a compact Hausdorff topology and $\psi : S \to X$ is a continuous homomorphism with dense image such that:

(**i.**) in X all right translates $x \longmapsto xy$ are continuous and

(**ii.**) the left translates $y \longmapsto \psi(s)y$ are continuous in X for all $s \in G$.

Definition. A semigroup compactification (X, ψ) of $\mathcal{G}$ is said to be universal w.r.t. a property $\mathcal{P}$ if

(i.) (X, ψ) has the property $\mathcal{P}$ and,

(ii.) whenever (Y, φ) is a semigroup compactification of $\mathcal{G}$ wich has the property $\mathcal{P}$, there exists a surjective continuous homomorphism $\pi : Y \to X$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\pi"] & Y \\ \mathcal{G}. \arrow[u, "\psi"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Example. If $\mathcal{G}$ is a discrete infinite group it is well known (...) that the Stone-Čech compactification $\beta\mathcal{G}$ with product defined by

$$x \cdot y := \lim_{s \to x} \lim_{t \to y} s + t$$

is the compactification of $\mathcal{G}$ which is universal w.r.t. the *joint continuity property.*

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- Given $s \in \mathcal{G}$ we denote by ρ_s [λ_s] the right [left] translation by s, defined by $\rho_s(t) := ts$ [$\lambda_s(t) := st$] for $t \in \mathcal{G}$.

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- Given a function $f : \mathcal{G} \to \mathcal{G}$ we write $\mathrm{L}_s(f)$ for $f \circ \lambda_s$.

- Given a subalgebra $\mathrm{F} \subseteq \mathrm{C}(\mathcal{G})$ we say that F is left introverted if for every $n \in \mathrm{F}^*$ and $f \in \mathrm{F}$ the function $n \cdot f$ defined by

$$(n \cdot f)(x) := \langle n, \mathrm{L}_x(f) \rangle \qquad (x \in \mathcal{G})$$

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(The algebra is *left–m–introverted* if this holds only for multiplicative functions.)

- Translation invariant subalgebras of $C(\mathcal{G})$ containing the constants which are left–m–introverted are called *admissible*.

Fact. There is a precise correspondance between admissible subalgebras of $\mathrm{C}(\mathcal{G})$ and universal semigroup compactifications of $\mathcal{G}$.

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- Given an admissible algebra F there exist a semigroup compactification $\mathcal{G}^{\mathrm{F}}$ which is universal with respect to the property that are precisely the functions in F which can be continuously extended to $\mathcal{G}^{\mathrm{F}}$.

A "concrete" way to construct $\mathcal{G}^{\mathrm{F}}$ is to regard it as the quotient $\beta(\mathcal{G}_d)/\sim$ where $x \sim y$ if, for every $f \in \mathrm{F}$ we have that $f^\beta(x) = f^\beta(y)$.

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Definition. The universal topological group compactification of $\mathcal{G}$ is called the AP–compactification (AP for almost periodic) and denoted by $\mathcal{G}^{\mathrm{AP}}$. The semigroup compactification of $\mathcal{G}$ universal w.r.t. the property of being a semitopological semigroup (i.e. having separately continuous multiplication) is called the WAP–compactification (WAP for weakly almost periodic) and denoted by $\mathcal{G}^{\mathrm{WAP}}$.

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The spaces $\mathrm{AP}(\mathcal{G})$ and $\mathrm{WAP}(G)$ have several interesting characterisations.

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- A function $f \in \mathrm{C}(\mathcal{G})$ is in $\mathrm{AP}(\mathcal{G})$ if and only if $L_{\mathcal{G}}(f)$ is relatively compact.

- (H. Bohr) A function in $C(\mathbf{R})$ is almost periodic if for every $\varepsilon > 0$ there esists $\ell_\varepsilon > 0$ such that interval of length ℓ_ε contains an element t such that $\|L_t(f) - f\| < \varepsilon$.

