Completeness properties of initial topologies

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A familiar line of investigation

Question schema

Let $f: X \to X$ be a self-map on a set X. Can I put a nice topology (or more generally, structure) on X which makes f continuous (or, more generally, structure preserving)?

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e.g. nice $=$ compact Hausdorff.

A familiar theorem

Theorem (Stone-Weierstrass)

Let X be a compact Hausdorff space, and $\mathcal{F}$ a ring of continuous functions $X \to \mathbb{R}$. If

1. *Every constant map is in $\mathcal{F}$;*
2. *For each pair of distinct points x, y in X there is an $f \in \mathcal{F}$ with $f(x) \neq f(y)$;*
3. *$\mathcal{F}$ is closed under uniformly convergent sequences;*

then $\mathcal{F} = C(X, \mathbb{R})$.

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then $\mathcal{F} = C(X, \mathbb{R})$.

Clearly each of these conditions is necessary.

A related question

Question

Suppose $\mathcal{F}$ is a collection of functions $X \to \mathbb{R}$, where X is a set. Let $\tau_{\mathcal{F}}$ denote the initinal topology on X generated by the functions in $\mathcal{F}$. When do we have $\mathcal{F} = C((X, \tau_{\mathcal{F}}), \mathbb{R})$?

The obvious necessary conditions

We clearly need a few obvious conditions on $\mathcal{F}$. These are encapsulated in the definition below. The definition says "$\mathcal{F}$ behaves like a set of continuous functions $X \to \mathbb{R}$" (on a functionally Hausdorff topology on X).

The obvious necessary conditions

We clearly need a few obvious conditions on $\mathcal{F}$. These are encapsulated in the definition below. The definition says "$\mathcal{F}$ behaves like a set of continuous functions $X \to \mathbb{R}$" (on a functionally Hausdorff topology on X).

Definition

Let X be a set and $\mathcal{F} \subseteq \mathbb{R}^X$. We say $\mathcal{F}$ is a real functional subring (on X) if

1. $\mathcal{F}$ is closed under pointwise addition, multiplication and subtraction (so in particular $\mathcal{F}$ forms an algebraic ring), as well as $f \mapsto |f|$;
2. $\mathcal{F}$ separates the points of X (i.e. for $x, y \in X$ with $x \neq y$ there is an $f \in \mathcal{F}$ with $f(x) \neq f(y)$);
3. $\mathcal{F}$ contains all the constant maps.
4. $\mathcal{F}$ is closed under uniformly convergent sequences.

Given a real functional subring on a set X, we denote the initial topology on X generated by $\mathcal{F}$ as $\tau_{\mathcal{F}}$.

Refined question

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But also...

Question

Given a real functional subring $\mathcal{F}$ on X, when is the initial topology $\tau_{\mathcal{F}}$ "nice"?

Pseudocompactness

Definition

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Questions

Suppose $\mathcal{F}$ is a real functional subring on X of functions with compact image. Is

- $(X, \tau_{\mathcal{F}})$ pseudocompact?
- $C((X, \tau_{\mathcal{F}}), \mathbb{R}) = \mathcal{F}$?

Answers

Theorem

No.

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More precisely.

Theorem

There is a real functional subring on a set X of functions with compact image such that $(X, \tau_{\mathcal{F}})$ is not pseudocompact.

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First example due to Suabedissen, an additional one found by L and Pitz. Maybe more are known!

Do we get *any* completeness like properties?

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Given a real functional subring $\mathcal{F}$ on X of functions with compact image, does $\tau_{\mathcal{F}}$ have any completeness properties?

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Answer

Yes. Its Baire.

Outline

Theorem

If $\mathcal{F}$ consists of maps with compact image then $(X, \tau_{\mathcal{F}})$ has a compactification with no non-trivial G_δ sets.

Non-trivial means here that any non-empty G_δ in the compactification meets the space X.

Outline

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Theorem (Oxtoby)

A space X is Baire if and only if Player I does not have a winning strategy in the Choquet game on X.

A compactification

Lemma

Suppose $\mathcal{F}$ is a real functional subring on X. Then $(X, \tau_{\mathcal{F}})$ is Tychonoff. In particular, the set

$$\mathcal{B} = \left\{ f^{-1}(U) : U \ \mathit{open\ in}\ [0,1],\ f \in \mathcal{F} \cap [0,1]^X \right\}$$

forms a base for $(X, \tau_{\mathcal{F}})$.

A compactification

Definition

Given a real functional subring $\mathcal{F}$ on a set X, write $\mathcal{F}_b$ for $\mathcal{F} \cap [0,1]^X$. The above lemma says that the diagonal map

$$\mathop{\Delta}_{f \in \mathcal{F}_b} f : X \to [0,1]^{\mathcal{F}_b}$$

is an embedding. This gives us a compactification of $(X, \tau_{\mathcal{F}})$, which we denote $\mathcal{F}X$, given by

$$\left(\mathop{\Delta}_{f \in \mathcal{F}_b} f, \overline{\left(\mathop{\Delta}_{f \in \mathcal{F}_b} f \right)(X)}^{[0,1]^{\mathcal{F}_b}} \right).$$

Observe also that the projection map π_f for $f \in \mathcal{F}_b$ is a continuous extension of f to $\mathcal{F}X$. When we write π_f we will mean $\pi_f \restriction \mathcal{F}X$, and not the map on the appropriate Tychonoff cube (unless noted otherwise).

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Theorem

Suppose $\mathcal{F}$ is a real functional subring on the set X. Then every $f \in \mathcal{F}_b$ has compact image if and only if $\mathcal{F}X \setminus X$ contains no non-empty G_δ subsets of $\mathcal{F}X$.

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We use the following technical lemma:

Lemma

Suppose $x \in \mathcal{F}X$ and C is closed in $\mathcal{F}X$ with $x \notin C$. Suppose $n \in \omega$. Then there is an $f \in \mathcal{F}_b$ with $ran(f) \subseteq \left[0, \frac{1}{2^n}\right]$

$$\pi_f(C) \subseteq \{0\}$$

and

$$\pi_f(x) = \frac{1}{2^n}.$$

Some details of the trickier direction

If $G = \bigcap_{n \in \omega} G_n$ is a non-trivial G_δ and $x \in G$, let for each $n \in \omega$, $f_n \in \mathcal{F}_b$ be such that

- $\mathrm{ran}\,(f_n) \subseteq \left[0, \frac{1}{2^n}\right]$
- $\pi_{f_n}(\mathcal{F}X \setminus G_n) \subseteq \{0\}$
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Let $f = \sum_{n\in\omega} f_n \in \mathcal{F}$. Observe that $\pi_f = \pi_{\sum_{n\in\omega} f_n} = \sum_{n\in\omega} \pi_{f_n}$ since the two functions agree on the dense set X.

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f does not have compact image!

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Proof idea

Transfer a winning strategy from the Choquet game on $\mathcal{F}X$.

Some details

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But this non-empty intersection meets X (its a G_δ!). But intersecting each element of our σ-compatible sequence with X is just our original μ-compatible sequence.

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But this non-empty intersection meets X (its a G_δ!). But intersecting each element of our σ-compatible sequence with X is just our original μ-compatible sequence.
Hence μ wins for Player II.

Some questions

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Thanks for listening!