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- (Wilhelm Maak) A functions in $\mathrm{C}(\mathcal{G})$ is in $\mathrm{AP}(G)$ if for every $\varepsilon > 0$ there exists a finite cover of $\mathcal{G}$ with the property that, whenever there exist $s, t \in \mathcal{G}$ and $a, b \in \mathcal{G}$ such that asb and atb are in the same member of the cover, then:

$$|f(csd) - f(ctd)| < \varepsilon \qquad (c, d \in \mathcal{G}).$$

- Solutions of second order linear differential equation are in $\mathrm{AP}(\mathbf{R})$.

Definition. A group $\mathcal{G}$ is said to be *unitarily representable* if it can be embedded into the unitary group of some Hilbert space (endowed with the strong operator topology).

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(**iii**) $\mathrm{AP}(\mathcal{G})$ determines the topology of $\mathcal{G}$.

Definition. A group $\mathcal{G}$ is said to be *reflexively representable* if it can be embedded into the unitary group of some Hilbert space (endowed with the strong operator topology).

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- **(i)** $\mathcal{G}$ is reflexively representable;
- **(ii)** $\mathcal{G}$ embeds into $\mathcal{G}^{\mathrm{WAP}}$;
- **(iii)** $\mathrm{WAP}(\mathcal{G})$ determines the topology of $\mathcal{G}$.

Our contribution.

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If $\mathcal{G}$ is a non–compact SIN group, then $\mathcal{G}^{\mathrm{WAP}}$ contains a topological copy of $\beta\kappa \setminus \kappa$, where κ is the compact covering number of $\mathcal{G}$.

This led to the question of whether every SIN group (or at least every Abelian one) is reflexively representable.

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The proof follows the lines of Raynaud's proof that c_0 cannot be uniformly embedded into ℓ_2.

Lemma 1. Let $\mathcal{G}$ be a topological group, $\phi \in$WAP$(\mathcal{G})$ and fix integers $n > k \geq 1$. Let $\langle x_{i_1,\ldots,i_k}\rangle$ and $\langle y_{i_{k+1},\ldots,i_n}\rangle$ be two multi-indexed sequences in $\mathcal{G}$. Fix n free ultrafilters $p_1, \ldots, p_n$ on . If π is a shuffle with cut k, then

$$p_1\text{–}\lim_{i_1} \ldots p_n\text{–}\lim_{i_n} \phi(x_{i_1,\ldots,i_k} + y_{i_{k+1},\ldots,i_n}) =$$

$$= p_{\pi(1)}\text{–}\lim_{i_{\pi(1)}} \ldots p_{\pi(n)}\text{–}\lim_{i_{\pi(n)}} \phi(x_{i_1,\ldots,i_k} + y_{i_{k+1},\ldots,i_n}).$$

Lemma 2. Let $\mathcal{G}$ be a metrizable group equipped with a translation invariant metric d and let e denote its identity. If the embedding $w : \mathcal{G} \to \mathcal{G}$ is a homeomorphism, then for every $\varepsilon > 0$ there exists a continuous weakly almost periodic function ϕ_ε and some $\delta_\varepsilon > 0$ such that $\phi_\varepsilon(e) = 0$ and, for every $x \in \mathcal{G}$,

$$|\phi_\varepsilon(x)| < \delta_\varepsilon \text{ implies } d(x, e) < \varepsilon.$$

Theorem. (S.F. & J. Galindo) The aditive group c_0 is not reflexively representable.

We won't give a proof but it follows the lines of Raynaud's proof that c_0 cannot be uniformly embedded into ℓ_2.

Proof of Theorem. Suppose towards a contradiction that c_0 embeds into its WAP-compactification and consider the weakly almost periodic function ϕ and the positive number $\delta > 0$ determined by $\varepsilon = 1/2$ in Lemma 1.

Since ϕ is continuous, there will be some $\alpha > 0$ such that $\|x - y\|_\infty < \alpha$ implies $|\phi(x - y)| < \delta/2$.

Fix k such that $1/k < \alpha$ and consider the vectors $s_n \in c_0$ defined so that their first n coordinates are $1/k$ and the rest are 0. The inequality $\| \sum_{j=1}^{2k} (-1)^j s_j \|_\infty < \alpha$ then holds.

Clearly, the same inequality will hold for every sequence of indices $n_1 < n_2 < \ldots < n_{2k}$:

$$\Big\|\sum_{j=1}^{k} s_{n_{2j}} - \sum_{j=1}^{k} s_{n_{2j-1}}\Big\|_\infty < \alpha.$$

We have thus that, for every $n_1, \ldots, n_{2k}$ (with $n_1 < n_2 < \ldots < n_{2k}$)

$$\Big|\phi\Big(\sum_{j=1}^{k} s_{n_{2j}} - \sum_{j=1}^{k} s_{n_{2j-1}}\Big)\Big| < \delta/2.$$

Taking p-limits along free ultrafilters $p_1, \ldots, p_{2k}$, we get

$$\Big| p_1\text{–}\lim_{n_1} p_2\text{–}\lim_{n_2} \ldots p_{2k}\text{–}\lim_{n_{2k}} \phi\Big(\sum_{j=1}^{k} s_{n_{2j}} - \sum_{j=1}^{k} s_{n_{2j-1}}\Big)\Big| \leq \delta/2.$$

The permutation sending $1 \to 1$, $2 \to 3$, $3 \to 5$, $\ldots$,k$\to$ $2k-1$, $k+1 \to 2$, $k+2 \to 4$,$\ldots$,2k$\to$ $2k$ is a shuffle.

By Lemma 2 (recall that ϕ is weakly almost periodic) the above limit equals

$$p_1\text{–}\lim_{n_1} p_3\text{–}\lim_{n_3} p_5\text{–}\lim_{n_5} \dots p_{2k-1}\text{–}\lim_{n_{2k-1}} p_2\text{–}\lim_{n_2} \dots$$

$$\dots p_{2k}\text{–}\lim_{n_{2k}} \Big| \phi\Big(\sum_{j=1}^{k} s_{n_{2j}} - \sum_{j=1}^{k} s_{n_{2j-1}} \Big) \Big|.$$

Hence for large enough $n_1 < n_3 < n_5 < \ldots < n_{2k-1} < n_2 < n_4 < \ldots < n_{2k}$ we have that

$$\left|\phi\Big(\sum_{j=1}^{k} s_{n_{2j}} - \sum_{j=1}^{k} s_{n_{2j-1}}\Big)\right| < \delta.$$

The election of ϕ and δ implies that

$$\Big\|\sum_{j=1}^{k} s_{n_{2j}} - \sum_{j=1}^{k} s_{n_{2j-1}}\Big\|_{\infty} < 1/2.$$

But, taking into account that $n_1 < n_3 < n_5 < \ldots < n_{2k-1} < n_2 < n_4 < \ldots < n_{2k}$, a moment's reflection shows that

$$\Big\|\sum_{j=1}^{k} s_{n_{2j}} - \sum_{j=1}^{k} s_{n_{2j-1}}\Big\|_\infty = \Big\|\sum_{j=1}^{k} s_j - \sum_{j=k+1}^{2k} s_j\Big\|_\infty = 1.$$

This is the desired contradiction.

Definition. A metric on $\mathcal{G}$ is *stable* if for every sequences $\langle x_n \rangle_n$, $\langle y_n \rangle_n$ in $\mathcal{G}$ and every pair of ultrafilters $p, q \in \beta$ we have that

$$p\text{–}\lim_n q\text{–}\lim_m d(x_n, y_m) = q\text{–}\lim_m p\text{–}\lim_n d(x_n, y_m).$$

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Corollary. The aditive groups T and J are not reflexively representable.